Chapter 2 The Sub-Region Variational Principles

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Abstract This chapter focuses on the developments of the variational principles which are usually considered as the theoretical basis for the finite element method. In this chapter, we will discuss the *sub-region variational principles* which are the results by the combination of the variational principles and the concept of sub-region interpolation. Following the introduction, the sub-region variational principles for various structural forms, i.e., 3D elastic body, thin plate, thick plate and shallow shell, are presented respectively. Finally, a *sub-region mixed energy partial derivative theorem* is also given.

Keywords variational principle, sub-region variational principle, sub-region mixed energy partial derivative theorem.

2.1 Introduction

Variational principles are usually considered as the theoretical basis for the finite element method. References [1-3] present systematical discussions on some of these variational principles. And, some advances and reviews on this field can be found in the references [4-8].

The sub-region variational principles for elasticity and structural mechanics have been proposed in the references [2, 9]. In the third edition of the reference [1] (1982), the contents of modified variational principles were supplemented. Though the expressions are different, they indeed have a close relationship with the sub-region variational principles.

The studies on the sub-region variational principles were promoted by the advances in the finite element method, and especially by the development of the incompatible element, the generalized conforming element, the hybrid element and the sub-region mixed element approaches. The sub-region generalized variational principles for 3D elasticity was proposed and extended to multi-region mixed energy principle in [2] and [9]. And, the sub-layer variational principle was also discussed in [10]. A review of the sub-region variational principles and their applications in the finite element method was given in [11]. For the elastic thin plate, its sub-region potential principle and sub-region complementary principle were presented in [2], and its sub-region mixed energy principle was given in [12]. For the thick plate and the shallow shell, their sub-region variational principles were proposed in [13] and [14], respectively. And, the reference [15] provided the *sub-region mixed energy partial derivative theorem*, a generalization of the famous *Castigliano first* and *second energy partial derivative theorems*.

From the viewpoint of structure forms, it can be seen that there are four types, 3D elasticity, thin plate, thick plate and shallow shell, as listed above. The sub-region variational principles of these structures and their energy functional expressions will be introduced in the following four sections, respectively.

From the viewpoint of independent field variables assumed in each sub-region, it can be found that three cases of regions are existing here: ① three-field-region (displacement field, strain field and stress field), ② two-field-region (displacement field and stress field), and ③ single-field-region (displacement field or stress field).

From the viewpoint of energy types, it can be seen that each sub-region can be assumed as either potential or complementary energy region. If all the regions are assumed as potential (or complementary) energy regions, the sub-region potential (or complementary) energy principle will be obtained. If some regions are assumed as potential energy regions, and the others are assumed as complementary energy regions, the sub-region mixed energy variational principle will be obtained.

The sub-region variational principle provides the theoretical basis for developing new finite element methods. For example, the generalized conforming element method described in Part Ⅱ of this book is based on the sub-region potential energy principle; and the sub-region mixed element method given in Part Ⅲ is based on the sub-region mixed energy principle.

2.2 The Sub-Region Variational Principle for Elasticity

This section will discuss the various forms[9,10] of the sub-region generalized variational principle used in elasticity problems. Firstly, let an elastic body be divided into two sub-regions, *a* and *b*, then the sub-region three-field generalized mixed, potential and complementary energy variational principles are discussed, respectively. Secondly, two special cases, the sub-region two-field and single-field generalized variational principles, are discussed. Finally, a general form of the multi-region variational principle is established.

2.2.1 The Sub-Region Three-Field Generalized Mixed Variational Principle for Elasticity

Let an elastic body be divided into two sub-regions a and b; V_a and V_b be the volumes of the regions a and b, respectively; S_a and S_b be the surfaces of a and b, respectively. Thus, both the surfaces S_a and S_b are composed of three parts:

$$S_a = S_{\sigma a} + S_{ua} + S_{ab}$$
$$S_b = S_{\sigma b} + S_{ub} + S_{ab}$$

Where S_{ab} is the interface between a and b; $S_{\sigma a}$ and $S_{\sigma b}$ are the boundaries with given tractions $\overline{T}_i (i=1,2,3)$; S_{ua} and S_{ub} are the boundaries with given displacements $\overline{u}_i (i=1,2,3)$. (see Fig. 2.1)

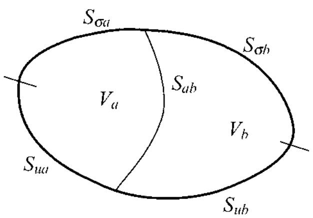

Figure 2.1 An elastic body divided into two sub-regions

In the sub-region three-field generalized mixed variational principle, the displacements, strains and stresses

$$u_i^{(a)}, \varepsilon_{ij}^{(a)}, \sigma_{ij}^{(a)}; \quad u_i^{(b)}, \varepsilon_{ij}^{(b)}, \sigma_{ij}^{(b)} \quad (i, j = 1,2,3)$$

in the regions a and b are all field variables. Then the corresponding functional Π_3 can be defined by

$$\Pi_3 = \Pi_{3\mathrm{p}}^{(a)} - \Pi_{3\mathrm{c}}^{(b)} + H_{\mathrm{pc}} \tag{2-1}$$

where $\Pi_{3\mathrm{p}}^{(a)}$ is named as the three-field generalized potential energy of the sub-region a (excluding the interface S_{ab}):

$$\Pi_{3\mathrm{p}}^{(a)} = \iiint_{V_a} \left[\tilde{U}(\varepsilon_{ij}) - \sigma_{ij} \left(\varepsilon_{ij} - \frac{1}{2} u_{i,j} - \frac{1}{2} u_{j,i} \right) - \overline{F}_i u_i \right] \mathrm{d}V - \iint_{S_{\sigma a}} \overline{T}_i u_i \mathrm{d}S - \iint_{S_{ua}} T_i \left(u_i - \overline{u}_i \right) \mathrm{d}S \tag{2-2}$$

in which $\tilde{U}(\varepsilon_{ij})$ denotes the strain energy density; $\overline{F}_i$ denotes the given body force; $u_{i,j}$ denotes the partial derivative of u_i with respect to x_j . $\Pi_{3c}^{(b)}$ is named as the three-field generalized complementary energy of the sub-region b (also excluding the interface S_{ab}):

$$\Pi_{3c}^{(b)} = \iiint_{V_b} \left[\sigma_{ij}\varepsilon_{ij} - \tilde{U}(\varepsilon_{ij}) + \left(\sigma_{ij,j} + \overline{F}_i\right) u_i \right] \mathrm{d}V - \iint_{S_{\sigma b}} \left(T_i - \overline{T}_i\right) u_i \mathrm{d}S - \iint_{S_{ub}} T_i \overline{u}_i \mathrm{d}S \tag{2-3}$$

H_{pc} is the mixed energy at the interface S_{ab}, and given by

$$H_{pc} = \iint_{S_{ab}} T_i^{(b)} u_i^{(a)} \mathrm{d}S \tag{2-4}$$

in which $T_i^{(b)}$ denotes the traction of the complementary energy region (sub-region b) at the interface S_{ab}:

$$T_i^{(b)} = \sigma_{ij}^{(b)} n_j^{(b)}$$

$n_j^{(b)}$ is the direction cosine of the outer normal of the region b at the interface S_{ab}; $u_i^{(a)}$ denotes the displacement of the potential energy region (sub-region a) at the interface S_{ab}.

