Chapter 20 Concluding Remarks

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Abstract This chapter presents a summary of the contributions of the whole book, including seven new achievements in the finite element method, five new element series with 108 new element models, and new solution strategies for five challenging problems.

Keywords finite element, new achievements, new element series, challenging problems.

20.1 Seven New Achievements in the Finite Element Method

Besides the Introduction and this chapter, this book uses eighteen chapters, three parts to illustrate seven new achievements in the finite element method. The contents of these seven achievements and the corresponding original literatures are listed in Table 20.1.

Table 20.1 Seven new achievements and their original literatures

Name (chapter no.)	New creations	Original literatures
1. Sub-region variational principle (Chap. 2)	1.1 New variational principles based on sub-region interpolation and relaxed continuity conditions at the interfaces are established 1.2 Sub-region mixed energy principle—the theoretical basis of sub-region mixed element method	[1 – 3]

(Continued)

Name (chapter no.)	New creations	Original literatures
	1.3 Sub-region potential energy principle—the theoretical basis of the generalized conforming element 1.4 Sub-region complementary energy principle—the theoretical basis of the stress hybrid element	
2. Variational principle with several adjustable parameters (Chap. 3)	2.1 An optimization space is available for the applications of the variational principles because of the existence of adjustable parameters 2.2 Several patterns of functional transformations are proposed 2.3 Variable-substitution-multiplier method is proposed	[4]
3. Generalized Conforming element (Chaps. 4 – 11)	3.1 Generalized conforming element theory, which can guarantee the convergence of the non-conforming elements, is proposed 3.2 Various new conforming schemes, such as Line conforming, Perimeter conforming, Point-Line-Perimeter conforming, SemiLoof conforming, etc., are proposed 3.3 New rational interpolation scheme for shear strains and mixed assumption method of strains-displacements, which can eliminate shear locking completely, are proposed 3.4 A new hybrid-enhanced procedure, which can improve the accuracy for stress solutions of displacement-based elements, is proposed 3.5 New membrane elements with additional rigid rotation DOFs are developed 3.6 New flat-shell and curved shell elements are constructed	[6]
4. Sub-region mixed element method (Chaps. 12, 13)	4.1 A new mixed finite element method in which a coupling mesh composed of conventional displacement-based and stress-based elements is proposed 4.2 High precision solution methods for crack and notch problems are developed	[7]
5. Analytical trial function method (Chaps. 14, 15)	5.1 A new finite element method, the analytical trial function method, is proposed. It uses the basic analytical solutions as its trial functions which exhibits complementarities of analytical and discrete methods 5.2 High precision solution method is developed for the singular stress problem	[8]
6. Quadrilateral area coordinate method (Chaps. 16, 17)	6.1 Area coordinate method is generalized from triangular elements to quadrilateral elements. Two quadrilateral area coordinate methods (QACM-Ⅰ and QACM-Ⅱ) are systematically established 6.2 A new tool for construction of quadrilateral elements is provided. Elements formulated by such tool are more insensitive to mesh distortion than those by the isoparametric coordinate method	[10]
7. Spline element method (Chaps. 18, 19)	7.1 A new method with both merits of high flexibility of finite element method and high smoothing of spline functions is proposed 7.2 New elements for the analysis of plate/shell structures and high-rise buildings are constructed	[11,12]

20.2 Five New Element Series with 108 New Element Models

Brief introduction on the theoretical achievements have been given in Table 20.1. On the basis of these developments, new element models, which possess excellent performance, are successfully developed.

In this book, five new element series with 108 new element models are developed and listed in Table 20.2.

Table 20.2 Five new element series with 108 new element models

	New element series		Chapter no.	Number of new element models	Remark
1	Generalized conforming element series	Thin plate elements	5,6,7	28	See Table 5.1
		Thick and composite plate elements	8,9,10	11	See Table 20.3
		Membrane and shell elements	11	15	See Table 20.4
2	Sub-region mixed element series		12,13	9	See Table 20.5
3	Analytical trial function element series		q	7	5 generalized conforming elements; 2 singular hybrid elements; See Table 20.6
4	Quadrilateral area coordinate element series		16,17	20	All are generalized conforming elements; See Table 17.1
5	Spline element series		18,19	18	See Table 20.7
Total				108 new element models	

Following are the supplementary explanations for the five new element series in Table 20.2.

