Chapter 7 Generalized Conforming Thin Plate Element Ⅲ— Perimeter-Point and Least-Square Conforming Schemes

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Abstract This chapter discusses the last two groups of the construction schemes for the generalized conforming thin plate element: perimeter-point conforming scheme and least-square conforming scheme. Five triangular and rectangular element models formulated by these schemes are presented in detail. Numerical examples show that these generalized conforming models also exhibit excellent performance in the analysis of thin plates. Furthermore, the generalized conforming element theory is applied to verify or improve the convergence of two famous non-conforming element models, ACM and BCIZ, and some valuable conclusions are obtained.

Keywords thin plate element, generalized conforming, perimeter-point conforming, least-square conforming.

7.1 Perimeter-Point Conforming Scheme—Elements LR12-1 and LR12-2

This chapter will take the rectangular generalized conforming elements LR12-1 and LR12-2[1] as the examples to illustrate the procedure for the combination scheme of perimeter and point conforming conditions. These two elements are both elements with $m = n = 12$: the number of the element DOFs $n = 12$, and the number of the unknown coefficients in the deflection field $m = 12$. The selected 12 conforming conditions include 3 point conforming conditions and 9 perimeter conforming conditions.

7.1.1 Element LR12-1

This rectangular thin plate element is also shown in Fig. 6.11. The element nodal displacement vector $\boldsymbol{q}^e$ is still composed of w_i, ψ_{xi}, ψ_{yi} ($i = 1,2,3,4$) at four corner nodes. And, the element deflection field w is assumed to be an incomplete quartic polynomial with 12 unknown coefficients, as shown in Eqs. (6-22) and (6-23), i.e.,

$$w = \boldsymbol{F}_\lambda \boldsymbol{\lambda} \tag{7-1}$$

where

$$\left.\begin{aligned} \boldsymbol{\lambda} &= [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5 \quad \lambda_6 \quad \lambda_7 \quad \lambda_8 \quad \lambda_9 \quad \lambda_{10} \quad \lambda_{11} \quad \lambda_{12}]^{\mathrm{T}} \\ \boldsymbol{F}_\lambda &= [1 \quad \xi \quad \eta \quad \xi^2 \quad \xi\eta \quad \eta^2 \quad \xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \quad \xi^3\eta \quad \xi\eta^3] \end{aligned}\right\} \tag{7-2}$$

In order to solve $\boldsymbol{\lambda}$, 12 conforming conditions are needed.

Firstly, 3 conforming conditions for the corner nodal deflections can be established:

$$\left.\begin{aligned} \sum_{i=1}^{4} w_i &= 4(\lambda_1 + \lambda_4 + \lambda_6) \\ \sum_{i=1}^{4} w_i \xi_i &= 4(\lambda_2 + \lambda_7 + \lambda_9) \\ \sum_{i=1}^{4} w_i \eta_i &= 4(\lambda_3 + \lambda_8 + \lambda_{10}) \end{aligned}\right\} \tag{7-3}$$

Secondly, the perimeter conforming condition (5-2c) is used, i.e.,

$$\iint_{A^e} \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \mathrm{d}A = \oint_{\partial A^e} (M_n \tilde{\psi}_n + M_{ns} \tilde{\psi}_s - Q_n \tilde{w}) \mathrm{d}s \tag{7-4}$$

where the weighting functions M_n, M_{ns} and Q_n are the boundary forces (bending moment, twisting moment and transverse shear force); M_x, M_y and M_{xy} are the internal moments within the element domain, which are assumed to satisfy the homogeneous equilibrium Eq. (5-3), i.e.,

$$\left.\begin{aligned} &\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = 0 \\ &Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \\ &Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \end{aligned}\right\} \tag{7-5}$$

For the rectangular elements, Eq. (7-4) can be written as

$$\iint_{A^e} \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \mathrm{d}A$$

$$= a\int_{-1}^{1}[(M_y\tilde{\psi}_y + M_{xy}\tilde{\psi}_x - Q_y\tilde{w})_{43} - (M_y\tilde{\psi}_y + M_{xy}\tilde{\psi}_x - Q_y\tilde{w})_{12}]\mathrm{d}\xi$$
$$+ b\int_{-1}^{1}[(M_x\tilde{\psi}_x + M_{xy}\tilde{\psi}_y - Q_x\tilde{w})_{23} - (M_x\tilde{\psi}_x + M_{xy}\tilde{\psi}_y - Q_x\tilde{w})_{14}]\mathrm{d}\eta \qquad (7\text{-}6)$$

The following equilibrium internal force fields

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi & \eta & \xi\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \dfrac{1}{a} & 0 & \dfrac{1}{a}\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{b} & \dfrac{1}{b}\xi & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{Bmatrix} \qquad (7\text{-}7)$$

are adopted, where β_1, β_2, $\cdots$, β_9 are 9 arbitrary parameters.

Since β_i is an arbitrary parameter, substituting Eq. (7-7) into Eq. (7-6), the 9 conforming conditions can be obtained as follows

$$\left.\begin{aligned}
&\iint_{A^e}\frac{\partial^2 w}{\partial x^2}\mathrm{d}A = b\int_{-1}^{1}(\tilde{\psi}_{x23} - \tilde{\psi}_{x14})\mathrm{d}\eta \\
&\iint_{A^e}\xi\frac{\partial^2 w}{\partial x^2}\mathrm{d}A = b\int_{-1}^{1}(\tilde{\psi}_{x23} + \tilde{\psi}_{x14})\mathrm{d}\eta - \frac{b}{a}\int_{-1}^{1}(\tilde{w}_{23} - \tilde{w}_{14})\mathrm{d}\eta \\
&\iint_{A^e}\eta\frac{\partial^2 w}{\partial x^2}\mathrm{d}A = b\int_{-1}^{1}(\tilde{\psi}_{x23} - \tilde{\psi}_{x14})\eta\mathrm{d}\eta \\
&\iint_{A^e}\xi\eta\frac{\partial^2 w}{\partial x^2}\mathrm{d}A = b\int_{-1}^{1}(\tilde{\psi}_{x23} + \tilde{\psi}_{x14})\eta\mathrm{d}\eta - \frac{b}{a}\int_{-1}^{1}(\tilde{w}_{23} - \tilde{w}_{14})\eta\mathrm{d}\eta \\
&\iint_{A^e}\frac{\partial^2 w}{\partial y^2}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{y43} - \tilde{\psi}_{y12})\mathrm{d}\xi \\
&\iint_{A^e}\xi\frac{\partial^2 w}{\partial y^2}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{y43} - \tilde{\psi}_{y12})\xi\mathrm{d}\xi \\
&\iint_{A^e}\eta\frac{\partial^2 w}{\partial y^2}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{y43} + \tilde{\psi}_{y12})\mathrm{d}\xi - \frac{a}{b}\int_{-1}^{1}(\tilde{w}_{43} - \tilde{w}_{12})\mathrm{d}\xi \\
&\iint_{A^e}\xi\eta\frac{\partial^2 w}{\partial y^2}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{y43} + \tilde{\psi}_{y12})\xi\mathrm{d}\xi - \frac{a}{b}\int_{-1}^{1}(\tilde{w}_{43} - \tilde{w}_{12})\xi\mathrm{d}\xi \\
&2\iint_{A^e}\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{x43} - \tilde{\psi}_{x12})\mathrm{d}\xi + b\int_{-1}^{1}(\tilde{\psi}_{y23} - \tilde{\psi}_{y14})\mathrm{d}\eta
\end{aligned}\right\} \qquad (7\text{-}8)$$

Then, λ_4, λ_5, $\cdots$, λ_{12} can be solved in turn from Eq. (7-8) as follows

$$\left.\begin{aligned}
\lambda_4 &= \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i \\
\lambda_7 &= -\frac{1}{8}\sum_{i=1}^{4}w_i\xi_i + \frac{a}{8}\sum_{i=1}^{4}\psi_{xi} + \frac{b}{24}\sum_{i=1}^{4}\psi_{yi}\xi_i\eta_i \\
\lambda_8 &= \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i\eta_i \\
\lambda_{11} &= -\frac{3}{20}\sum_{i=1}^{4}w_i\xi_i\eta_i + \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\eta_i + \frac{b}{40}\sum_{i=1}^{4}\psi_{yi}\xi_i \\
\lambda_6 &= \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i \\
\lambda_9 &= \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i\eta_i \\
\lambda_{10} &= -\frac{1}{8}\sum_{i=1}^{4}w_i\eta_i + \frac{a}{24}\sum_{i=1}^{4}\psi_{xi}\xi_i\eta_i + \frac{b}{8}\sum_{i=1}^{4}\psi_{yi} \\
\lambda_{12} &= -\frac{3}{20}\sum_{i=1}^{4}w_i\xi_i\eta_i + \frac{a}{40}\sum_{i=1}^{4}\psi_{xi}\eta_i + \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i \\
\lambda_5 &= \frac{11}{20}\sum_{i=1}^{4}w_i\xi_i\eta_i - \frac{3a}{20}\sum_{i=1}^{4}\psi_{xi}\eta_i - \frac{3b}{20}\sum_{i=1}^{4}\psi_{yi}\xi_i
\end{aligned}\right\} \tag{7-9}$$