The sub-region three-field generalized mixed variational principle can be described as follows.

The functional stationary condition

$$\delta\Pi_3 = \delta\Pi_{3p}^{(a)} - \delta\Pi_{3c}^{(b)} + \delta H_{pc} = 0 \tag{2-5}$$

is equivalent to the whole system of equations of the elastic body with sub-regions, including equilibrium differential equation:

$$\sigma_{ij,j} + \overline{F}_i = 0 \quad (\text{in } V) \tag{2-6}$$

strain-displacement relations (geometrical equation)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (\text{in } V) \tag{2-7}$$

stress-strain relations (constitutive equation)

$$\sigma_{ij} = \frac{\partial \tilde{U}}{\partial \varepsilon_{ij}} \quad (\text{in } V) \tag{2-8}$$

boundary conditions of tractions

$$T_i = \sigma_{ij} n_j = \overline{T}_i \quad (\text{on } S_\sigma) \tag{2-9}$$

boundary conditions of displacements

$$u_i = \overline{u}_i \quad (\text{on } S_u) \tag{2-10}$$

and continuous conditions at the interface

$$T_i^{(a)} = -T_i^{(b)} \quad (\text{on } S_{ab}) \tag{2-11}$$

$$u_i^{(a)} = u_i^{(b)} \quad (\text{on } S_{ab}) \tag{2-12}$$

In order to demonstrate the equivalency between the functional stationary condition (2-5) and the Eqs. (2-6) – (2-12), the variation $\delta \Pi_{3\text{p}}^{(a)}$ of Eq. (2-2) is firstly developed:

$$\delta \Pi_{3\text{p}}^{(a)} = \iiint_{V_a} \left[\left(\frac{\partial \tilde{U}}{\delta \varepsilon_{ij}} - \sigma_{ij} \right) \delta \varepsilon_{ij} - \left(\varepsilon_{ij} - \frac{1}{2} u_{i,j} - \frac{1}{2} u_{j,i} \right) \delta \sigma_{ij} + \sigma_{ij} \delta u_{i,j} - \overline{F}_i \delta u_i \right] \mathrm{d}V$$
$$- \iint_{S_{\sigma a}} \overline{T}_i \delta u_i \mathrm{d}S - \iint_{S_{ua}} [T_i \delta u_i + (u_i - \overline{u}_i) \delta T_i] \mathrm{d}S$$

Since

$$\iiint_{V_a} \sigma_{ij} \delta u_{i,j} \mathrm{d}V = \iint_{S_a = S_{\sigma a} + S_{ua} + S_{ab}} T_i \delta u_i \mathrm{d}S - \iiint_{V_a} \sigma_{ij,j} \delta u_i \mathrm{d}V$$

we have

$$\delta \Pi_{3\text{p}}^{(a)} = \iiint_{V_a} \left[\left(\frac{\partial \tilde{U}}{\delta \varepsilon_{ij}} - \sigma_{ij} \right) \delta \varepsilon_{ij} - \left(\varepsilon_{ij} - \frac{1}{2} u_{i,j} - \frac{1}{2} u_{j,i} \right) \delta \sigma_{ij} - (\sigma_{ij,j} + \overline{F}_i) \delta u_i \right] \mathrm{d}V$$
$$+ \iint_{S_{\sigma a}} (T_i - \overline{T}_i) \delta u_i \mathrm{d}S - \iint_{S_{ua}} (u_i - \overline{u}_i) \delta T_i \mathrm{d}S + \iint_{S_{ab}} T_i^{(a)} \delta u_i^{(a)} \mathrm{d}S \tag{2-13}$$

Secondly, the variation $\delta \Pi_{3\text{c}}^{(b)}$ of Eq. (2-3) can be written as:

$$\delta \Pi_{3\text{c}}^{(b)} = \iiint_{V_b} \left[\left(\sigma_{ij} - \frac{\partial \tilde{U}}{\delta \varepsilon_{ij}} \right) \delta \varepsilon_{ij} + (\sigma_{ij,j} + \overline{F}_i) \delta u_i + \varepsilon_{ij} \delta \sigma_{ij} + u_i \delta \sigma_{ij,j} \right] \mathrm{d}V$$
$$- \iint_{S_{\sigma b}} [(T_i - \overline{T}_i) \delta u_i + u_i \delta T_i] \mathrm{d}S - \iint_{S_{ub}} \overline{u}_i \delta T_i \mathrm{d}S$$

Since

$$\iiint_{V_b} u_i \delta \sigma_{ij,j} \mathrm{d}V = \iint_{S_b = S_{\sigma b} + S_{ub} + S_{ab}} u_i \delta T_i \mathrm{d}S - \iiint_{V_b} \frac{1}{2} (u_{i,j} + u_{j,i}) \delta \sigma_{ij} \mathrm{d}V$$

we have

$$\delta\Pi_{3c}^{(b)} = \iiint\limits_{V_b}\left[\left(\sigma_{ij} - \frac{\partial\tilde{U}}{\delta\varepsilon_{ij}}\right)\delta\varepsilon_{ij} + (\sigma_{ij,j} + \bar{F}_i)\delta u_i + \left(\varepsilon_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}\right)\delta\sigma_{ij}\right]\mathrm{d}V$$
$$- \iint\limits_{S_{\sigma b}}(T_i - \bar{T}_i)\delta u_i \mathrm{d}S + \iint\limits_{S_{ub}}(u_i - \bar{u}_i)\delta T_i \mathrm{d}S + \iint\limits_{S_{ab}} u_i^{(b)}\delta T_i^{(b)}\mathrm{d}S \tag{2-14}$$

Thirdly, the variation δH_{pc} of Eq. (2-4) is

$$\delta H_{\mathrm{pc}} = \iint\limits_{S_{ab}}(T_i^{(b)}\delta u_i^{(a)} + u_i^{(a)}\delta T_i^{(b)})\mathrm{d}S \tag{2-15}$$

Finally, the substitution of Eqs. (2-13), (2-14) and (2-15) into (2-5) yields

$$\delta\Pi_3 = \iiint\limits_{V}\left[\left(\frac{\partial\tilde{U}}{\delta\varepsilon_{ij}} - \sigma_{ij}\right)\delta\varepsilon_{ij} - \left(\varepsilon_{ij} - \frac{1}{2}u_{i,j} - \frac{1}{2}u_{j,i}\right)\delta\sigma_{ij} - (\sigma_{ij,j} + \bar{F}_i)\delta u_i\right]\mathrm{d}V$$
$$+ \iint\limits_{S_\sigma}(T_i - \bar{T}_i)\delta u_i \mathrm{d}S - \iint\limits_{S_u}(u_i - \bar{u}_i)\delta T_i \mathrm{d}S + \iint\limits_{S_{ab}}[(T_i^{(a)} + T_i^{(b)})\delta u_i^{(a)}$$
$$+ (u_i^{(a)} - u_i^{(b)})\delta T_i^{(b)}]\mathrm{d}S = 0 \tag{2-16}$$

Equations (2-6)−(2-12) can be derived from the functional stationary condition (2-16), and vice versa. Thus, the equivalency is proved.