20.2.1 Generalized Conforming Element Series

Generalized conforming element series are the main part of this book, so eight chapters (Part Ⅱ, Chaps. 4 – 11) are used to discuss them.

The generalized conforming element series can be divided into three groups: thin plate elements, thick and composite plate elements, membrane/shell elements. Some remarks are given as follows.

1. Generalized conforming thin plate element series

Generalized conforming elements were cradled in the field of the thin plate. The initial generalized conforming elements (elements TGC-9 and RGC-12) are all thin plate bending elements. Therefore, it can be seen that the generalized conforming element theory can exhibit its advantages most easily in the thin plate problem. This is because the construction of the exactly conforming thin plate element, which belongs to C_1 continuity problems, is a difficult task. But, the generalized conforming element theory can solve this difficulty simply by its own advantages, and both convergence and convenience can be insured. Consequently, it has been paid attention to by some researchers, and successfully generalized to other fields. In general, new theories and methods are firstly born in special areas, and then, are generalized and applied in other areas.

The detailed discussions on generalized conforming thin plate element series can be referred to the content given by Table 5.1. There are 28 element models, which can be classified as five generalized conforming schemes: Line conforming, Line-point conforming, SemiLoof conforming, Perimeter-point conforming and Least square schemes. The wide applicability and the multiplicity of construction schemes are also the features and advantages of the generalized conforming element theory.

2. Generalized conforming thick plate and composite plate element series

The detailed discussions on generalized conforming thick and composite plate element series can be referred to the content given by Table 20.3. There are 11 elements, in which 8 are the thick plate elements, 2 are composite plate elements, and 1 is piezoelectric composite plate element.

Table 20.3 Index of generalized conforming thick and composite plate elements (11 elements)

Symbol of element			Degenerated case	Original reference	Chapter & section no.
Thick plate element	By assuming (ψ, γ)	1. Triangular element TMT	Thin plate element DKT	[13]	8.5
		2.Quadrilateral element TMQ	Thin plate element DKQ	[14]	8.5
		3.Quadrilateral element ARS-Q12	Thin plate element DKQ	[15]	9.3
	By assuming (w, γ)	4. Triangular element TCGC-T9	Thin plate element GPL-T9	[16]	8.6
		5. Triangular element TSL-T9	Thin plate element LSL-T9	[17]	11.5.3
	From thin plate element to thick plate element	6. Rectangular element LFR1	Thin plate element LR12-2	[18]	8.7
		7. Rectangular element LFR2	Thin plate element ACM	[18]	
		8. Rectangular element GACM	Thin plate element ACM	[19]	
Composite plate element		9. Quadrilateral element TMQ20	Thick plate element TMQ	[20]	9.3
		10.Quadrilateral element CTMQ20	Thick plate element ARS-Q12	[21]	
Piezoelectric composite plate element		11.Quadrilateral element CTMQE	Composite plate element CTMQ20	[22]	10.3

During the studies on the generalized conforming thick plate and composite plate elements, developments for two key problems have been achieved:

(1) The shear locking problem—According to the generalized conforming element theory, countermeasures for solving shear locking are proposed. The key point of these countermeasures is the rational interpolation method for shear strains of the thick plates. On the basis of this point, three schemes can be used, such as no. 1 – 3 elements constructed by assuming (ψ, γ), no. 4 and 5 elements constructed by assuming (w, γ), and no. 6 – 8 elements constructed by the transition from thin plate elements to thick plate elements.

(2) The precisions for stress solutions, including transverse shear stresses at the interfaces of laminated composite plate, are improved—On the basis of the displacement solutions calculated by the generalized conforming elements, a hybrid-enhanced post-processing procedure for stress solutions is used.

3. Generalized conforming membrane and shell element series

Membrane element is the model for plane stress or plane strain problems, and also can be treated as the component of the shell element.

The generalized conforming membrane and shell element series involve 8 membrane elements and 7 shell elements, i.e., totally 15 models. Their detailed information can be found in Table 20.4, in which the membrane elements are classified as elements with or without drilling freedoms, and the shell elements are classified as flat-shell element and shallow shell element.