Substituting the above equation into Eq. (7-3), λ_1, λ_2 and λ_3 can be obtained

$$\left.\begin{aligned}
\lambda_1 &= \frac{1}{4}\sum_{i=1}^{4}w_i - \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i - \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i \\
\lambda_2 &= \frac{3}{8}\sum_{i=1}^{4}w_i\xi_i - \frac{a}{8}\sum_{i=1}^{4}\psi_{xi} - \frac{b}{6}\sum_{i=1}^{4}\psi_{yi}\xi_i\eta_i \\
\lambda_3 &= \frac{3}{8}\sum_{i=1}^{4}w_i\eta_i - \frac{a}{6}\sum_{i=1}^{4}\psi_{xi}\xi_i\eta_i - \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}
\end{aligned}\right\} \tag{7-10}$$

Thus, all the coefficients in $\boldsymbol{\lambda}$ have been obtained. Then, the shape functions and the element stiffness matrix can be derived from them.

7.1.2 Element LR12-2

The construction procedure of this element is basically similar to that of the element LR12-1. Only the selected equilibrium internal force fields are different from Eq. (7-7), and replaced by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi^2 & \eta^2 \\ 0 & \dfrac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{2}{b}\eta \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{b} & 0 & \dfrac{2}{a}\xi & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{Bmatrix} \tag{7-11}$$

Following a similar procedure, all the coefficients in $\boldsymbol{\lambda}$ can be obtained, in which λ_1, λ_2, λ_3, λ_4, λ_6, λ_7, λ_8, λ_9, λ_{10} are the same, and still given by Eqs. (7-9) and (7-10); the other three unknown coefficients are as follows

$$\left.\begin{aligned} \lambda_5 &= \frac{1}{2}\sum_{i=1}^{4} w_i\xi_i\eta_i - \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\eta_i - \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i \\ \lambda_{11} &= -\frac{1}{8}\sum_{i=1}^{4} w_i\xi_i\eta_i + \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\eta_i \\ \lambda_{12} &= -\frac{1}{8}\sum_{i=1}^{4} w_i\xi_i\eta_i + \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i \end{aligned}\right\} \tag{7-12}$$

Example 7.1 Central deflection and central moment of the simply-supported and the clamped square plates (the side length is L) subjected to uniform load.

The results by the elements LR12-1 and LR12-2 are listed in Tables 7.1 and 7.2. For comparison, the results by the element ACM[2] are also given. The Poisson's ratio is 0.3.

Table 7.1 The central deflection of square plates subjected to uniform load

Mesh (1/4 plate)	Simply-supported			Clamped		
	LR12-1	LR12-2	ACM	LR12-1	LR12-2	ACM
2×2	0.4051 (−0.3%)	0.4052 (−0.3%)	0.3939 (−3.0%)	0.1238 (−2.0%)	0.1243 (−1.7%)	0.1403 (11.0%)
4×4	0.406 16 (−0.02%)	0.406 17 (−0.02%)	0.4033 (−0.7%)	0.1260 (−0.4%)	0.1261 (−0.4%)	0.1304 (4.0%)
8×8	0.406 23 (−0.001%)	0.406 23 (−0.001%)	0.4056 (−0.2%)	0.126 45 (−0.06%)	0.126 46 (−0.05%)	0.1275 (0.8%)
Analytical	0.406 235($qL^4/100D$)			0.126 53($qL^4/100D$)		

Table 7.2 The central moment of square plates subjected to uniform load

Mesh (1/4 plate)	Simply-supported			Clamped		
	LR12-1	LR12-2	ACM	LR12-1	LR12-2	ACM
2×2	0.512 45 (7.0%)	0.512 23 (7.0%)	0.521 69 (8.9%)	0.255 23 (11.4%)	0.253 23 (10.5%)	0.277 83 (11.3%)
4×4	0.487 30 (1.8%)	0.487 32 (1.8%)	0.489 20 (2.2%)	0.236 89 (3.4%)	0.236 96 (3.4%)	0.240 50 (5.0%)
8×8	0.480 98 (0.4%)	0.480 98 (0.4%)	0.481 66 (0.6%)	0.231 09 (0.8%)	0.231 10 (0.8%)	0.231 91 (1.2%)
Analytical	0.478 86($qL^2/10$)			0.229 05($qL^2/10$)		

7.2 The Application of Perimeter Conforming Conditions —Verification for the Convergence of the Element ACM

This section will use the perimeter conforming conditions under the constant stress state to verify the convergence of the non-conforming elements. And, as a typical example, the convergence of the well-known element ACM[2] will be verified.

The element ACM is a non-conforming rectangular thin plate element, which is constructed by the conventional point conforming scheme, i.e., 12 unknown coefficients are determined by 12 point conforming conditions about w, ψ_x, ψ_y at the corner nodes. It can be seen from the deflection field finally determined that the deflection w along the element boundary is exactly compatible, while the normal slope $\frac{\partial w}{\partial n}$ is not. Though the element ACM belongs to the non-conforming elements, it still exhibits good convergence in applications and has been proved in theory that it can pass the patch test.

By starting from the generalized conforming theory, this section will verify the convergence of the element ACM from another point of view. At the same time, this example can also be used to illustrate one of the applications of the generalized conforming theory, that is, the generalized conforming theory can be used to verify or improve the convergence of other non-conforming elements.

7.2.1 Derivation of Element ACM from Symmetry

The rectangular thin plate element ACM, proposed by Adini, Clough and Melosh[2], is a non-conforming element with 12 DOFs. Its element deflection field is assumed to be an incomplete quartic polynomial, as shown in Eqs. (7-1) and (7-2) (i.e. Eqs. (6-22) and (6-23)), which involves 12 unknown coefficients that will be determined by 12 point conforming conditions about w, ψ_x and ψ_y at the corner nodes. Since the solution procedure for the 12 unknown coefficients is quite complicated, here we will simplify it by using the symmetry (refer to Sect. 6.2).

Firstly, the 12 unknown coefficients and their basis functions in Eq. (7-1) have already been classified as 4 groups. And, each group contains 3 unknown coefficients, in turn; they are (λ_1, λ_4, λ_6); (λ_2, λ_7, λ_9); (λ_3, λ_8, λ_{10}); (λ_5, λ_{11}, λ_{12}), as shown in Eq. (6-28).

Secondly, the 12 point conforming conditions at the corner nodes can be recombined. They are classified as 4 independent groups, and each group contains 3 equations and 3 unknown coefficients. Thereupon, the original simultaneous equations with 12 unknowns decompose to be four independent equation groups each with 3 unknowns, which greatly simplifies the problem. The 4 equation groups are listed as follows.