It should be pointed out that, in the expression (2-4) for the mixed energy H_{pc} at the interface S_{ab}, T_i is indicated as belonging to the sub-region b (complementary energy region), and u_i as belonging to the sub-region a (potential energy region). If H_{pc} is defined as

$$\iint\limits_{S_{ab}} T_i^{(a)}u_i^{(b)}\mathrm{d}S \quad \text{or} \quad \iint\limits_{S_{ab}} T_i^{(a)}u_i^{(a)}\mathrm{d}S \quad \text{or} \quad \iint\limits_{S_{ab}} T_i^{(b)}u_i^{(b)}\mathrm{d}S,$$

incorrect results will appear. The reason is that the field variables of the sub-regions a and b are all independent variables, they do not previously satisfy the continuous conditions (2-11) and (2-12) at the interface.

The variational principle discussed above is a kind of unconditioned variational principle. "Unconditioned" has two meanings: ① Firstly, the three variables u_i, ε_{ij}, and σ_{ij} within each sub-region are all independent and have no relation with each other; ② Secondly, at the interface S_{ab}, the variables from the two regions are also independent, they are not required in advance to satisfy the continuous conditions (2-11) and (2-12).

2.2.2 The Transformation Between $\Pi_{3\mathrm{p}}^{(a)}$ and $\Pi_{3\mathrm{c}}^{(a)}$

In Fig. 2.1, the three-field generalized potential energy $\Pi_{3\mathrm{p}}^{(a)}$ and the three-field generalized complementary energy $\Pi_{3c}^{(a)}$ of the sub-region a (excluding the interface S_{ab}) have the following transformation relationship:

$$\Pi_{3p}^{(a)} + \Pi_{3c}^{(a)} = \iint_{S_{ab}} T_i^{(a)} u_i^{(a)} \mathrm{d}S \tag{2-17}$$

The sub-region three-field generalized variational principle has three forms: sub-region mixed energy, sub-region potential energy and sub-region complementary energy. One form can be easily transformed to the other two by using the relation (2-17).

Following is the demonstration of Eq. (2-17). Firstly, the expression of $\Pi_{3c}^{(a)}$ can be written as:

$$\Pi_{3c}^{(a)} = \iiint_{V_a} \left[\sigma_{ij}\varepsilon_{ij} - \tilde{U}(\varepsilon_{ij}) + \left(\sigma_{ij,j} + \overline{F}_i\right) u_i \right] \mathrm{d}V - \iint_{S_{\sigma a}} (T_i - \overline{T}_i) u_i \mathrm{d}S - \iint_{S_{ua}} T_i \overline{u}_i \mathrm{d}S \tag{2-18}$$

Then, the sum of Eqs. (2-2) and (2-18) can be obtained:

$$\Pi_{3p}^{(a)} + \Pi_{3c}^{(a)} = \iiint_{V_a} \left[\frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) + \sigma_{ij,j}u_i \right] \mathrm{d}V - \iint_{S_{\sigma a}+S_{ua}} T_i u_i \mathrm{d}S$$

Since

$$\iiint_{V_a} \left[\frac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) + \sigma_{ij,j}u_i \right] \mathrm{d}V = \iiint_{V_a} (\sigma_{ij}u_i)_{,j} \mathrm{d}V = \iint_{S_a = S_{\sigma a}+S_{ua}+S_{ab}} T_i u_i \mathrm{d}S$$

The substitution of this equation into the previous one will yield Eq. (2-17).

Two special cases can be derived from Eq. (2-17):

Special case 1: when there are no sub-regions in the whole body, $S_{ab} = 0$. Then we have

$$\Pi_{3p} + \Pi_{3c} = 0 \tag{2-19a}$$

Special case 2: when the sub-region a is surrounded by other regions, $S_{\sigma a} = 0$, $S_{ua} = 0$, $S_a = S_{ab}$. Then we have

$$\Pi_{3p}^{(a)} + \Pi_{3c}^{(a)} = \iint_{S_a} T_i^{(a)} u_i^{(a)} \mathrm{d}S \tag{2-19b}$$

2.2.3 The Sub-Region Three-Field Generalized Potential and Complementary Energy Principles for Elasticity

Now, by using Eq. (2-17), the functional of the sub-region three-field generalized potential and complementary principles can be derived from the functional of the sub-region three-field generalized mixed variational principle.

1. The sub-region three-field generalized potential energy principle

In the expression (2-1) of the sub-region three-field generalized mixed variational principle, the sub-region a is represented by the generalized potential energy while the sub-region b is represented by the generalized complementary energy. Here, we require the sub-region b given by the generalized potential energy, too. Then, from Eq. (2-17), we have

$$\Pi_{3c}^{(b)} = -\Pi_{3p}^{(b)} + \iint_{S_{ab}} T_i^{(b)} u_i^{(b)} \mathrm{d}S$$

Substitution of this equation into (2-1) yields

$$\Pi_3 = \Pi_{3p}^{(a)} + \Pi_{3p}^{(b)} + H_{pp} \tag{2-20}$$

where H_{pp} is the additional term of the potential energy at the interface S_{ab}:

$$H_{pp} = \iint_{S_{ab}} T_i^{(b)} (u_i^{(a)} - u_i^{(b)}) \mathrm{d}S \tag{2-21a}$$

Equations (2-20) and (2-21a) are the functional expressions of the sub-region three-field generalized potential energy principle. It can be shown that the stationary condition $\delta\Pi_3 = 0$ of this functional is equivalent to all equations, boundary conditions and interface continuous conditions of the elastic body with sub-regions. Another expression of H_{pp} can also be obtained by interchanging a and b in Eq. (2-21a):

$$H_{pp} = \iint_{S_{ab}} T_i^{(a)} (u_i^{(b)} - u_i^{(a)}) \mathrm{d}S \tag{2-21b}$$

If the continuous condition (2-12) at the interface S_{ab} is satisfied in advance, $H_{pp} = 0$.

2. The sub-region three-field generalized complementary energy principle

In Eq. (2-1), if we require that the sub-region a is given by the generalized complementary energy, the substitution of (2-17) into (2-1) will yield

$$\Pi_3 = -\Pi_{3c}^{(a)} - \Pi_{3c}^{(b)} + H_{cc} \tag{2-22}$$

where H_{cc} is the additional term of the complementary energy at the interface S_{ab}:

$$H_{cc} = \iint_{S_{ab}} (T_i^{(a)} + T_i^{(b)}) u_i^{(a)} \mathrm{d}S \tag{2-23a}$$

Equations (2-22) and (2-23a) are the functional expressions of the sub-region three-field generalized complementary energy principle. Another expression of H_{cc} can also be obtained by interchanging a and b in Eq. (2-23a):

$$H_{\mathrm{cc}} = \iint_{S_{ab}} (T_i^{(a)} + T_i^{(b)}) u_i^{(b)} \mathrm{d}S \tag{2-23b}$$

If the continuous condition (2-11) at the interface S_{ab} is satisfied in advance, $H_{\mathrm{cc}} = 0$.

2.2.4 The Sub-Region Two-Field and Single-Field Variational Principle for Elasticity

The functional expression for the three forms of the sub-region three-field generalized variational principle have been given by Eqs. (2-1), (2-20) and (2-22), respectively. Now, we discuss two special cases.

