Chapter 4 Generalized Conforming Element Theory

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Abstract As the beginning of Part Ⅱ, this chapter discusses the fundamental theory and existing construction modes of generalized conforming finite element method. First, by discussion on the different characters of conforming and nonconforming elements, the background and need for the development of the generalized conforming element are described. Second, as an example, the earliest pattern of the generalized conforming element and its excellent performance are exhibited. Third, some theoretical features of the generalized conforming element, including duality of its variational principle basis, flexibility, multiformity and convergence, are discussed in detail.

Keywords finite element, generalized conforming element, conforming, nonconforming, convergence.

4.1 Introduction

This chapter discusses the fundamental theory and construction modes of generalized conforming finite element method[1−3]. The applications of generalized conforming element method for thin plate, thick plate, laminated composite plate, piezoelectric laminated composite plate, membrane and thin shell will be introduced in Chap. 5 to Chap. 11, respectively.

Generalized conforming element method is a new technique which was developed from the basis of comparison and analysis of conforming and nonconforming element methods. The core problem, which is also the main difference between these three types of displacement-based elements, is the requirement for the displacement compatibility between two adjacent elements.

Generalized conforming element is a kind of limit conforming element which can ensure convergence: for a coarse mesh, it belongs to nonconforming elements; and for a refined mesh divided by infinite elements, it approaches conforming models.

Variational principle corresponding to the generalized conforming element possesses duality: it starts from the sub-region potential energy principle, and ends with the minimum potential energy principle.

The feature of the generalized conforming method is that it is a combination of energy method and weighted residual method: if the essential generalized conforming conditions are satisfied, the point conforming conditions, line conforming conditions, perimeter conforming conditions can be flexibly applied. Therefore, the generalized conforming method has both flexibility and multiformity.

4.2 Conforming and Nonconforming Elements—Some Consideration about "Conforming"

We have known that the earliest finite element model is displacement-based model, and the earliest displacement-based element is conforming (compatible) element. The theoretical basis of these conforming elements is the minimum potential energy principle, in which displacement is taken as the field variable of energy functional. And the displacement fields of two adjacent elements are required to be conforming exactly at the interface. The name *conforming element* reflects this main feature.

Though the conforming element possesses the longest history, and has an important advantage of ensuring convergence, there are still some embarrassing problems left to solve. For examples, the requirement of exactly conforming is not easy to be satisfied; the performance of the element may be over-stiff in some occasions, and so on. When the conforming element method is used to deal with thin plate/shell problem, which is a kind of C_1-continuity problem (i.e., the displacement and its first derivative are both required to be compatible between two adjacent elements), the above shortcomings will be especially noticeable. Thereupon, the nonconforming element method was proposed[4] for overcoming these disadvantages. In this method, only the nodal conforming conditions are required while the exact compatibility between the displacement fields of adjacent elements is not required. And the minimum potential energy principle is still taken as its theoretical start point. Elements constructed by this strategy can exhibit some merits: the relaxed conforming conditions are easier to satisfy; and the accuracy of some nonconforming elements is much better than that of conforming ones. However, some nonconforming models can not converge to correct solutions, which is a fatal defect.

Table 4.1 The contradistinction between conforming and nonconforming elements

	Conforming element	Nonconforming element
Compatibility requirement	Exact compatibility	Inexact compatibility
Variational principle	Minimum potential energy principle: total potential energy Π_p = stationary	
Advantages	Simple functional Π_p (contains only single variable: displacement)	
	Convergence ensured	Easily assumed displacement field; and better accuracy in some occasions
Disadvantages	Exact compatibility is not easy to satisfy; element performance is often over-stiff	Convergence can not be ensured

Generalized conforming element is developed on the basis of conforming and nonconforming elements. To understand its background, a comparison should be performed first between conforming and nonconforming elements (Table 4.1). Then, the "puzzle of compatibility" is analyzed and taken into account.

Puzzle of "conforming conditions"

Puzzles: two questions "why"?

Why the accuracy of conforming elements is not as good as that of some nonconforming elements?
Why some nonconforming elements are not convergent?

Solutions: The requirement of conformity should be moderate and appropriate.