Table 20.4 Index of generalized conforming membrane and shell elements (15 elements)

<table>
<tr><th colspan="2">Symbol of element</th><th>Original reference</th><th>Chapter & section no.</th><th>Remark</th></tr>
<tr><td>Membrane element</td><td>1. Isoparametric element GC-Q6</td><td>[23]</td><td>11.2</td><td>It degenerates to Wilson Q6 element in rectangular case</td></tr>
<tr><td rowspan="3">Membrane element with drilling freedom</td><td>2. Rectangular element GR12
3. Rectangular element GR12M</td><td>[24]
[24]</td><td>11.3</td><td>Without internal parameter
With two internal parameters</td></tr>
<tr><td>4. Quadrilateral element GQ12
5. Quadrilateral element GQ12M</td><td>[25]
[25]</td><td>11.3</td><td>Without internal parameter
With two internal parameters</td></tr>
<tr><td>6. Triangular element GT9
7. Triangular element GT9M
8. Triangular element GT9M8</td><td>[26]
[26]
[27]</td><td>11.4</td><td>Without internal parameter
With one internal parameter
With eight internal parameters</td></tr>
<tr><td>Flat-shell element</td><td>9. Rectangular element GCR24
10. Triangular element GST18
11. Triangular element GST18M
12. Triangular element GMST18</td><td>[28]
[29]
[29]
[17]</td><td>11.5</td><td>GCR24 = GR12 + GPL-R12
GST18 = GT9 + GPL-T9
GST18M = GT9M8 + GPL-T9
GMST18 = GT9 + TSL-T9</td></tr>
</table>

(Continued)

Symbol of element		Original reference	Chapter & section no.	Remark
Shallow shell element	13. Rectangular element GC-S20	[30]	11.6	For eliminating membrane locking, generalized conforming conditions between membrane strains and displacements are used
	14. Triangular element SST21	[31]	11.7	For eliminating membrane locking, tangential freedoms at mid-side nodes are introduced
	15. Rectangular element SSR28	[32]	11.9	For eliminating membrane locking, tangential freedoms at mid-side nodes are introduced

During the studies on the generalized conforming membrane and shell elements, developments for three key problems have been achieved:

(1) Rational definition of nodal drilling freedom in membrane element—On the basis of the analysis of the disadvantages existing in two early definitions for nodal rotation freedom, a new more rational definition is proposed.

(2) Double improvements on flat-shell element—Flat-shell element is composed of two parts: membrane element and plate bending element. Here, these two parts are both upgraded: the membrane element part is replaced by the new model with newly defined drilling freedoms, and the plate bending element part is replaced by the generalized conforming element with excellent performance.

(3) Elimination of the membrane locking phenomenon in shallow shell element—By adding the tangential displacement DOFs at the mid-side nodes, the orders of the tangential and normal displacements can match with each other, so that membrane locking will be eliminated.

20.2.2 Sub-Region Mixed Element Series

The sub-region mixed element method is a new finite element method derived from the sub-region mixed variational principle. Its feature is that a coupling mesh composed of conventional displacement-based and singular stress-based elements is used, so that both advantages from the two models can be obtained. It specializes in the analysis of crack and notch problems which contain singular stress points.

This book presents a detailed introduction to the sub-region mixed element method for the analyses of 4 different crack problems and 5 different V-notch problems. And, the index of the new elements for these 9 problems is given in Table 20.5.

Table 20.5 Index of sub-region mixed elements (9 elements)

Symbol of element		Original reference	Chapter & section no.
Crack problem	1. SRM-C1 2D crack of mode Ⅰ	[7]	12.3
	2. SRM-C2 2D crack of mixed mode	[33]	12.3
	3. SRM-C3 Crack in thick plate	[34]	12.4
	4. SRM-C4 Surface crack in 3D body	[35]	12.5
Notch problem	5. SRM-V1 Plane V-notch	[36]	13.2
	6. SRM-V2 Bi-material plane notch	[37]	13.3
	7. SRM-V3 Bi-material anti-plane notch	[38]	13.4
	8. SRM-V4 V-notch in thick plate	[39]	13.5
	9. SRM-V5 3D V-notch	[40]	13.6

20.2.3 Analytical Trial Function Element Series

By taking the analytical solutions as trial functions, three developments for the finite element method have been achieved:

(1) For the membrane element, excellent models which can still keep high precision in distorted mesh are successfully constructed.

(2) For the thick plate element, shear locking phenomenon can be eliminated from the outset.

(3) For the crack and notch problems, high precision hybrid elements with singular point are developed.

The index of the new models is given in Table 20.6.