1. The sub-region two-field generalized variational principle

By employing the relationship

$$\tilde{V}(\sigma_{ij}) = \sigma_{ij}\varepsilon_{ij} - \tilde{U}(\varepsilon_{ij}) \tag{2-24}$$

between the strain energy density $\tilde{U}(\varepsilon_{ij})$ and the strain complementary energy density $\tilde{V}(\sigma_{ij})$, the variable ε_{ij} in the three-field generalized potential energy $\Pi_{3\mathrm{p}}^{(a)}$ and the three-field generalized complementary energy $\Pi_{3\mathrm{c}}^{(a)}$ of the sub-region a (excluding the interface S_{ab}) can be eliminated. Thus, the two-field (displacement u_i, stress σ_{ij}) generalized potential energy $\Pi_{2\mathrm{p}}^{(a)}$ and the two-field generalized complementary energy $\Pi_{2\mathrm{c}}^{(a)}$ can be obtained:

$$\Pi_{2\mathrm{p}}^{(a)} = \iiint_{V_a} \left[\frac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij} - \tilde{V}(\sigma_{ij}) - \bar{F}_i u_i \right] \mathrm{d}V - \iint_{S_{\sigma a}} \bar{T}_i u_i \mathrm{d}S - \iint_{S_{ua}} T_i (u_i - \bar{u}_i) \mathrm{d}S \tag{2-25}$$

$$\Pi_{2\mathrm{c}}^{(a)} = \iiint_{V_a} \left[\tilde{V}(\sigma_{ij}) + (\sigma_{ij,j} + \bar{F}_i) u_i \right] \mathrm{d}V - \iint_{S_{\sigma a}} (T_i - \bar{T}_i) u_i \mathrm{d}S - \iint_{S_{ua}} T_i \bar{u}_i \mathrm{d}S \tag{2-26}$$

From Eqs. (2-1), (2-20) and (2-22), the functional expressions of the sub-region two-field generalized mixed energy, potential energy and complementary energy principle can be written as follows:

$$\Pi_2 = \Pi_{2\mathrm{p}}^{(a)} - \Pi_{2\mathrm{c}}^{(b)} + H_{\mathrm{pc}} \tag{2-27}$$

$$\Pi_2 = \Pi_{2\mathrm{p}}^{(a)} + \Pi_{2\mathrm{p}}^{(b)} + H_{\mathrm{pp}} \tag{2-28}$$

$$\Pi_2 = -\Pi_{2\mathrm{c}}^{(a)} - \Pi_{2\mathrm{c}}^{(b)} + H_{\mathrm{cc}} \tag{2-29}$$

where H_{pc}, H_{pp} and H_{cc} are still given by Eqs. (2-4), (2-21) and (2-23), respectively.

2. The sub-region single-field generalized variational principle

Now, we discuss the case where each sub-region has only a single independent variable. If the sub-region a is a potential energy region, only the displacement $u_i^{(a)}$ will be taken as the independent variable. Thus, the $\Pi_{3\mathrm{p}}^{(a)}$ in Eq. (2-2) and the $\Pi_{2\mathrm{p}}^{(a)}$ in Eq. (2-25) will transform to the single-field potential energy $\Pi_{1\mathrm{p}}^{(a)}$ of the region a:

$$\Pi_{1\mathrm{p}}^{(a)} = \iiint_{V_a} [\tilde{U}(u_i) - \bar{F}_i u_i]\mathrm{d}V - \iint_{S_{\sigma a}} \bar{T}_i u_i \mathrm{d}S - \iint_{S_{ua}} T_i (u_i - \bar{u}_i)\mathrm{d}S \tag{2-30a}$$

If $u_i^{(a)}$ satisfies the displacement boundary condition (2-10) on S_{ua} in advance, then we have

$$\Pi_{1\mathrm{p}}^{(a)} = \iiint_{V_a} [\tilde{U}(u_i) - \bar{F}_i u_i]\mathrm{d}V - \iint_{S_{\sigma a}} \bar{T}_i u_i \mathrm{d}S \tag{2-30b}$$

If the sub-region a is a complementary energy region, only the stress $\sigma_{ij}^{(a)}$ will be taken as the independent variable, and $\sigma_{ij}^{(a)}$ should satisfy the equilibrium differential Eq. (2-6) in advance. Thus, the $\Pi_{3\mathrm{c}}^{(a)}$ in Eq. (2-18) and the $\Pi_{2\mathrm{c}}^{(a)}$ in Eq. (2-26) will transform to the single-field complementary energy $\Pi_{1\mathrm{c}}^{(a)}$ of the region a:

$$\Pi_{1\mathrm{c}}^{(a)} = \iiint_{V_a} \tilde{V}(\sigma_{ij})\mathrm{d}V - \iint_{S_{ua}} T_i \bar{u}_i \mathrm{d}S - \iint_{S_{\sigma a}} (T_i - \bar{T}_i) u_i \mathrm{d}S \tag{2-31a}$$

If $\sigma_{ij}^{(a)}$ satisfies the boundary condition (2-9) on $S_{\sigma a}$ in advance, then we have

$$\Pi_{1\mathrm{c}}^{(a)} = \iiint_{V_a} \tilde{V}(\sigma_{ij})\mathrm{d}V - \iint_{S_{ua}} T_i \bar{u}_i \mathrm{d}S \tag{2-31b}$$

From Eqs. (2-1), (2-20) and (2-22), or (2-27), (2-28) and (2-29), the functional expressions of the sub-region single-field generalized mixed energy, potential energy and complementary energy principle can be written as follows:

$$\Pi_1 = \Pi_{1\mathrm{p}}^{(a)} - \Pi_{1\mathrm{c}}^{(b)} + H_{\mathrm{pc}} \tag{2-32}$$

$$\Pi_1 = \Pi_{1\mathrm{p}}^{(a)} + \Pi_{1\mathrm{p}}^{(b)} + H_{\mathrm{pp}} \tag{2-33}$$

$$\Pi_1 = -\Pi_{1\mathrm{c}}^{(a)} - \Pi_{1\mathrm{c}}^{(b)} + H_{\mathrm{cc}} \tag{2-34}$$

where H_{pc}, H_{pp} and H_{cc} are still given by Eqs. (2-4), (2-21) and (2-23), respectively. $T_i^{(a)}$ or $T_i^{(b)}$ in Eq. (2-21), and $u_i^{(a)}$ or $u_i^{(b)}$ in Eq. (2-23), can be treated as Lagrange multipliers.

2.2.5 The General Form of the Multi-Region Variational Principle for Elasticity

From the above discussions, a general form of the multi-region variational principle can be obtained.

Let an elastic body be divided into several sub-regions (see Fig. 2.2). Each sub-region can be arbitrarily appointed as potential energy region or complementary energy region, and each region can be three-field region, or two-field region or single-field region. The interfaces between two adjacent regions are of three types, S_{pc}, S_{pp} and S_{cc}: ① one side of S_{pc} is the potential energy region, while the other side is the complementary one; ② both sides of S_{pp} are potential energy regions; and ③ both sides of S_{cc} are complementary energy regions.