(1) Combination conditions belonging to the SS group (3 conditions)

$$\sum_{i=1}^{4}(w-\tilde{w})_i = 0 \tag{7-A1}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial x}-\tilde{\psi}_x\right)_i \xi_i = 0 \tag{7-A2}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial y}-\tilde{\psi}_y\right)_i \eta_i = 0 \tag{7-A3}$$

Substitution of Eq. (7-1) into the above equations yields

$$\lambda_1+\lambda_4+\lambda_6 = \frac{1}{4}\sum_{i=1}^{4} w_i \tag{7-13a}$$

$$\lambda_4 = \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i \tag{7-13b}$$

$$\lambda_6 = \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i \tag{7-13c}$$

(2) Combination conditions belonging to the SA group (3 conditions)

$$\sum_{i=1}^{4}(w-\tilde{w})_i\xi_i = 0 \tag{7-B1}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial x}-\tilde{\psi}_x\right)_i = 0 \tag{7-B2}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial y}-\tilde{\psi}_y\right)_i \xi_i \eta_i = 0 \tag{7-B3}$$

We can obtain

$$\lambda_2 + \lambda_7 + \lambda_9 = \frac{1}{4}\sum_{i=1}^{4} w_i \xi_i \tag{7-14a}$$

$$\lambda_2 + 3\lambda_7 + \lambda_9 = \frac{a}{4}\sum_{i=1}^{4} \psi_{xi} \tag{7-14b}$$

$$\lambda_9 = \frac{b}{8}\sum_{i=1}^{4} \psi_{yi} \xi_i \eta_i \tag{7-14c}$$

(3) Combination conditions belonging to the AS group (3 conditions)

$$\sum_{i=1}^{4}(w-\tilde{w})_i \eta_i = 0 \tag{7-C1}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial x}-\tilde{\psi}_x\right)_i \xi_i \eta_i = 0 \tag{7-C2}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial y}-\tilde{\psi}_y\right)_i = 0 \tag{7-C3}$$

We can obtain

$$\lambda_3 + \lambda_8 + \lambda_{10} = \frac{1}{4}\sum_{i=1}^{4} w_i \eta_i \tag{7-15a}$$

$$\lambda_8 = \frac{a}{8}\sum_{i=1}^{4} \psi_{xi} \xi_i \eta_i \tag{7-15b}$$

$$\lambda_3 + \lambda_8 + 3\lambda_{10} = \frac{b}{4}\sum_{i=1}^{4} \psi_{yi} \tag{7-15c}$$

(4) Combination conditions belonging to the AA group (3 conditions)

$$\sum_{i=1}^{4}(w-\tilde{w})_i \xi_i \eta_i = 0 \tag{7-D1}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial x}-\tilde{\psi}_x\right)_i \eta_i = 0 \tag{7-D2}$$

$$\sum_{i=1}^{4}\left(\frac{\partial w}{\partial y}-\tilde{\psi}_y\right)_i \xi_i = 0 \tag{7-D3}$$

We can obtain

$$\lambda_5+\lambda_{11}+\lambda_{12}=\frac{1}{4}\sum_{i=1}^{4}w_i\xi_i\eta_i \tag{7-16a}$$

$$\lambda_5+3\lambda_{11}+\lambda_{12}=\frac{a}{4}\sum_{i=1}^{4}\psi_{xi}\eta_i \tag{7-16b}$$

$$\lambda_5+\lambda_{11}+3\lambda_{12}=\frac{b}{4}\sum_{i=1}^{4}\psi_{yi}\xi_i \tag{7-16c}$$

From the above four groups of simultaneous equations with 3 unknowns, the solutions can be easily obtained:

$$\left.\begin{aligned}
\lambda_1&=\frac{1}{4}\sum_{i=1}^{4}w_i-\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i-\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i\\
\lambda_4&=\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i\\
\lambda_6&=\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i\\
\lambda_2&=\frac{3}{8}\sum_{i=1}^{4}w_i\xi_i-\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}-\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i\eta_i\\
\lambda_7&=-\frac{1}{8}\sum_{i=1}^{4}w_i\xi_i+\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\\
\lambda_9&=\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i\eta_i\\
\lambda_3&=\frac{3}{8}\sum_{i=1}^{4}w_i\eta_i-\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i\eta_i-\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\\
\lambda_8&=\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i\eta_i\\
\lambda_{10}&=-\frac{1}{8}\sum_{i=1}^{4}w_i\eta_i+\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\\
\lambda_5&=\frac{1}{2}\sum_{i=1}^{4}w_i\xi_i\eta_i-\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\eta_i-\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i\\
\lambda_{11}&=-\frac{1}{8}\sum_{i=1}^{4}w_i\xi_i\eta_i+\frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\eta_i\\
\lambda_{12}&=-\frac{1}{8}\sum_{i=1}^{4}w_i\xi_i\eta_i+\frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\xi_i
\end{aligned}\right\} \tag{7-17}$$

Finally, substitution of the above solutions into Eq. (7-1) yields the element deflection field and its shape functions:

$$w = \sum_{i=1}^{4}(N_i w_i + N_{xi}\psi_{xi} + N_{yi}\psi_{yi}) \tag{7-18}$$

where the shape functions are

$$\left.\begin{aligned} N_i &= \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(2+\xi_i\xi+\eta_i\eta-\xi^2-\eta^2) \\ N_{xi} &= -\frac{a}{8}\xi_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\xi^2) \\ N_{yi} &= -\frac{b}{8}\eta_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\eta^2) \end{aligned}\right\} \quad (i=1,2,3,4) \tag{7-19}$$

Once the shape functions are obtained, the element stiffness matrix can be derived following the conventional procedure.

7.2.2 The Fundamental Conforming Conditions for Verifying Convergence

According to the generalized conforming theory, the fundamental conditions that ensure convergence are Eq. (4-7) or Eq. (4-8), i.e.,

$$H = 0 \quad \text{(corresponding to constant strain and rigid body displacement states)} \tag{7-20}$$

The above equation is called the fundamental generalized conforming conditions.

Under the limit state that a mesh is refined by the infinite elements, the element strain will tend to be constant. For the thin plate bending problem, the element displacement field in the limit state involves only 6 DOFs, i.e., 3 corresponding to rigid body displacement modes and 3 corresponding to constant strain states. Thereby, the fundamental generalized conforming condition (7-20) should be composed of 6 conforming conditions.

The 3 conditions which the element should satisfy in the rigid body displacement mode can be selected as

$$\sum_{i=1}^{p}(w-\tilde{w})_i = 0, \quad \sum_{i=1}^{p}(w-\tilde{w})_i x_i = 0, \quad \sum_{i=1}^{p}(w-\tilde{w})_i y_i = 0 \tag{7-21}$$

They denote the 3 combination conditions of the point conforming conditions about deflections at the corner nodes, where p is the number of the corner nodes. Besides, they can also be selected as

$$\sum_{i=1}^{p}(w-\tilde{w})_i=0,\quad \sum_{i=1}^{p}\left(\frac{\partial w}{\partial x}-\psi_x\right)_i=0,\quad \sum_{i=1}^{p}\left(\frac{\partial w}{\partial y}-\psi_y\right)_i=0 \tag{7-22}$$

They denote the combination point conforming conditions about w, ψ_x and ψ_y at the corner nodes. For the rigid body displacement modes, the above two sets of equations are equivalent to each other; but for non-rigid body displacement modes, they are not equivalent anymore.

When an element is under constant strain states, the perimeter conforming condition (5-2c) should also be satisfied, in which the weighting functions can be selected according to the constant internal force states. If internal moments are constants and transverse shear forces are zero, Eq. (5-2c) will be simplified as:

$$M_x\iint_{A^e}\frac{\partial^2 w}{\partial x^2}\mathrm{d}A+M_y\iint_{A^e}\frac{\partial^2 w}{\partial y^2}\mathrm{d}A+2M_{xy}\iint_{A^e}\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}A-\sum_{i=1}^{p}(M_{ni}\int_0^{d_i}\tilde{\psi}_{ni}\mathrm{d}s+M_{nsi}\int_0^{d_i}\tilde{\psi}_{si}\mathrm{d}s)=0 \tag{7-23}$$

If the constant internal force fields are assumed to be those in Eq. (5-24), the perimeter conforming conditions given in Eq. (5-26) are obtained, i.e.,

$$\left.\begin{aligned}
&\iint_{A^e}\frac{\partial^2 w}{\partial x^2}\mathrm{d}A=\oint_{\partial A^e}(l^2\tilde{\psi}_n-lm\tilde{\psi}_s)\mathrm{d}s\\
&\iint_{A^e}\frac{\partial^2 w}{\partial y^2}\mathrm{d}A=\oint_{\partial A^e}(m^2\tilde{\psi}_n+lm\tilde{\psi}_s)\mathrm{d}s\\
&\iint_{A^e}2\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}A=\oint_{\partial A^e}[2lm\tilde{\psi}_n+(l^2-m^2)\tilde{\psi}_s]\mathrm{d}s
\end{aligned}\right\} \tag{7-24}$$

The 6 conditions given in Eq. (7-23) or Eq. (7-24) and Eq. (7-21) or Eq. (7-22) are the fundamental conforming conditions for verifying the convergence of the non-conforming thin plate elements.

7.2.3 Verification for the Convergence of Element ACM

Now, we use the 6 fundamental conforming conditions to verify the convergence of the element ACM.