The requirement of conforming element is too severe—It is difficult to be satisfied and leads to over-stiff performance.
Since the threshold level is too high, it excludes many excellent element models as unacceptable ones.
The requirement of nonconforming element is over-relaxed—it can not ensure the convergence.
Example: The nonconforming thin plate element BCIZ[4] can not ensure the convergence.
Source: The minimum potential energy principle is not a proper principle to formulate the nonconforming element.
⇒A moderate and rational requirement of conformity is needed—This is the background for creation of the generalized conforming element.

4.3 The First Pattern of Generalized Conforming Element —Replacing Nodal Conforming by Line Conforming Conditions

The first pattern of generalized conforming element was proposed in 1987[1], in which the new concepts of generalized conforming element and generalized conforming conditions were established by replacing nodal conforming conditions at element nodes by line conforming conditions of average displacement along each element side. As an example, the triangular generalized conforming thin plate bending element TGC in references [1, 2] utilized the following generalized conforming conditions of average displacements (average deflection, tangent and normal average rotations) along each element side:

$$\left.\begin{aligned} &\int_{S_k} (w-\tilde{w})\mathrm{d}s = 0 \\ &\int_{S_k} \left(\frac{\partial w}{\partial n} - \tilde{\psi}_n\right)\mathrm{d}s = 0 \\ &\int_{S_k} \left(\frac{\partial w}{\partial s} - \tilde{\psi}_s\right)\mathrm{d}s = 0 \end{aligned}\right\} \quad \text{(along each element side } S_k\text{)} \tag{4-1}$$

where w is the deflection function of the element; $\tilde{w}$, $\tilde{\psi}_n$ and $\tilde{\psi}_s$ are the deflection, tangent and normal rotations along each element side, respectively.

In a traditional pattern, the following nodal conforming conditions

$$\left.\begin{aligned} &(w-\tilde{w})_j = 0 \\ &\left(\frac{\partial w}{\partial x} - \tilde{\psi}_x\right)_j = 0 \\ &\left(\frac{\partial w}{\partial y} - \tilde{\psi}_y\right)_j = 0 \end{aligned}\right\} \quad \text{(at node } j\text{)} \tag{4-2}$$

at element nodes are usually used.

The main difference between the generalized conforming element method and the traditional pattern of nonconforming element method is that the former uses Eq. (4-1) instead of Eq. (4-2). In other words, the traditional method emphasizes nodal conforming conditions at nodes, so it is hard to simultaneously satisfy line conforming conditions along each side. On the contrary, the generalized conforming element method pays attention to average displacement conforming conditions of each element side, so it can ensure a kind of limit conforming conditions along each side when the mesh is refined gradually. This is the reason why the generalized conforming element method can ensure the convergence, but traditional nonconforming element scheme can not.

In reference [1], the performance of the generalized conforming element TGC was compared with those of other famous element models, i.e., DKT[5, 6], HSM[5, 7], BCIZ[4] and HCT[8]. The variations of computation errors with mesh number N for central deflection w_C and central bending moment M_{yC} of a clamped square plate subjected to uniform load are plotted in Fig. 4.1 and Fig. 4.2. It can be seen that the precision of element TGC is the best.

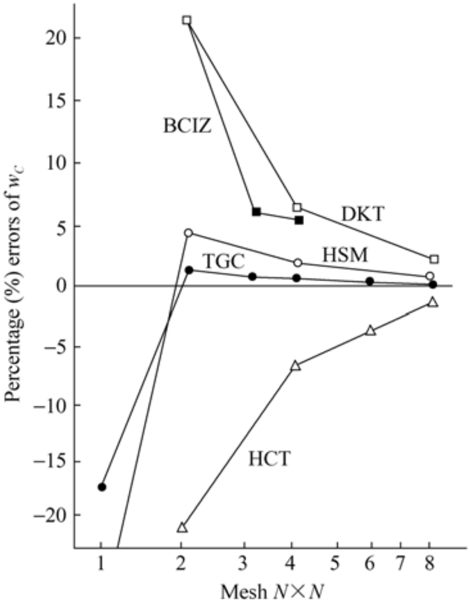

Figure 4.1 Errors for central deflection of a clamped square plate subjected to uniform load