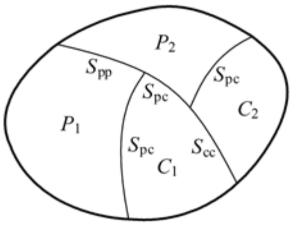

Figure 2.2 An elastic body divided into multi-regions

The general form of the functional for multi-region variational principle can be written as

$$\Pi = \sum_{V_p} \Pi_p - \sum_{V_c} \Pi_c + \sum_{S_{pc}} H_{pc} + \sum_{S_{pp}} H_{pp} + \sum_{S_{cc}} H_{cc} \tag{2-35}$$

The meanings of the terms on the right-side of this equation are as follows:

The first term denotes the sum of the potential (or generalized potential) energy Π_p of each potential energy region V_p, where Π_p can be Π_{1p} or Π_{2p} or Π_{3p}, which is given by Eqs. (2-30), (2-25) and (2-2), respectively.

The second term denotes the sum of the complementary (or generalized complementary) energy Π_c of each complementary energy region V_c, where Π_c can be Π_{1c} or Π_{2c} or Π_{3c}, which is given by Eqs. (2-31), (2-26) and (2-3), respectively.

The third term denotes the sum of the additional term H_{pc} on the interface S_{pc}, in which H_{pc} is given by Eq. (2-4). The fourth term denotes the sum of the additional term H_{pp} on the interface S_{pp}, in which H_{pp} is given by Eq. (2-21). The fifth term denotes the sum of the additional term H_{cc} on the interface S_{cc}, in which H_{cc} is given by Eq. (2-23).

It can be shown that the stationary condition

$$\delta\Pi = 0$$

of the functional Π in Eq. (2-35) is equivalent to all equations, boundary conditions and interface continuous conditions of the elastic body with multi-regions.

If all regions are potential energy regions, the functional of the sub-region potential (or generalized potential) energy principle can be obtained from Eq. (2-35):

$$\Pi = \sum_{V_{\mathrm{p}}} \Pi_{\mathrm{p}} + \sum_{S_{\mathrm{pp}}} H_{\mathrm{pp}} \tag{2-36}$$

It can be seen that Eqs. (2-20), (2-28) and (2-33) are all special cases of (2-36).

If all regions are complementary energy regions, the functional of the sub-region complementary (or generalized complementary) energy principle can be obtained from Eq. (2-35):

$$\Pi = -\sum_{V_{\mathrm{c}}} \Pi_{\mathrm{c}} + \sum_{S_{\mathrm{cc}}} H_{\mathrm{cc}} \tag{2-37}$$

It can be seen that Eqs. (2-22), (2-29) and (2-34) are all special cases of (2-37).

Incidentally, the interface S_{ab} can vest in V_a (or V_b), and then, the additional terms H_{pc}, H_{pp} and H_{cc} on S_{ab} will vest in the energy terms $\Pi_{\mathrm{p}}^{(a)}$ and $\Pi_{\mathrm{c}}^{(a)}$ of V_a (or the energy terms $\Pi_{\mathrm{p}}^{(b)}$ and $\Pi_{\mathrm{c}}^{(b)}$ of V_b) as new additional terms. Several cases are discussed as follows:

Firstly, if we assume V_a as potential energy region, when S_{ab} is not included, the potential or generalized potential energy of V_a can be written as

$$\Pi_{\mathrm{p}}^{(a)} = \iiint_{V_a} I_{\mathrm{p}} \mathrm{d}V - \iint_{S_{\sigma a}} \bar{T}_i u_i \mathrm{d}S - \iint_{S_{ua}} T_i (u_i - \bar{u}_i) \mathrm{d}S$$

where I_{p} denotes the integrand in volume terms of Eqs. (2-30) or (2-25) or (2-2). Now, if S_{ab} vests in V_a, the new additional terms of $\Pi_{\mathrm{p}}^{(a)}$ can be derived as follows:

(1) If the adjacent region V_b is a potential region, S_{ab} can be dealt with in the same manner as S_{ua}. Let $\bar{u}_i = u_i^{(b)}$, so the new additional term in $\Pi_{\mathrm{p}}^{(a)}$ is

$$-\iint_{S_{ab}} T_i^{(a)} (u_i^{(a)} - u_i^{(b)}) \mathrm{d}S$$

From Eq. (2-21b), it can be seen that this new additional term is just H_{pp}.

(2) If the adjacent region V_b is a complementary region, S_{ab} can be dealt with in the same manner as $S_{\sigma a}$. Let $\bar{T}_i = -T_i^{(b)}$, so the new additional term in $\Pi_{\mathrm{p}}^{(a)}$ is

$$-\iint_{S_{ab}} (-T_i^{(b)}) u_i^{(a)} \mathrm{d}S$$

From Eq. (2-4), it can be seen that this new additional term is just H_{pc}.

Secondly, if we assume V_b as complementary energy region, when S_{ab} is not included, the complementary or generalized complementary energy of V_b can be written as

$$-\Pi_{\mathrm{c}}^{(b)} = -\iiint\limits_{V_b} I_{\mathrm{c}} \mathrm{d}V + \iint\limits_{S_{\sigma b}} (T_i - \overline{T}_i) u_i \mathrm{d}S + \iint\limits_{S_{ub}} T_i \overline{u}_i \mathrm{d}S$$

where I_{c} denotes the integrand in volume terms of Eqs. (2-31) or (2-26) or (2-3). Now, if S_{ab} vests in V_b, the new additional terms of $-\Pi_{\mathrm{c}}^{(b)}$ can be derived as follows:

(3) If the adjacent region V_a is a potential region, the new additional term will be $\iint\limits_{S_{ab}} T_i^{(b)} u_i^{(a)} \mathrm{d}S$, i.e. H_{pc}.

(4) If the adjacent region V_a is a complementary region, the new additional term will be $\iint\limits_{S_{ab}} (T_i^{(b)} + T_i^{(a)}) u_i^{(b)} \mathrm{d}S$, i.e. H_{cc} in Eq. (2-23b).

2.2.6 Some Remarks

The general form of the sub-region generalized variational principle for small displacement elasticity problems is presented in this section, and Eq. (2-35) is its functional expression. Its universality is due to the following reasons:

(1) Each sub-region can be independently specified as potential and complementary energy regions, and the sub-region potential energy, complementary energy and mixed variational principle are three special forms of the general form.

(2) The field variables in each region can be specified independently. The sub-region single-field, two-field, three-field and their mixed forms are all special cases of the general form.

(3) The displacement and traction conditions on each interface can be relaxed partly or completely. It is not necessary to satisfy them in advance.

Various finite element models can all be regarded as the special applications of this principle. For example, the sub-region potential energy principle and its functional (2-36) are the theoretical basis of the generalized conforming elements and the hybrid-displacement elements; the sub-region complementary energy principle and its functional (2-37) are the theoretical basis of the hybrid-stress elements; the sub-region mixed energy principle and its functional (2-1) are the theoretical basis of the sub-region mixed elements.

Besides, there are still some other points worthy of being paid attention to:

(1) By using the relation (2-17), the transformation between the different forms of the variational principle can be performed conveniently.

(2) The general form (2-35) of the functional for the multi-region variational principle establishes a bridge linking the various forms of the variational principle.