Firstly, the element ACM has already satisfied the 12 point conforming conditions about w, ψ_x and ψ_y at the corner nodes, thereby, Eq. (7-21) or Eq. (7-22) are satisfied naturally. In fact, the 3 conditions (7-A1), (7-B1) and (7-C1) selected previously are the same as those expressions in Eq. (7-21); and another 3 conditions (7-A1), (7-B2) and (7-C3) are the same as those expressions in Eq. (7-22).

Secondly, verification for the perimeter conforming conditions (7-24) is performed. For the rectangular elements, Eq. (7-24) can be written as

$$\left.\begin{aligned}&\iint_{A^e}\frac{\partial^2 w}{\partial x^2}\mathrm{d}A = b\int_{-1}^{1}(\tilde{\psi}_{x23}-\tilde{\psi}_{x14})\mathrm{d}\eta\\&\iint_{A^e}\frac{\partial^2 w}{\partial y^2}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{y43}-\tilde{\psi}_{y12})\mathrm{d}\xi\\&\iint_{A^e}2\frac{\partial^2 w}{\partial x\partial y}\mathrm{d}A = a\int_{-1}^{1}(\tilde{\psi}_{x43}-\tilde{\psi}_{x12})\mathrm{d}\xi + b\int_{-1}^{1}(\tilde{\psi}_{y23}-\tilde{\psi}_{y14})\mathrm{d}\eta\end{aligned}\right\}\tag{7-25}$$

Substituting Eqs. (7-1) and (7-2) into w at the left side of the above equations, and substituting the corresponding interpolation formulae into the boundary rotations at the right sides, we obtain

$$\left.\begin{aligned}&\lambda_4 = \frac{a}{8}\sum_{i=1}^{4}\psi_{xi}\xi_i\\&\lambda_6 = \frac{b}{8}\sum_{i=1}^{4}\psi_{yi}\eta_i\\&\lambda_5+\lambda_{11}+\lambda_{12} = \frac{1}{4}\sum_{i=1}^{4}w_i\xi_i\eta_i\end{aligned}\right\}\tag{7-26}$$

The above 3 equations are the previous Eqs. (7-13b,c) and (7-16a), respectively, and thereby, have been satisfied.

Since the element ACM has already satisfied the 6 fundamental conforming conditions, so, the element ACM is convergent. Actually, it is also a generalized conforming element.

7.3 Super-Basis Perimeter-Point Conforming Scheme — Verification and Improvement of the Element BCIZ

This section will introduce the construction procedure of the super-basis thin plate element formulated by the combination scheme of the perimeter and point conforming conditions. The no. 25 and 26 elements GCⅡ-T9 and LT9 in Table 5.1 belong to this element group. They are both triangular thin plate elements, in which $n = 9$ and $m = 12$, but the 12 conforming conditions used by them are different: the element GCⅡ-T9 adopts 9 point conforming conditions and 3 perimeter conforming conditions, while the element LT9 adopts 3 point conforming conditions and 9 perimeter conforming conditions.

The conventional scheme that the non-conforming elements usually adopt is: let $m=n$, and only the point conforming conditions at the corner nodes are used. Such non-conforming elements sometimes are convergent (such as the rectangular thin plate element ACM), sometimes are not convergent (such as the triangular thin plate element BCIZ). The super-basis generalized conforming element scheme is an improved strategy for the non-conforming elements. For example, the super-basis element GCIII-R12 in Sect. 6.4 is an improvement on the element ACM (though the element ACM possesses convergence, its accuracy can be improved further), and the super-basis element GCII-T9 in this section is an improvement on the element BCIZ (the number of the basis functions in the element GCII-T9 is more than the number of the element DOFs, thereby, the shortcoming that the element BCIZ cannot ensure convergence will be eliminated). In this section, we firstly apply the fundamental conforming conditions of the generalized conforming element, especially, the 3 perimeter conforming conditions in Eq. (5-26) under constant stress state, to verify the convergence of the element BCIZ; then, by employing the concept of the super-basis elements, we make the perimeter conforming condition (5-26) satisfied, consequently, a new element GC II-T9 is constructed.

7.3.1 Formulations of Element BCIZ

The triangular thin plate element BCIZ is a famous non-conforming model proposed in the past[3]. And, reference [3] is one of the earliest literatures which pointed out the limitation of the conforming elements and the rationality of the non-conforming elements.

Before verifying the convergence of the element BCIZ, its formulations are introduced briefly as follows.

The element BCIZ has 9 DOFs (Fig. 6.1), and the element nodal displacement vector $\boldsymbol{q}^e$ is given by Eq. (6-1). The assumed element deflection field w is given by Eqs. (6-2) and (6-3), i.e.,

$$
\begin{aligned}
w = {} & \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \lambda_4 L_1 L_2 \left(L_1 + \frac{1}{2} L_3 \right) + \lambda_5 L_2 L_3 \left(L_2 + \frac{1}{2} L_1 \right) \\
& + \lambda_6 L_3 L_1 \left(L_3 + \frac{1}{2} L_2 \right) + \lambda_7 L_1 L_3 \left(L_1 + \frac{1}{2} L_2 \right) \\
& + \lambda_8 L_2 L_1 \left(L_2 + \frac{1}{2} L_3 \right) + \lambda_9 L_3 L_2 \left(L_3 + \frac{1}{2} L_1 \right)
\end{aligned}
\tag{7-27}
$$

By applying the 9 point conforming conditions about w, ψ_x and ψ_y at the corner nodes, we obtain

$$\left.\begin{aligned}
w_1 &= \lambda_1 \\
w_2 &= \lambda_2 \\
w_3 &= \lambda_3 \\
\psi_{x1} &= \frac{1}{2A}(b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_2\lambda_4 + b_3\lambda_7) \\
\psi_{x2} &= \frac{1}{2A}(b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_3\lambda_5 + b_1\lambda_8) \\
\psi_{x3} &= \frac{1}{2A}(b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_1\lambda_6 + b_2\lambda_9) \\
\psi_{y1} &= \frac{1}{2A}(c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 + c_2\lambda_4 + c_3\lambda_7) \\
\psi_{y2} &= \frac{1}{2A}(c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 + c_3\lambda_5 + c_1\lambda_8) \\
\psi_{y3} &= \frac{1}{2A}(c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3 + c_1\lambda_6 + c_2\lambda_9)
\end{aligned}\right\} \tag{7-28}$$

Then $\boldsymbol{\lambda}$ can be solved

$$\left.\begin{aligned}
\lambda_1 &= w_1 \\
\lambda_2 &= w_2 \\
\lambda_3 &= w_3 \\
\lambda_4 &= w_1 - w_2 + c_3\psi_{x1} - b_3\psi_{y1} \\
\lambda_5 &= w_2 - w_3 + c_1\psi_{x2} - b_1\psi_{y2} \\
\lambda_6 &= w_3 - w_1 + c_2\psi_{x3} - b_2\psi_{y3} \\
\lambda_7 &= w_1 - w_3 - c_2\psi_{x1} + b_2\psi_{y1} \\
\lambda_8 &= w_2 - w_1 - c_3\psi_{x2} + b_3\psi_{y2} \\
\lambda_9 &= w_3 - w_2 - c_1\psi_{x3} + b_1\psi_{y3}
\end{aligned}\right\} \tag{7-29}$$

The above equation can be rewritten as

$$\boldsymbol{\lambda} = \hat{\boldsymbol{A}}\boldsymbol{q}^e$$

Substitution of Eq. (7-29) into (7-27) yields

$$w = \boldsymbol{N}\boldsymbol{q}^e = \sum_{i=1}^{3}(N_i w_i + N_{xi}\psi_{xi} + N_{yi}\psi_{yi}) \tag{7-30}$$

in which the 3 shape functions related to the node 1 are

$$\left.\begin{aligned} N_1 &= L_1 + L_1^2L_2 + L_1^2L_3 - L_2^2L_1 - L_3^2L_1 \\ N_{x1} &= c_3\left(L_1^2L_2 + \frac{1}{2}L_1L_2L_3\right) - c_2\left(L_1^2L_3 + \frac{1}{2}L_1L_2L_3\right) \\ N_{y1} &= -b_3\left(L_1^2L_2 + \frac{1}{2}L_1L_2L_3\right) + b_2\left(L_1^2L_3 + \frac{1}{2}L_1L_2L_3\right) \end{aligned}\right\} \tag{7-31}$$

By the permutation of 1, 2 and 3, the other 6 shape functions can be obtained. The element curvature field is