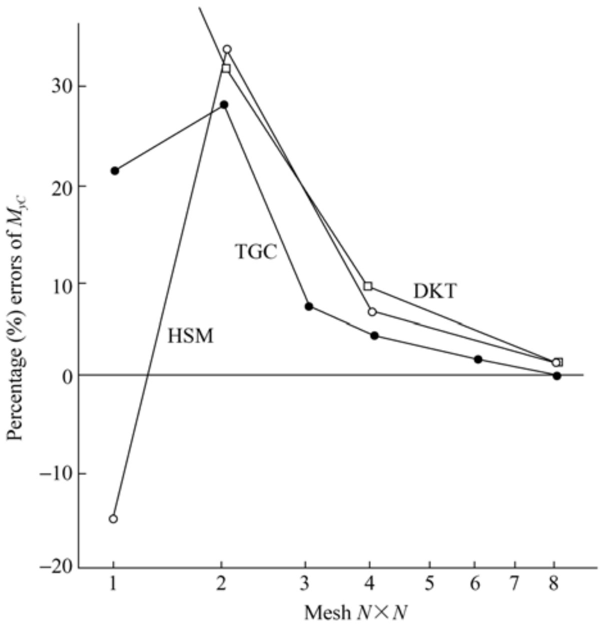

Figure 4.2 Errors for central moment of a clamped square plate subjected to uniform load

4.4 The Variational Basis of Generalized Conforming Element—Duality

Variational principle is the starting point for deriving finite element method. The variational principles corresponding to conforming element, nonconforming element and generalized conforming element as an example for a thin plate bending problem are listed as follows. A clearer understanding for the characters and advantages of generalized conforming element may be obtained by contrasting these three types of elements.

4.4.1 The Variational Principle Corresponding to Conforming Element

Compatibility requirement: In thin plate conforming element, the defection field $w(x, y)$ must be exactly compatible with deflection $\tilde{w}$, normal rotation $\tilde{\psi}_n$ and tangent rotation $\tilde{\psi}_s$ along each element side, i.e., at each point of element boundary ∂A^e the following conforming conditions

$$w = \tilde{w}, \quad \frac{\partial w}{\partial n} = \tilde{\psi}_n, \quad \frac{\partial w}{\partial s} = \tilde{\psi}_s \quad \text{(at each point of } \partial A^e\text{)} \tag{4-3}$$

must be satisfied precisely.

Variational principle used: The minimum potential energy is the starting point of the conforming element, and its functional Π_p can be written as

$$\Pi_\mathrm{p} = \sum_e \Pi_\mathrm{p}^e = \sum_e \iint_{A^e} \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}\mathrm{d}A \tag{4-4}$$

where $\sum_e$ denotes the sum of each element e; D is the bending stiffness of thin plate; μ is the Poisson's ratio. Since in the potential energy Π_p displacement w is the only field variable, Π_p is called a single-field functional. Compared with multi-field functionals, single-field functional is particularly simple.

Disadvantage & advantage: Since the conforming requirement is too severe, it leads to over-stiff performance and the displacement field is difficult to assume. However, the functional is quite simple, and convergence can be guaranteed.

4.4.2 The Variational Principle Corresponding to Nonconforming Element

Compatibility requirement: In nonconforming element, the element deflection field $w(x, y)$ is not required to be exactly compatible, so the conforming conditions (4-3) between elements are relaxed or partially relaxed.

Variational principle used: In order to ensure convergence, the sub-region potential energy principle[9, 10] (refer to Chap. 2) or modified potential energy principle[11] must be employed.

Its functional is the sub-region potential energy Π_{mp}

$$\Pi_{\mathrm{mp}} = \sum_{e} (\Pi_{\mathrm{p}}^{e} + H) \tag{4-5}$$

where H is the additional energy corresponding to incompatible displacement along element boundary

$$H = \oint_{\partial A^e} \left[M_n \left(\frac{\partial w}{\partial n} - \tilde{\psi}_n \right) + M_{ns} \left(\frac{\partial w}{\partial s} - \tilde{\psi}_s \right) - Q_n (w - \tilde{w}) \right] \mathrm{d}s \tag{4-6}$$

in which M_n, M_{ns} and Q_n are the Lagrange multipliers, denoting boundary forces. It is worthy of note that, the functional Π_{mp} contains two field variables, displacement and boundary force, which is much more complicated than Π_{p} with single-field variable. For simplicity, some authors still use potential energy functional Π_{p} instead of sub-region potential energy functional Π_{mp} when they are constructing nonconforming elements, thereby convergence can not be ensured.