2.3 The Sub-Region Variational Principle for Elastic Thin Plate

This section will discuss the sub-region variational principle for elastic thin plate[2,12,16]. The thin plate variational principle with relaxed continuity requirements has been discussed in [16]. And, the multi-region potential and complementary energy generalized variational principles were given by [2]. Reference [12] proposed the multi-region mixed energy generalized variational principle of thin plate, considered the thin plate multi-region potential and complementary energy generalized variational principles as its special cases, and gave out the transformation relations between generalized potential energy and generalized complementary energy in the sub-regions. By using these relations, transformation between different functionals of the variational principle can be performed conveniently.

The sequence of presentation used in the previous section is adopted again here: firstly, the case with two sub-regions is discussed; secondly, from the three-field principle, the two-field and single-field principles are obtained; finally, the general form of the multi-region variational principle is given.

2.3.1 The Sub-Region Three-Field Generalized Mixed Variational Principle for Thin Plate

1. The description of the sub-regions and the boundaries for thin plate

Let an elastic thin plate be divided into two sub-regions a and b (Fig. 2.3), and Ω_a and Ω_b represent the domains of the regions a and b, respectively. The outer

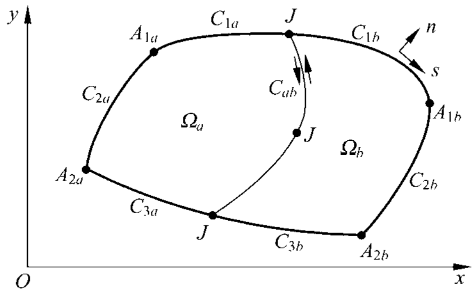

Figure 2.3 A thin plate divided into two sub-regions

boundaries C_a and C_b of the regions a and b are both composed of three parts:

$$C_a = C_{1a} + C_{2a} + C_{3a}$$
$$C_b = C_{1b} + C_{2b} + C_{3b}$$

where C_{1a} and C_{1b} are the fixed boundaries (the deflection $\overline{w}$ and the normal rotation $\bar{\phi}_n$ on the boundaries are specified); C_{2a} and C_{2b} are the simply-supported boundaries (the deflection $\bar{w}$ and the normal moment $\bar{M}_n$ on the boundaries are specified); and C_{3a} and C_{3b} are the free boundaries (the normal moment $\bar{M}_n$ and the equivalent shear force $\bar{V}_n$ on the boundaries are specified).

The corner points A_a and A_b on the outer boundaries of the regions a and b are composed of two corner point types:

$$A_a = A_{1a} + A_{2a}, \quad A_b = A_{1b} + A_{2b}$$

where A_{1a} and A_{1b} are the corner points where the deflection $\bar{w}$ is specified; A_{2a} and A_{2b} are the corner points where the concentrated force $\bar{R}$ is specified.

The interface of the two regions is C_{ab}, on which the node J is also composed of two node types:

$$J = J_1 + J_2$$

where J_1 is the node where the deflection $\bar{w}$ is specified; J_2 is the node where the concentrated force $\bar{R}$ is specified.

(x, y) are the Cartesian co-ordinates within the mid-surface of the thin plate; n is the outer normal of the boundary; s is the tangent of the boundary, and its positive direction is shown in Fig. 2.3.

2. The key points of the sub-region three-field generalized variational principle

(1) The field variables

Both regions a and b possess three field variables:

Deflections:

$$w(a), \quad w(b)$$

Bending and twisting moments:

$$\boldsymbol{M}^{(a)} = [M_x \quad M_y \quad M_{xy}]^{\mathrm{T}(a)}$$
$$\boldsymbol{M}^{(b)} = [M_x \quad M_y \quad M_{xy}]^{\mathrm{T}(b)}$$

Curvatures:

$$\boldsymbol{\kappa}^{(a)} = [\kappa_x \quad \kappa_y \quad 2\kappa_{xy}]^{\mathrm{T}(a)}$$
$$\boldsymbol{\kappa}^{(b)} = [\kappa_x \quad \kappa_y \quad 2\kappa_{xy}]^{\mathrm{T}(b)}$$

These field variables are not required to satisfy any conditions in advance within the domain and on the boundaries and interfaces.

(2) Definition of the functional

Let the region a be the potential energy region, and the region b the complementary energy region. Then, the definition of the functional is

$$\Pi_3 = \Pi_{3\mathrm{p}}^{(a)} - \Pi_{3\mathrm{c}}^{(b)} + H_{\mathrm{pc}} + G_{1\mathrm{pc}} + G_{2\mathrm{pc}} \qquad (2\text{-}38)$$

where $\Pi_{3p}^{(a)}$ is the three-field generalized potential energy of the region *a* (excluding the interface C_{ab} and the node *J*):

$$\Pi_{3p}^{(a)}=\iint_{\Omega_a}\left[\tilde{U}(\boldsymbol{\kappa})-qw-\left(\frac{\partial^2 w}{\partial x^2}+\kappa_x\right)M_x-\left(\frac{\partial^2 w}{\partial y^2}+\kappa_y\right)M_y-2\left(\frac{\partial^2 w}{\partial x\partial y}+\kappa_{xy}\right)M_{xy}\right]\mathrm{d}x\mathrm{d}y$$
$$-\int_{C_{1a}+C_{2a}}\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)(w-\bar{w})\mathrm{d}s-\int_{C_{3a}}\bar{V}_n w\mathrm{d}s+\int_{C_{1a}}M_n\left(\frac{\partial w}{\partial n}-\bar{\psi}_n\right)\mathrm{d}s$$
$$+\int_{C_{2a}+C_{3a}}\bar{M}_n\frac{\partial w}{\partial n}\mathrm{d}s-\sum_{A_{1a}}\Delta M_{ns}(w-\bar{w})-\sum_{A_{2a}}\bar{R}w \tag{2-39}$$

Here, q is the density of the normal load; $\tilde{U}(\boldsymbol{\kappa})$ is the density of the strain energy:

$$\tilde{U}(\boldsymbol{\kappa})=\frac{D}{2}[(\kappa_x+\kappa_y)^2+2(1-\mu)(\kappa_{xy}^2-\kappa_x\kappa_y)] \tag{2-40}$$

where $D=\dfrac{Eh^3}{12(1-\mu^2)}$ is the bending stiffness of the plate; E is the Young's modulus; h is the thickness; μ is the Poisson's ratio; M_n, M_{ns},and Q_n are the normal bending moment, twisting moment and transverse shear force on the boundary, respectively; ΔM_{ns} is the increment of the twisting moment at two sides of the corner node on the boundary.