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ 2\kappa_{xy} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} w \tag{7-32}$$

By using the transformation of second-order derivatives between the Cartesian coordinate system and the area coordinate system:

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \frac{1}{A}\boldsymbol{ts}\partial_2 \tag{7-33}$$

where

$$\boldsymbol{t} = \frac{1}{4A}\begin{bmatrix} b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \\ 2b_1c_1 & 2b_2c_2 & 2b_3c_3 \end{bmatrix} \tag{7-34}$$

$$\boldsymbol{s} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \tag{7-35}$$

$$\partial_2 = \left[\begin{array}{ccc:ccc} \dfrac{\partial^2}{\partial L_1^2} & \dfrac{\partial^2}{\partial L_2^2} & \dfrac{\partial^2}{\partial L_3^2} & \dfrac{\partial^2}{\partial L_1 \partial L_2} & \dfrac{\partial^2}{\partial L_2 \partial L_3} & \dfrac{\partial^2}{\partial L_3 \partial L_1} \end{array}\right]^{\mathrm{T}} \tag{7-36}$$

Thus, the expressions of the curvatures can be obtained as follows

$$\boldsymbol{\kappa} = -\frac{1}{A}\boldsymbol{tsH\lambda} \tag{7-37}$$

where

$$\boldsymbol{H} = \left[\begin{array}{ccc:ccc:ccc}
0 & 0 & 0 & 2L_2 & 0 & 0 & 2L_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 2L_3 & 0 & 0 & 2L_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2L_1 & 0 & 0 & 2L_2 \\
\hdashline
0 & 0 & 0 & 2L_1+\frac{1}{2}L_3 & \frac{1}{2}L_3 & \frac{1}{2}L_3 & \frac{1}{2}L_3 & 2L_2+\frac{1}{2}L_3 & \frac{1}{2}L_3 \\
0 & 0 & 0 & \frac{1}{2}L_1 & 2L_2+\frac{1}{2}L_1 & \frac{1}{2}L_1 & \frac{1}{2}L_1 & \frac{1}{2}L_1 & 2L_3+\frac{1}{2}L_1 \\
0 & 0 & 0 & \frac{1}{2}L_2 & \frac{1}{2}L_2 & 2L_3+\frac{1}{2}L_2 & 2L_1+\frac{1}{2}L_2 & \frac{1}{2}L_2 & \frac{1}{2}L_2
\end{array}\right] \tag{7-38}$$

And, the curvature field can also be rewritten as

$$\boldsymbol{\kappa} = \boldsymbol{Bq}^e \tag{7-39}$$

where

$$\boldsymbol{B} = -\frac{1}{A}\boldsymbol{tsH\hat{A}} \tag{7-40}$$

Finally, the element stiffness matrix can be obtained

$$\boldsymbol{K}^e = \iint_{A^e} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{DB}\mathrm{d}A \tag{7-41}$$

in which $\boldsymbol{D}$ is the elastic matrix of thin plate.

7.3.2 Verification for the Convergence of the Element BCIZ

According to the generalized conforming element theory, the fundamental conditions which ensure the convergence of the non-conforming elements are given by Eq. (7-20). They involve 6 conforming conditions, such as Eqs. (7-22) and (7-23), which should be satisfied when the element is under the rigid body displacement and constant strain states. Now, we apply these 6 fundamental conforming conditions to verify the convergence of the element BCIZ.

Firstly, the element BCIZ has already satisfied the 9 point conforming conditions about w, ψ_x and ψ_y at the corner nodes, thereby, the point conforming conditions in Eq. (7-22), which should be satisfied when the element is under rigid body displacement states, are satisfied naturally.

Secondly, we check the perimeter conforming conditions given in Eq. (7-23) which should be satisfied when the element is under constant strain states, i.e.,

$$\boldsymbol{M}^{\mathrm{T}} \iint_{A^e} \boldsymbol{\kappa} \mathrm{d}A = -\sum_{i=1}^{3} \left(M_{ni} \int_0^{d_i} \tilde{\psi}_{ni} \mathrm{d}s + M_{nsi} \int_0^{d_i} \tilde{\psi}_{si} \mathrm{d}s \right) \tag{7-42}$$

where the 3 constant internal force states are usually assumed to be those in Eq. (5-24). But, for the triangular element, they would better be assumed as follows

$$\left.\begin{aligned} M_{n1} &= \frac{4A}{d_1^2}\alpha_1 \\ M_{n2} &= \frac{4A}{d_2^2}\alpha_2 \\ M_{n3} &= \frac{4A}{d_3^2}\alpha_3 \end{aligned}\right\} \tag{7-43}$$

where the arbitrary parameters α_i ($i = 1,2,3$) are corresponding to the constant internal force states in which the normal moment of ith side is not zero while the normal moments of the other two sides are zero. The twisting moment M_{nsi} along each element side and the internal moments M_x, M_y, M_{xy} corresponding to this constant internal force state are as follows

$$\left.\begin{aligned} M_{ns1} &= r_1\alpha_1 - \alpha_2 + \alpha_3 \\ M_{ns2} &= \alpha_1 + r_2\alpha_2 - \alpha_3 \\ M_{ns3} &= -\alpha_1 + \alpha_2 + r_3\alpha_3 \end{aligned}\right\} \tag{7-44}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{1}{A} \begin{bmatrix} -c_2c_3 & -c_3c_1 & -c_1c_2 \\ -b_2b_3 & -b_3b_1 & -b_1b_2 \\ \frac{1}{2}(b_2c_3 + b_3c_2) & \frac{1}{2}(b_3c_1 + b_1c_3) & \frac{1}{2}(b_1c_2 + b_2c_1) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{7-45}$$

in which r_1, r_2 and r_3 are given by Eq. (6-58). Equation (7-45) can be written as

$$\boldsymbol{M} = (\boldsymbol{t}^{-1})^{\mathrm{T}} \boldsymbol{\alpha} \tag{7-46}$$

Now, we substitute the above constant internal force state into Eq. (7-42).

Firstly, the substitution of Eqs. (7-46) and (7-37) into the left side of Eq. (7-42) yields

$$\boldsymbol{M}^{\mathrm{T}}\iint_{A^e}\boldsymbol{\kappa}\mathrm{d}A=-\frac{1}{A}\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{s}\left(\iint_{A^e}\boldsymbol{H}\mathrm{d}A\right)\boldsymbol{\lambda}$$

$$=-\boldsymbol{\alpha}^{\mathrm{T}}\frac{1}{6}\left[\begin{array}{ccc:ccc:ccc}0&0&0&-1&3&-5&-1&-5&3\\0&0&0&-5&-1&3&3&-1&-5\\0&0&0&3&-5&-1&-5&3&-1\end{array}\right]\boldsymbol{\lambda}$$

$$=-\boldsymbol{\alpha}^{\mathrm{T}}\left(\frac{1}{6}\begin{bmatrix}-1&3&-5\\-5&-1&3\\3&-5&-1\end{bmatrix}\begin{Bmatrix}\lambda_4\\\lambda_5\\\lambda_6\end{Bmatrix}+\frac{1}{6}\begin{bmatrix}-1&-5&3\\3&-1&-5\\-5&3&-1\end{bmatrix}\begin{Bmatrix}\lambda_7\\\lambda_8\\\lambda_9\end{Bmatrix}\right) \tag{7-47}$$

Secondly, consider the right side of Eq. (7-42). The integrations of rotations along the boundary line are as follows

$$\left.\begin{aligned}\int_0^{d_1}\tilde{\psi}_{n1}\mathrm{d}s&=-\frac{b_1}{2}(\psi_{x2}+\psi_{x3})-\frac{c_1}{2}(\psi_{y2}+\psi_{y3})\\\int_0^{d_2}\tilde{\psi}_{n2}\mathrm{d}s&=-\frac{b_2}{2}(\psi_{x3}+\psi_{x1})-\frac{c_2}{2}(\psi_{y3}+\psi_{y1})\\\int_0^{d_3}\tilde{\psi}_{n3}\mathrm{d}s&=-\frac{b_3}{2}(\psi_{x1}+\psi_{x2})-\frac{c_3}{2}(\psi_{y1}+\psi_{y2})\end{aligned}\right\} \tag{7-48}$$

$$\left.\begin{aligned}\int_0^{d_1}\tilde{\psi}_{s1}\mathrm{d}s&=-w_2+w_3\\\int_0^{d_2}\tilde{\psi}_{s2}\mathrm{d}s&=-w_3+w_1\\\int_0^{d_3}\tilde{\psi}_{s3}\mathrm{d}s&=-w_1+w_2\end{aligned}\right\} \tag{7-49}$$