Advantage & disadvantage: The deflection field is easy to assume; however, the functional Π_{mp} is more complicated. If the simplified functional Π_{p} is improperly employed, convergence will not be guaranteed.

4.4.3 The Variational Principle Corresponding to Generalized Conforming Element

Duality is the feature of generalized conforming element and its variational principle.

Generalized conforming element possesses duality. It is a kind of limit conforming element:

$$\begin{cases} \text{For a coarse mesh—It belongs to nonconforming element} \\ \text{For a refined mesh divided by infinite elements—It tends to be conforming element} \end{cases}$$

The variational principle corresponding to generalized conforming element also possesses duality. It is a kind of degenerated potential energy principle:

$$\begin{cases} \text{Start point—sub-region potential energy principle: } \varPi_{\mathrm{mp}} = \varPi_{\mathrm{p}} + H = \text{stationary} \\ \text{End-result—degenerated potential energy principle: } \varPi_{\mathrm{p}} = \text{stationary} \end{cases}$$

The degenerate condition introduced—generalized conforming condition:

$$H \to 0(\varPi_{\mathrm{mp}} \to \varPi_{\mathrm{p}})$$

The duality of generalized conforming element and its variational principle can also be described in details as follows:

Compatibility requirement: Since the conforming conditions (4-3) are relaxed, the defection field is easy to assume.

The initial variational principle: In order to ensure convergence, the sub-region potential energy principle is taken as the starting point. However, its functional $\varPi_{\mathrm{mp}}$ is more complicated because it is a two-field -functional.

The degenerate conditions introduced: For simplicity, the two-field-functional $\varPi_{\mathrm{mp}}$ is replaced by its degenerate form—the single-field-functional $\varPi_{\mathrm{p}}$. Therefore, following generalized conforming conditions are introduced:

$$H \to 0 \quad \text{(for any refined mesh divided by infinite elements)} \tag{4-7}$$

or

$$H = 0 \quad \text{(for any constant strain or rigid-body displacement state)} \tag{4-8}$$

When the curvatures of the displacement field tend to be constants, the generalized conforming conditions (4-8) can be written as

$$\oint_{\partial A^e} \left[M_n \left(\frac{\partial w}{\partial n} - \tilde{\psi}_n \right) + M_{ns} \left(\frac{\partial w}{\partial s} - \tilde{\psi}_s \right) - Q_n (w - \tilde{w}) \right] \mathrm{d}s = 0 \tag{4-9}$$

(M_n, M_{ns} and Q_n are corresponding to the constant stress state)

The ending variational principle: Since the generalized conforming conditions (4-8) are introduced, the two-field-functional $\varPi_{\mathrm{mp}}$ will degenerate to the single-field-functional $\varPi_{\mathrm{p}}$. Then the final variational principle used in practice is the degenerate potential energy principle

$$\varPi_{\mathrm{p}} = \text{stationary} \tag{4-10}$$

Advantages: The deflection field is easy to assume, and the variational principle used in practice is still the single-field-functional $\varPi_{\mathrm{p}}$, which is very simple; since the sub-region potential energy principle is taken as the starting point, convergence can be guaranteed. Hereby, both advantages of convenience and convergence are available because of generalized conforming conditions (4-7) or (4-8).

4.4.4 Some Discussions

The key problems for constructing displacement-based elements are how to rationally deal with compatibility problem between elements and how to rationally select a corresponding variational principle.

Conforming element starts from the minimum potential principle. Thereby, the displacement between elements must be exactly compatible. For conforming models, the functional Π_{p} is simple (since Π_{p} is a single-field-functional) while the displacement field is difficult to assume.

Nonconforming element does not require exact compatibility. Thereby, it must start from the sub-region potential energy principle. For nonconforming models, the displacement field is easy to assume, but functional Π_{mp} is complicated (since Π_{mp} is a multi-field-functional). Some workers still take the minimum potential principle as the starting point. This strategy is illegal and leads to non-convergence.