$\Pi_{3c}^{(b)}$ is the three-field generalized complementary energy of the region *b* (excluding the interface C_{ab} and the node *J*):

$$\Pi_{3c}^{(b)}=\iint_{\Omega_b}\left[M_x\kappa_x+M_y\kappa_y+2M_{xy}\kappa_{xy}-\tilde{U}(\boldsymbol{\kappa})+\left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+q\right)w\right]\mathrm{d}x\mathrm{d}y$$
$$-\int_{C_{1b}+C_{2b}}\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)\bar{w}\mathrm{d}s-\int_{C_{3b}}\left(Q_n+\frac{\partial M_{ns}}{\partial s}-\bar{V}_n\right)w\mathrm{d}s+\int_{C_{1b}}M_n\bar{\psi}_n\mathrm{d}s$$
$$+\int_{C_{2b}+C_{3b}}(M_n-\bar{M}_n)\frac{\partial w}{\partial n}\mathrm{d}s-\sum_{A_{1b}}\Delta M_{ns}\bar{w}-\sum_{A_{2b}}(\Delta M_{ns}-\bar{R})w \tag{2-41}$$

H_{pc}, G_{1pc}, G_{2pc} are the additional energy terms on the interface C_{ab} and the nodes J_1 and J_2:

$$H_{pc}=\int_{C_{ab}}\left[M_n^{(b)}\left(\frac{\partial w}{\partial n}\right)^{(a)}+\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)^{(b)}w^{(a)}\right]\mathrm{d}s \tag{2-42}$$

$$G_{1pc}=\sum_{J_1}[-(\Delta M_{ns})^{(a)}(w^{(a)}-\bar{w})+(\Delta M_{ns})^{(b)}\bar{w}] \tag{2-43}$$

$$G_{2pc}=\sum_{J_2}[(\Delta M_{ns})^{(b)}-\bar{R}]w^{(a)} \tag{2-44}$$

(3) Stationary condition

The stationary condition of the functional is

$$\delta\Pi_3 = \delta\Pi_{3\mathrm{p}}^{(a)} - \delta\Pi_{3\mathrm{c}}^{(b)} + \delta H_{\mathrm{pc}} + \delta G_{1\mathrm{pc}} + \delta G_{2\mathrm{pc}} = 0 \tag{2-45}$$

which is equivalent to all field equations, boundary conditions, interface conditions, and conditions at the corner points and nodes, including:

The field equations within Ω_a and Ω_b:

$$\left.\begin{aligned} &M_x = D(\kappa_x + \mu\kappa_y), \quad M_y = D(\kappa_y + \mu\kappa_x) \\ &M_{xy} = D(1-\mu)\kappa_{xy} \\ &\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \\ &\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q = 0 \end{aligned}\right\} \tag{2-46}$$

The boundary conditions on C_a and C_b:

$$\left.\begin{aligned} &Q_n + \frac{\partial M_{ns}}{\partial s} = \bar{V}_n && (\text{on} \quad C_{3a} + C_{3b}) \\ &w = \bar{w} && (\text{on} \quad C_{1a} + C_{2a} + C_{1b} + C_{2b}) \\ &M_n = \bar{M}_n && (\text{on} \quad C_{2a} + C_{3a} + C_{2b} + C_{3b}) \\ &\frac{\partial w}{\partial n} = \bar{\psi}_n && (\text{on} \quad C_{1a} + C_{1b}) \end{aligned}\right\} \tag{2-47}$$

The interface conditions on C_{ab}:

$$\left.\begin{aligned} &M_n^{(a)} = M_n^{(b)} \\ &\left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)^{(a)} = -\left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)^{(b)} \\ &\left(\frac{\partial w}{\partial n}\right)^{(a)} = -\left(\frac{\partial w}{\partial n}\right)^{(b)} \\ &w^{(a)} = w^{(b)} \end{aligned}\right\} \tag{2-48}$$

The conditions at the corner points:

$$\left.\begin{aligned} &w = \bar{w} && (\text{at} \quad A_{1a} + A_{1b}) \\ &\Delta M_{ns} = \bar{R} && (\text{at} \quad A_{2a} + A_{2b}) \end{aligned}\right\} \tag{2-49}$$

The conditions at the nodes on the interface:

$$\left.\begin{aligned} &w^{(a)} = \overline{w} && (\text{at } J_1) \\ &w^{(b)} = \overline{w} && (\text{at } J_1) \\ &w^{(a)} = w^{(b)} && (\text{at } J_2) \\ &(\Delta M_{ns})^{(a)} + (\Delta M_{ns})^{(b)} = \overline{R} && (\text{at } J_2) \end{aligned}\right\} \tag{2-50}$$

The proof of the above equivalent equations is given in Appendix A.

2.3.2 The Sub-Region Three-Field Generalized Potential and Complementary Energy Principles for Thin Plate

1. The transformation relation between $\varPi_{3\mathrm{p}}^{(a)}$ and $\varPi_{3\mathrm{c}}^{(a)}$

The three-field generalized potential energy $\varPi_{3\mathrm{p}}^{(a)}$ and the three-field generalized complementary energy $\varPi_{3\mathrm{c}}^{(a)}$ of the region a (excluding the interface C_{ab} and the node J) have the following transformation relation:

$$\varPi_{3\mathrm{p}}^{(a)} + \varPi_{3\mathrm{c}}^{(a)} = \int_{C_{ab}} \left[-M_n^{(a)} \left(\frac{\partial w}{\partial n} \right)^{(a)} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(a)} w^{(a)} \right] \mathrm{d}s + \sum_{J_1+J_2} w^{(a)} (\Delta M_{ns})^{(a)} \tag{2-51}$$

Proof From Eqs. (2-39) and (2-41), replacing b by a in Eq. (2-41), we have

$$\begin{aligned} \varPi_{3\mathrm{p}}^{(a)} + \varPi_{3\mathrm{c}}^{(a)} = & \iint_{\Omega_a} \left[-\left(\frac{\partial^2 w}{\partial x^2} M_x + \frac{\partial^2 w}{\partial y^2} M_y + 2\frac{\partial^2 w}{\partial x \partial y} M_{xy} \right) \right. \\ & \left. + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} \right) w \right] \mathrm{d}x\mathrm{d}y \\ & - \int_{C_{1a}+C_{2a}+C_{3a}} \left[\left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) w - M_n \frac{\partial w}{\partial n} \right] \mathrm{d}s - \sum_{A_{1a}+A_{2a}} (\Delta M_{ns}) w \end{aligned} \tag{2-52}$$

By using integration by parts, the following relation can be obtained:

$$\begin{aligned} \iint_{\Omega_a} \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} \right) w \mathrm{d}x\mathrm{d}y = & \iint_{\Omega_a} \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \mathrm{d}x\mathrm{d}y \\ & - \int_{C_{1a}+C_{2a}+C_{3a}+C_{ab}} \left[M_n \frac{\partial w}{\partial n} - \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) w \right] \mathrm{d}s + \sum_{A_{1a}+A_{2a}+J_1+J_2} w \Delta M_{ns} \end{aligned} \tag{2-53}$$

Substitution of Eq. (2-53) into Eq. (2-52) yields Eq. (2-51). □

If the whole domain is not divided into sub-regions, C_{ab}, J_1 and J_2 will no longer exist, so we have

$$\Pi_{3\mathrm{p}}^{(a)} + \Pi_{3\mathrm{c}}^{(a)} = 0 \tag{2-54}$$

2. The sub-region three-field generalized potential energy principle

In the functional expression (2-38) of the sub-region three-field generalized mixed variational principle, the region a represents the generalized potential energy region, and the region b represents the generalized complementary energy region. Now, if the region b is changed to represent the generalized potential energy region, then from Eq. (2-51), we have

$$\Pi_{3\mathrm{c}}^{(b)} = -\Pi_{3\mathrm{p}}^{(b)} + \int_{C_{ab}} \left[-M^{(b)} \left(\frac{\partial w}{\partial n} \right)^{(b)} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(b)} w^{(b)} \right] \mathrm{d}s + \sum_{J_1+J_2} w^{(b)} (\Delta M_{ns})^{(b)}$$