Substitution of Eqs. (7-43), (7-44), (7-48) and (7-49) into the right side of Eq. (7-42) yields

$$\begin{aligned}&-\sum_{i=1}^{3}\left(M_{ni}\int_0^{d_i}\tilde{\psi}_{ni}\mathrm{d}s+M_{nsi}\int_0^{d_i}\tilde{\psi}_{si}\mathrm{d}s\right)\\&=\alpha_1[-2w_1+(1+r_1)w_2+(1-r_1)w_3+\frac{2Ab_1}{d_1^2}(\psi_{x2}+\psi_{x3})+\frac{2Ac_1}{d_1^2}(\psi_{y2}+\psi_{y3})]\\&\quad+\alpha_2[(1-r_2)w_1-2w_2+(1+r_2)w_3+\frac{2Ab_2}{d_2^2}(\psi_{x3}+\psi_{x1})+\frac{2Ac_2}{d_2^2}(\psi_{y3}+\psi_{y1})]\\&\quad+\alpha_3[(1+r_3)w_1+(1-r_3)w_2-2w_3+\frac{2Ab_3}{d_3^2}(\psi_{x1}+\psi_{x2})+\frac{2Ac_3}{d_3^2}(\psi_{y1}+\psi_{y2})]\end{aligned} \tag{7-50}$$

By employing Eq. (7-28), $\boldsymbol{q}^e$ on the right side of the above equation can be expressed in terms of $\boldsymbol{\lambda}$

$$-\sum_{i=1}^{3}(M_{ni}\int_0^{d_i}\tilde{\psi}_{ni}\mathrm{d}s+M_{nsi}\int_0^{d_i}\tilde{\psi}_{si}\mathrm{d}s)$$

$$=\boldsymbol{\alpha}^{\mathrm{T}}\left(\begin{bmatrix}0 & -\frac{1}{2}(1-r_1) & 1\\ 1 & 0 & -\frac{1}{2}(1-r_2)\\ -\frac{1}{2}(1-r_3) & 1 & 0\end{bmatrix}\begin{Bmatrix}\lambda_4\\ \lambda_5\\ \lambda_6\end{Bmatrix}\right.$$

$$\left.+\begin{bmatrix}0 & 1 & -\frac{1}{2}(1+r_1)\\ -\frac{1}{2}(1+r_2) & 0 & 1\\ 1 & -\frac{1}{2}(1+r_3) & 0\end{bmatrix}\begin{Bmatrix}\lambda_7\\ \lambda_8\\ \lambda_9\end{Bmatrix}\right) \tag{7-51}$$

Substituting Eqs. (7-47) and (7-51) into the two sides of Eq. (7-42), and eliminating the arbitrary parameters $\boldsymbol{\alpha}^{\mathrm{T}}$, we obtain

$$\begin{bmatrix}1 & -3 & 5\\ 5 & 1 & -3\\ -3 & 5 & 1\end{bmatrix}\begin{Bmatrix}\lambda_4\\ \lambda_5\\ \lambda_6\end{Bmatrix}+\begin{bmatrix}1 & 5 & -3\\ -3 & 1 & 5\\ 5 & -3 & 1\end{bmatrix}\begin{Bmatrix}\lambda_7\\ \lambda_8\\ \lambda_9\end{Bmatrix}$$

$$=\begin{bmatrix}0 & -3(1-r_1) & 6\\ 6 & 0 & -3(1-r_2)\\ -3(1-r_3) & 6 & 0\end{bmatrix}\begin{Bmatrix}\lambda_4\\ \lambda_5\\ \lambda_6\end{Bmatrix}+\begin{bmatrix}0 & 6 & -3(1+r_1)\\ -3(1+r_2) & 0 & 6\\ 6 & -3(1+r_3) & 0\end{bmatrix}\begin{Bmatrix}\lambda_7\\ \lambda_8\\ \lambda_9\end{Bmatrix}$$

i.e.,

$$\begin{bmatrix}1 & -3r_1 & -1\\ -1 & 1 & -3r_2\\ -3r_3 & -1 & 1\end{bmatrix}\begin{Bmatrix}\lambda_4-\lambda_8\\ \lambda_5-\lambda_9\\ \lambda_6-\lambda_7\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} \tag{7-52}$$

If $\boldsymbol{\lambda}$ is expressed in terms of $\boldsymbol{q}^e$, we obtain

$$\begin{bmatrix}1 & -3r_1 & -1\\ -1 & 1 & -3r_2\\ -3r_3 & -1 & 1\end{bmatrix}\begin{Bmatrix}2w_1-2w_2+c_3(\psi_{x1}+\psi_{x2})-b_3(\psi_{y1}+\psi_{y2})\\ 2w_2-2w_3+c_1(\psi_{x2}+\psi_{x3})-b_1(\psi_{y2}+\psi_{y3})\\ 2w_3-2w_1+c_2(\psi_{x3}+\psi_{x1})-b_2(\psi_{y3}+\psi_{y1})\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix} \tag{7-53}$$

Equations (7-52) and (7-53) are the 3 perimeter conforming conditions for verifying the convergence of the element BCIZ. When the components in the element DOFs $\boldsymbol{q}^e$ are 9 arbitrary parameters, Eq. (7-53) is not satisfied. Thereby, the convergence of the element BCIZ cannot be guaranteed.

7.3.3 Element GCⅡ-T9—an Improvement on the Element BCIZ

The triangular thin plate element GCⅡ-T9[4] is an improved model of the element BCIZ.

As described above, the reason why the element BCIZ cannot ensure convergence is that the perimeter conforming condition (7-42) is not satisfied. Therefore, an improvement scheme is proposed as follows: on the basis of the assumed element deflection field given by Eq. (7-27), 3 new unknown coefficients and basis functions are supplemented; and on the basis of the 9 point conforming conditions used, 3 perimeter conforming conditions given in Eq. (7-42) are also supplemented. Since the condition (7-42) has already been satisfied, the convergence can be ensured. This element obtained is called GCⅡ-T9.

The construction procedure of the element GCⅡ-T9 is as follows.

The element deflection field is assumed to be composed of two parts:

$$w = \overline{w}(\text{BCIZ}) + \hat{w} \tag{7-54}$$

where $\overline{w}(\text{BCIZ})$ is the assumed deflection field (7-27) of the element BCIZ with 9 unknown coefficients; $\hat{w}$ is the additional deflection field with 3 new unknown coefficients:

$$\hat{w} = \lambda_{10} L_1^2 L_2^2 + \lambda_{11} L_2^2 L_3^2 + \lambda_{12} L_3^2 L_1^2 \tag{7-55}$$

$\hat{w}$ has the following characteristic: at three corner nodes, $\hat{w}$, $\dfrac{\partial \hat{w}}{\partial x}$ and $\dfrac{\partial \hat{w}}{\partial y}$ are all zero.

The assumed deflection field in Eq. (7-54) contains 12 unknown coefficients, while the number of the element DOFs is still 9, so the new element is a super-basis element.

In order to solve the 12 unknown coefficients, 12 conforming conditions are needed.

Firstly, 9 point conforming conditions about w, ψ_x and ψ_y at three corner nodes are used. Because of the characteristic of the additional deflection $\hat{w}$ mentioned above, we know that λ_{10}, λ_{11} and λ_{12} will not appear in these conditions. So, the first 9 unknown coefficients $\lambda_1, \lambda_2, \cdots, \lambda_9$ can be solved just by these 9 conditions, as shown in Eq. (7-29), and are the same as those in the element BCIZ.

Secondly, the 3 new unknown coefficients λ_{10}, λ_{11} and λ_{12} will be solved by applying 3 perimeter conforming conditions given in Eq. (7-42). And, the weighting functions given in Eq. (7-42) are still the 3 constant internal force states shown in Eqs. (7-43) – (7-46).

For the items at the right side of Eq. (7-42), the derivation results are still given by Eqs. (7-50) and (7-51).