Generalized conforming element does not require exact compatibility either. Thereby, in theory, it also must start from the multi-field-functional Π_{mp}. On the other hand, since the generalized conforming conditions (4-8) are introduced, functional Π_{mp} will return to its degenerate form—single-field-functional Π_{p} in practice. Thus, for generalized conforming models, the displacement field is easy to assume while the functional Π_{p} is very simple in operation. It represents the best of both worlds and is never illegal. Thus, it can be concluded that generalized conforming method is the first successful attempt for applying the degenerate functional.

4.5 The Synthesis of Energy Method and Weighted Residual Method—Flexibility

Generalized conforming element method can be looked upon as a combination of energy method and weighted residual method[12]. In fact, generalized conforming condition (4-9) can be looked upon as a weighted residual equation, in which boundary forces M_n, M_{ns} and Q_n are weighting functions, i.e.,

$$\oint_{\partial A^e}\left[M_n\left(\frac{\partial w}{\partial n}-\tilde{\psi}_n\right)+M_{ns}\left(\frac{\partial w}{\partial s}-\tilde{\psi}_s\right)-Q_n(w-\tilde{w})\right]\mathrm{d}s=0 \tag{4-11}$$

(M_n, M_{ns} and Q_n are weighting functions)

If weighting functions M_n, M_{ns} and Q_n are assumed to be arbitrary functions, the weighted residual Eq. (4-11) will be equivalent to the boundary conforming conditions (4-3), which corresponds to the exactly compatible case.

If weighting functions M_n, M_{ns} and Q_n are assumed to contain only n arbitrary parameters, the weighted residual Eq. (4-11) will be equivalent to n conforming conditions in integral form. Generally, this corresponds to the approximate conforming case. If different weighting functions are selected, the corresponding weighted residual equation will represent conforming conditions with different physical meanings. Table 4.2 lists physical meanings of several commonly used weighting functions and their corresponding weighted residual equations.

Table 4.2 The physical meanings of weighting functions and their weighted residual equations in common use

Weighting functions in common use	Physical meanings of the weighted residual equations
Concentrated force / Concentrated couple } at a point	Point conforming { deflection / rotation
Uniform load / High order load } along a side	Line conforming { average / high order
Constant stress / High order stress } along perimeter	Perimeter conforming { constant stress / high order stress

It can be seen from Table 4.2 and Eq. (4-11) that, unlike the traditional method in which only the nodal conforming conditions are used, generalized conforming element method possesses great flexibility. It allows to choose various conditions, including point conforming, line conforming, perimeter conforming conditions and their combination forms. For convergence, the conditions employed in the generalized conforming element method should at least contain the fundamental generalized conforming conditions. i.e., the generalized conforming conditions or their equivalent conditions for constant strain and rigid-body displacement states of an element.

Since generalized conforming conditions can be explained as the weighted residual Eq. (4-11) along element boundary perimeter ∂A^e, then when establishing generalized conforming conditions, one can flexibly employ five conventional classical methods in a weighted residual method:

Collocation method, Sub-domain method, Least square method, Galerkin method, Method of moment

In fact, the usual methods for constructing generalized conforming elements in literature are the applications and generalizations of the above classical weighted residual methods, for example,

Line conforming method—Application of Sub-domain method

Perimeter conforming method—Applied on the boundary perimeter, which is similar to the method of moments

Least square conforming method—Application of Least square method

Point conforming method—In the conventional point conforming method, the element nodes are taken as collocation points; besides, there is still SemiLoof point conforming method in which the Gauss points at element boundary are taken as collocation points. All these belong to applications of Collocation method.

The combination forms of the above methods can also be used.

4.6 The Convergence of Generalized Conforming Element

From the viewpoint of mechanics, if the fundamental generalized conforming conditions, i.e. the generalized conforming conditions for constant strain and rigid-body displacement states of an element, have been already satisfied when constructing generalized conforming elements, the convergence can be guaranteed. Numerous numerical examples have demonstrated this.

For convergence of the generalized conforming element TGC[1], Shi Zhongci et al. presented a strict mathematical proof[13]. Moreover, Shi Zhongci also discussed the accuracy of the generalized conforming element TGC[14], and pointed out that the accuracy of element TGC is higher than those of elements BCIZ[4] and Specht[15].

In reference [16], Shi Zhongci proposed the FEM test and its test conditions for testing the convergence of nonconforming elements. By applying the test conditions, the convergence and uniqueness for the solutions of the line conforming and perimeter conforming modes were demonstrated[17].

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