Table 20.6 Index of analytical trial function elements (7 elements)

Symbol of element		Original reference	Chapter & section no.	Remark
4-node membrane element	1. ATF-Q4a (point conforming) 2. ATF-Q4b (perimeter-point conforming) 3. ATF-Q4θ (with drilling freedoms)	[41]	14.2	
Thick plate element	4. quadrilateral element ATF-MQ	[42]	14.5	
	5. triangular element GPLM	(this book)	14.6	Generalization of thin plate element GPL
Singular hybrid element	6. ATF-MS (for crack problem) 7. ATF-VN (for notch problem)	[43] [44]	15.3 15.7	

20.2.4 Quadrilateral Area Coordinate Element Series

The establishment of the quadrilateral area coordinate theory provides a new tool for the construction of the quadrilateral element models. Compared with those traditional isoparametric models, the elements formulated by such new tool are more insensitive to various mesh distortions. Up to date, more than 20 models have been developed, including membrane elements, membrane elements with drilling freedoms, thin plate elements, thick plate elements, laminated composite plate element, and so on. The index of these elements can be found in Table 17.1.

20.2.5 Spline Element Series

Spline interpolation—possesses the highest order continuity among the piecewise polynomial with the same order, and the most smoothing curve can be obtained.

Spline element method is a new finite element method which uses spline functions as the interpolation functions in the sub-regions. It possesses high smoothing and precision, which is suitable for complicated structures. This book introduces 18 spline element models, and their detailed information can be referred to Table 20.7. Some materials about triangular, sector and other spline elements can be referred to references [11] and [12].

Table 20.7 Index of spline elements (18 elements)

Symbol of element		Original reference	Chapter & section no.
Beam element	1. Quadratic spline beam element 2. Cubic spline beam element	[11]	18.2
Membrane element	3. Bi-quadratic spline rectangular element 4. Bi-cubic spline rectangular element	[11]	18.3
Membrane element for high-rise building	5. spline rectangular element TB-12 6. spline rectangular element TB-13 7. spline rectangular element TB-23	[45]	18.4 18.5
Thin plate element	8. quadratic spline rectangular element 9. cubic spline rectangular element	[11]	19.1
Thin-thick plate element	10. quadratic spline rectangular element 11. cubic spline rectangular element	[11]	19.2
Thin shallow shell element	12. spline element R-QQQ 13. spline element R-CQQ 14. spline element R-CCC	[11]	19.3
Thick shallow shell element	15. quadratic spline rectangular element 16. cubic spline rectangular element	[46]	19.4.1
Thick axisymmetric shell element	17. quadratic spline sector element 18. cubic spline sector element	[47]	19.4.2

20.3 New Solution Strategies for Five Challenging Problems

During the developments of many subjects, some plain and simple problems are always encountered and solved firstly, while a few unsolved challenging problems have to be left, even for a long time. The appearance of novel and effective solution strategies for the challenging problems is a symbol of advance in related subjects. For example:

(1) The shear-locking problem in thick plate element and its countermeasures

The shear-locking phenomenon of the thick plate element is caused by false shear strain when analyzing a thin plate. Two effective solution strategies proposed recently are: ① the rational interpolation method for shear strain[13,16] (1998, locking-free shear strain field is obtained by using the Timoshenko's beam formulae); and ② the analytical trial function method based on the analytical solutions of the thick plate theory[42].

(2) The sensitivity problem to mesh distortion and its countermeasures

Two effective solution strategies are proposed recently. ① Instead of isoparametric coordinates, the quadrilateral area coordinates are used in the interpolation polynomial of the element displacement field[48–51]; ② The analytical trial function method is used[41].

(3) The non-convergence problem of non-conforming elements and its countermeasure

The reason for the appearance of this problem is that the minimum potential energy principle is irrationally used to construct the non-conforming element. The key procedure of the generalized conforming element[6] is the introduction of the fundamental generalized conforming conditions, which makes the application of the minimum potential energy principle rational, so that the convergence can be ensured.

(4) The accuracy loss problem of stress solutions by displacement-based elements and its countermeasure

The accuracy loss of stress solutions by displacement-based elements is caused by differential operations. Its countermeasure is the hybrid-enhanced post-processing procedure[21], in which the stresses are computed by using Hellinger-Reissner variational principle, i.e., the differential operations are replaced by a macro method.

(5) The singular stress problem and its countermeasures

In order to improve the computational accuracy near a singular stress point, the analytical solutions with singular terms should be used adequately. Therefore, the effective countermeasures are the sub-region mixed element method[7] and the analytical trial function method[8].

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