For the items at the left side of Eq. (7-42), the items related to the new unknown coefficients λ_{10}, λ_{11}, λ_{12} should be supplemented on the basis of Eq. (7-47), i.e.,

$$\boldsymbol{M}^{\mathrm{T}}\iint\boldsymbol{\kappa}\,\mathrm{d}A=-\boldsymbol{\alpha}^{\mathrm{T}}\left(\frac{1}{6}\begin{bmatrix}-1&3&-5\\-5&-1&3\\3&-5&-1\end{bmatrix}\begin{Bmatrix}\lambda_4\\\lambda_5\\\lambda_6\end{Bmatrix}+\frac{1}{6}\begin{bmatrix}-1&-5&3\\3&-1&-5\\-5&3&-1\end{bmatrix}\begin{Bmatrix}\lambda_7\\\lambda_8\\\lambda_9\end{Bmatrix}\right.$$
$$\left.+\frac{1}{6}\begin{bmatrix}0&2&0\\0&0&2\\2&0&0\end{bmatrix}\begin{Bmatrix}\lambda_{10}\\\lambda_{11}\\\lambda_{12}\end{Bmatrix}\right)\tag{7-56}$$

Substitution of Eqs. (7-56) and (7-51) into the two sides of Eq. (7-42) yields

$$\begin{bmatrix}0&2&0\\0&0&2\\2&0&0\end{bmatrix}\begin{Bmatrix}\lambda_{10}\\\lambda_{11}\\\lambda_{12}\end{Bmatrix}=\begin{bmatrix}1&-3r_1&-1\\-1&1&-3r_2\\-3r_3&-1&1\end{bmatrix}\begin{Bmatrix}\lambda_4-\lambda_8\\\lambda_5-\lambda_9\\\lambda_6-\lambda_7\end{Bmatrix}$$

From this relation, we obtain

$$\begin{Bmatrix}\lambda_{10}\\\lambda_{11}\\\lambda_{12}\end{Bmatrix}=\frac{1}{2}\begin{bmatrix}-3r_3&-1&1\\1&-3r_1&-1\\-1&1&-3r_2\end{bmatrix}\begin{Bmatrix}\lambda_4-\lambda_8\\\lambda_5-\lambda_9\\\lambda_6-\lambda_7\end{Bmatrix}\tag{7-57}$$

Equation (7-57) can also be expressed in terms of $\boldsymbol{q}^e$,

$$\begin{Bmatrix}\lambda_{10}\\\lambda_{11}\\\lambda_{12}\end{Bmatrix}=\begin{bmatrix}-(1+3r_3)&\frac{1}{2}(c_2-3r_3c_3)&-\frac{1}{2}(b_2-3r_3b_3)\\2&\frac{1}{2}(c_3-c_2)&-\frac{1}{2}(b_3-b_2)\\-(1-3r_2)&-\frac{1}{2}(c_3+3r_2c_2)&\frac{1}{2}(b_3+3r_2b_2)\end{bmatrix}\begin{Bmatrix}w_1\\\psi_{x1}\\\psi_{y1}\end{Bmatrix}$$
$$+\begin{bmatrix}-(1-3r_3)&-\frac{1}{2}(c_1+3r_3c_3)&\frac{1}{2}(b_1+3r_3b_3)\\-(1+3r_1)&\frac{1}{2}(c_3-3r_1c_1)&-\frac{1}{2}(b_3-3r_1b_1)\\2&\frac{1}{2}(c_1-c_3)&-\frac{1}{2}(b_1-b_3)\end{bmatrix}\begin{Bmatrix}w_2\\\psi_{x2}\\\psi_{y2}\end{Bmatrix}$$

$$
+\begin{bmatrix} 2 & \frac{1}{2}(c_2-c_1) & -\frac{1}{2}(b_2-b_1) \\ -(1-3r_1) & -\frac{1}{2}(c_2+3r_1c_1) & \frac{1}{2}(b_2+3r_1b_1) \\ -(1+3r_2) & \frac{1}{2}(c_1-3r_2c_2) & -\frac{1}{2}(b_1-3r_2b_2) \end{bmatrix} \begin{Bmatrix} w_3 \\ \psi_{x3} \\ \psi_{y3} \end{Bmatrix} \tag{7-58}
$$

After the 12 unknown coefficients are determined, the element deflection field and its shape functions can then be obtained. All the shape functions are composed of two parts, for examples, for the node 1, we have

$$
N_1 = \bar{N}_1 + \hat{N}_1, \quad N_{x1} = \bar{N}_{x1} + \hat{N}_{x1}, \quad N_{y1} = \bar{N}_{y1} + \hat{N}_{y1} \tag{7-59}
$$

where $\bar{N}_1$, $\bar{N}_{x1}$, $\bar{N}_{y1}$ are the parts related to $\bar{w}$(BCIZ), and are the same as those given in Eq. (7-31). $\hat{N}_1$, $\hat{N}_{x1}$ and $\hat{N}_{y1}$ are the parts related to $\hat{w}$:

$$
\left.\begin{aligned}
\hat{N}_1 &= -(1+3r_3)L_1^2L_2^2 + 2L_2^2L_3^2 - (1-3r_2)L_3^2L_1^2 \\
\hat{N}_{x1} &= \frac{1}{2}(c_2-3r_3c_3)L_1^2L_2^2 + \frac{1}{2}(c_3-c_2)L_2^2L_3^2 - \frac{1}{2}(c_3+3r_2c_2)L_3^2L_1^2 \\
\hat{N}_{y1} &= -\frac{1}{2}(b_2-3r_3b_3)L_1^2L_2^2 - \frac{1}{2}(b_3-b_2)L_2^2L_3^2 + \frac{1}{2}(b_3+3r_2b_2)L_3^2L_1^2
\end{aligned}\right\} \tag{7-60}
$$

The other 6 shape functions can be obtained by permutation.

After the shape functions are determined, the element stiffness matrix can then be obtained.

Example 7.2 The central deflection w_C and central moment M_C of a square plate (the side length is L) subjected to uniform load q and central concentrated load P. The Poisson's ratio is 0.3. Meshes A and B in Fig. 6.2 and mesh C in Fig. 6.13 are used. The results by the element GCⅡ-T9 are given in Tables 7.3 and 7.4.

In the tables, $\alpha = w_C/(qL^4/100D)$, $\beta = w_C/(PL^2/10D)$ and $\alpha_1 = M_{xC}/(qL^2)$. And, the numbers in parentheses are the relative errors. From Tables 7.3 and 7.4, two points can be concluded:

(1) The precision of the element GCⅡ-T9 is very high, is better than the Discrete Kirchhoff Theory element DKT and the stress hybrid element HSM.

(2) For mesh C, the element BCIZ cannot pass the patch test, and cannot converge to correct solutions, either; but the computational results of the element GCⅡ-T9 are convergent under this mesh, even better than those obtained by the meshes A and B.

Table 7.3 The central deflection and central moment coefficients of a simply-supported plate (GCII-T9, Mesh A)

Mesh (1/4 plate)	Deflection coefficient (uniform load) α	Deflection coefficient (concentrated load) β	Moment coefficient (uniform load) α_1
2×2	0.4119(1.41%)	0.1174(1.18%)	0.0499(4.35%)
4×4	0.4085(0.57%)	0.1167(0.58%)	0.0488(1.88%)
8×8	0.4068(0.15%)	0.1162(0.14%)	0.0480(0.21%)
Analytical	0.4062	0.1160	0.0479

Table 7.4 The central deflection coefficients of simply-supported and clamped plates by using meshes A, B and C (GCII-T9, Mesh 8×8)

support \ mesh	α (uniform load)			β (concentrated load)		
	Mesh A	Mesh B	Mesh C	Mesh A	Mesh B	Mesh C
Simply-supported	0.4068 (0.15%)	0.4068 (0.15%)	0.4056 (−0.15%)	0.1162 (0.14%)	0.1182 (1.9%)	0.1163 (0.28%)
Clamped	0.1291 (2.5%)	0.1274 (1.1%)	0.1277 (1.4%)	0.5666 (1.2%)	0.5812 (3.8%)	0.5666 (1.2%)

7.4 Least-Square Scheme—Elements LSGC-R12 and LSGC-T9

This section will introduce the construction procedure of the thin plate element formulated by the least-square scheme. The no. 27 and 28 elements LSGC-R12 and LSGC-T9 in Table 5.1 belong to this element group.

7.4.1 Rectangular Element LSGC-R12[5] — an Improvement on the Element ACM

Rectangular thin plate element LSGC-R12 is a super-basis element by improving the element ACM[2] using the least-square scheme. The element DOFs are still the 12 conventional DOFs at the corner nodes. And, the element deflection field is assumed to be composed of two parts:

$$w = \overline{w}(\text{ACM}) + \hat{w} \tag{7-61}$$

where $\overline{w}(\text{ACM})$ is the deflection field (7-1) of the element ACM, and contains 12 unknown coefficients $\lambda_1, \lambda_2, \cdots, \lambda_{12}$; $\hat{w}$ is the additional deflection field with 2 new unknown coefficients

$$\hat{w} = \lambda_{13}\eta(\eta^2 - 1)(\xi^2 - 1)^2 + \lambda_{14}\xi(\xi^2 - 1)(\eta^2 - 1)^2 \tag{7-62}$$

$\hat{w}$ possesses the following characteristics:

(1) At 4 corner nodes, $\hat{w}$, $\dfrac{\partial \hat{w}}{\partial x}$ and $\dfrac{\partial \hat{w}}{\partial y}$ are all zero;

(2) Along 4 element sides, $\hat{w}$ identically equals to zero, but $\dfrac{\partial \hat{w}}{\partial n}$ does not equal to zero.

In order to solve the 14 unknown coefficients, 14 conforming conditions are needed.

Firstly, the 12 point conforming conditions about w, ψ_x and ψ_y at the corner nodes are applied. According to the characteristic (1) of $\hat{w}$, these 12 conditions do not contain λ_{13} and λ_{14}, thus the 12 unknown coefficients $\lambda_1, \lambda_2, \cdots, \lambda_{12}$ can just be solved, and are the same as those of the element ACM.

Since the displacement field $\overline{w}$ of the element ACM is exactly compatible with the deflection $\tilde{w}$ along the element boundary (but incompatible with the normal slope $\tilde{\psi}_n$ along the element boundary), and the value of $\hat{w}$ along the boundary identically equals to zero, so the total displacement $w = \overline{w} + \hat{w}$ is also compatible with the boundary deflection $\tilde{w}$.

Secondly, the conforming conditions about the normal slope along the element boundary also need to be considered, and then, they are used to determine the other residual 2 unknown coefficients λ_{13} and λ_{14}.

According to the least-square method, the following 2 conditions can be obtained:

$$\left.\begin{aligned} \frac{\partial}{\partial \lambda_{13}} \oint_{\partial A^e} \left(\frac{\partial \overline{w}}{\partial n} + \frac{\partial \hat{w}}{\partial n} - \tilde{\psi}_n \right)^2 \mathrm{d}s = 0 \\ \frac{\partial}{\partial \lambda_{14}} \oint_{\partial A^e} \left(\frac{\partial \overline{w}}{\partial n} + \frac{\partial \hat{w}}{\partial n} - \tilde{\psi}_n \right)^2 \mathrm{d}s = 0 \end{aligned}\right\} \tag{7-63}$$

From this equation, we obtain

$$\left.\begin{aligned} \lambda_{13} &= \frac{9}{128} a \sum_{i=1}^{4} \xi_i \eta_i \psi_{xi} \\ \lambda_{14} &= \frac{9}{128} b \sum_{i=1}^{4} \xi_i \eta_i \psi_{yi} \end{aligned}\right\} \tag{7-64}$$

After the 14 unknown coefficients are determined, the element deflection field and its shape functions can be obtained

$$w = \sum_{i=1}^{4} (N_i w_i + N_{xi}\psi_{xi} + N_{yi}\psi_{yi}) \tag{7-65}$$

where the shape functions are composed of two parts

$$\left.\begin{aligned} N_i &= \bar{N}_i + \hat{N}_i \\ N_{xi} &= \bar{N}_{xi} + \hat{N}_{xi} \\ N_{yi} &= \bar{N}_{yi} + \hat{N}_{yi} \end{aligned}\right\} \quad (i = 1,2,3,4) \tag{7-66}$$

in which $\bar{N}_i$, $\bar{N}_{xi}$ and $\bar{N}_{yi}$ are the parts related to $\bar{w}$, and the same as those given in Eq. (7-19). $\hat{N}_i$, $\hat{N}_{xi}$ and $\hat{N}_{yi}$ are the parts related to $\hat{w}$:

$$\left.\begin{aligned} \hat{N}_i &= 0 \\ \hat{N}_{xi} &= \frac{9}{128} a\xi_i\eta_i\eta(\eta^2-1)(\xi^2-1)^2 \\ \hat{N}_{yi} &= \frac{9}{128} b\xi_i\eta_i\xi(\xi^2-1)(\eta^2-1)^2 \end{aligned}\right\} \quad (i = 1,2,3,4) \tag{7-67}$$

Once the shape functions are obtained, the element stiffness matrix can be derived.

7.4.2 Triangular Element LSGC-T9[5] — an Improvement on the Element BCIZ

Triangular thin plate element LSGC-T9 is a super-basis element by improving the element BCIZ[3] using the least-square scheme. The element DOFs are still the 9 conventional DOFs at the corner nodes. And, the element deflection field is assumed to be composed of two parts:

$$w = \bar{w}(\mathrm{BCIZ}) + \hat{w} \tag{7-68}$$

where $\bar{w}$(BCIZ) is the deflection field (7-27) of the element BCIZ, and contains 9 unknown coefficients λ_1, λ_2, $\cdots$, λ_9; $\hat{w}$ is the additional deflection field with 3 new unknown coefficients:

$$\hat{w} = \lambda_{10}L_1^2L_2^2L_3 + \lambda_{11}L_2^2L_3^2L_1 + \lambda_{12}L_3^2L_1^2L_2 \tag{7-69}$$

The first 9 unknown coefficients can be solved by the point conforming conditions about w, ψ_x, ψ_y at the corner nodes, as shown in Eq. (7-29). And, the residual 3 unknown coefficients λ_{10}, λ_{11} and λ_{12} can be solved by the following least-square conditions:

$$\frac{\partial}{\partial\lambda_i}\oint_{\partial A^e}\left(\frac{\partial\bar{w}}{\partial n} + \frac{\partial\hat{w}}{\partial n} - \tilde{\psi}_n\right)^2 \mathrm{d}s = 0 \quad (i = 10,11,12) \tag{7-70}$$

After the determination of all the unknown coefficients, the shape functions and element stiffness matrix can then be derived.

Example 7.3 The central deflection and central moment of square plates (the side length is L) subjected to uniform load q and central concentrated load P. The Poisson's ratio $\mu = 0.3$.

The results by the elements LSGC-R12 and LSGC-T9 are given in Tables 7.5 and 7.6. And, mesh B in Fig. 6.2 and mesh C in Fig. 6.13 are used for triangular element.

Table 7.5 The central deflection and moment of square plate subjected to uniform load

	Mesh (1/4plate)	Clamped				Simply-supported			
		Rectangular elements		Triangular element LSGC-T9		Rectangular elements		Triangular element LSGC-T9	
		ACM	LSGC-R12	Mesh B	Mesh C	ACM	LSGC-R12	Mesh B	Mesh C
w	2×2	0.1403	0.1222	0.1025	0.1028	0.4328	0.3976	0.3949	0.3918
	4×4	0.1332	0.1241	0.1203	0.1212	0.4129	0.4042	0.4036	0.4032
	8×8	0.1275	0.1262	0.1250	0.1252	0.4081	0.4056	0.4063	0.4055
	Analytical	$0.1265(qL^4/100D)$				$0.4062(qL^4/100D)$			
M	2×2	0.2778	0.2132	0.2030	0.1624	0.5217	0.4442	0.4734	0.4456
	4×4	0.2405	0.2233	0.2240	0.2241	0.4892	0.4689	0.4810	0.4733
	8×8	0.2319	0.2275	0.2287	0.2267	0.4817	0.4762	0.4829	0.4754
	Analytical	$0.2291(qL^2/10)$				$0.4789(qL^2/10)$			

Table 7.6 The central deflection of square plate subjected to central concentrated load

Mesh (1/4 plate)	Clamped				Simply-supported			
	Rectangular elements		Triangular element LSGC-T9		Rectangular elements		Triangular element LSGC-T9	
	ACM	LSGC-R12	Mesh B	Mesh C	ACM	LSGC-R12	Mesh B	Mesh C
2×2	0.6135	0.5324	0.4327	0.4482	1.2327	1.1243	1.0622	1.0803
4×4	0.5803	0.5516	0.5207	0.5296	1.1829	1.1501	1.1278	1.1326
8×8	0.5673	0.5585	0.5494	0.5514	1.1674	1.1570	1.1520	1.1515
Analytical	$0.5612(PL^2/100D)$				$1.160(PL^2/100D)$			

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