Substitution of this equation into (2-38) yields

$$\Pi_3 = \Pi_{3\mathrm{p}}^{(a)} + \Pi_{3\mathrm{p}}^{(b)} + H_{\mathrm{pp}} + G_{1\mathrm{pp}} + G_{2\mathrm{pp}} \tag{2-55}$$

where H_{pp}, $G_{1\mathrm{pp}}$ and $G_{2\mathrm{pp}}$ are the additional potential energy terms on the interface C_{ab} and the nodes J_1 and J_2:

$$H_{\mathrm{pp}} = \int_{C_{ab}} \left\{ M_n^{(b)} \left[\left(\frac{\partial w}{\partial n} \right)^{(a)} + \left(\frac{\partial w}{\partial n} \right)^{(b)} \right] + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(b)} (w^{(a)} - w^{(b)}) \right\} \mathrm{d}s \tag{2-56a}$$

$$G_{1\mathrm{pp}} = -\sum_{J_1} [(\Delta M_{ns})^{(a)} (w^{(a)} - \overline{w}) + (\Delta M_{ns})^{(b)} (w^{(b)} - \overline{w})] \tag{2-57}$$

$$G_{2\mathrm{pp}} = \sum_{J_2} [(\Delta M_{ns})^{(b)} (w^{(a)} - w^{(b)}) - \overline{R} w^{(a)}] \tag{2-58a}$$

Equations (2-55), (2-56a), (2-57) and (2-58a) are the functional expressions of the sub-region three-field generalized potential energy principle. It can be shown that the stationary conditions of this functional is equivalent to all field equations, boundary conditions, interface conditions, corner point and node conditions of the thin plate with sub-regions.

Other expressions of H_{pp} and $G_{2\mathrm{pp}}$ can also be obtained by interchanging a and b in Eqs. (2-56a) and (2-58a):

$$H_{\mathrm{pp}} = \int_{C_{ab}} \left\{ M_n^{(a)} \left[\left(\frac{\partial w}{\partial n} \right)^{(a)} + \left(\frac{\partial w}{\partial n} \right)^{(b)} \right] + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(a)} (w^{(b)} - w^{(a)}) \right\} \mathrm{d}s \tag{2-56b}$$

$$G_{2\mathrm{pp}} = \sum_{J_2} [(\Delta M_{ns})^{(a)} (w^{(b)} - w^{(a)}) - \bar{R} w^{(b)}] \qquad \text{(2-58b)}$$

If the displacement continuous conditions on the interface C_{ab} and the nodes J_1 and J_2 are satisfied in advance, then from Eqs. (2-56), (2-57) and (2-58), we can obtain

$$H_{\mathrm{pp}} = 0$$
$$G_{1\mathrm{pp}} = 0$$
$$G_{2\mathrm{pp}} = -\sum_{J_2} \bar{R} w^{(a)} \quad \text{or} \quad G_{2\mathrm{pp}} = -\sum_{J_2} \bar{R} w^{(b)}$$

3. The sub-region three-field generalized complementary energy principle

In Eq. (2-38), if we require that the sub-region a is given by the generalized complementary energy, the substitution of (2-51) into (2-38) will yield

$$\Pi_3 = -\Pi_{3\mathrm{c}}^{(a)} - \Pi_{3\mathrm{c}}^{(b)} + H_{\mathrm{cc}} + G_{1\mathrm{cc}} + G_{2\mathrm{cc}} \qquad \text{(2-59)}$$

where H_{cc}, $G_{1\mathrm{cc}}$ and $G_{2\mathrm{cc}}$ are the additional complementary energy terms on the interface C_{ab} and the nodes J_1 and J_2:

$$H_{\mathrm{cc}} = \int_{C_{ab}} \left\{ (M_n^{(b)} - M_n^{(a)}) \left(\frac{\partial w}{\partial n} \right)^{(a)} + \left[\left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(b)} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(a)} \right] w^{(a)} \right\} \mathrm{d}s$$

(2-60a)

$$G_{1\mathrm{cc}} = \sum_{J_1} [(\Delta M_{ns})^{(b)} + (\Delta M_{ns})^{(a)}] \bar{w} \qquad \text{(2-61)}$$

$$G_{2\mathrm{cc}} = \sum_{J_2} \{ [(\Delta M_{ns})^{(b)} + (\Delta M_{ns})^{(a)} - \bar{R}] w^{(a)} \} \qquad \text{(2-62a)}$$

Equations (2-59), (2-60a), (2-61) and (2-62a) are the functional expressions of the sub-region three-field generalized complementary energy principle. Other expressions of H_{cc} and $G_{2\mathrm{cc}}$ can also be obtained by interchanging a and b in Eqs. (2-60a) and (2-62a):

$$H_{\mathrm{cc}} = \int_{C_{ab}} \left\{ (M_n^{(a)} - M_n^{(b)}) \left(\frac{\partial w}{\partial n} \right)^{(b)} + \left[\left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(a)} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right)^{(b)} \right] w^{(b)} \right\} \mathrm{d}s$$

(2-60b)

$$G_{2\mathrm{cc}} = \sum_{J_2} \{ [(\Delta M_{ns})^{(b)} + (\Delta M_{ns})^{(a)} - \bar{R}] w^{(b)} \} \qquad \text{(2-62b)}$$

If the traction conditions on the interface C_{ab} and the node J_2 are satisfied in advance, then from Eqs. (2-60) and (2-62), we can obtain:

$$H_{\mathrm{cc}} = 0, \quad G_{2\mathrm{cc}} = 0$$

2.3.3 The Sub-Region Two-Field and Single-Field Variational Principle for Thin Plate

1. The sub-region two-field generalized variational principle

By using the relation between the strain energy density $\tilde{U}(\boldsymbol{\kappa})$ and the strain complementary energy density $\tilde{V}(\boldsymbol{M})$:

$$\tilde{V}(\boldsymbol{M}) = M_x \kappa_x + M_y \kappa_y + 2M_{xy}\kappa_{xy} - \tilde{U}(\boldsymbol{\kappa}) \tag{2-63}$$

the variable $\boldsymbol{\kappa}$ in the three-field generalized potential energy $\Pi_{3\mathrm{p}}^{(a)}$ and generalized complementary energy $\Pi_{3\mathrm{c}}^{(a)}$ of the region a (excluding the interface C_{ab} and the nodes J_1 and J_2) can be eliminated. Thereby, the two-field (displacement field w and internal moment field $\boldsymbol{M}$) generalized potential energy $\Pi_{2\mathrm{p}}^{(a)}$ and generalized complementary energy $\Pi_{2\mathrm{c}}^{(a)}$ can be written as follows:

$$\begin{aligned}
\Pi_{2\mathrm{p}}^{(a)} = & \iint_{\Omega_a}\left[-\left(\frac{\partial^2 w}{\partial x^2}M_x + \frac{\partial^2 w}{\partial y^2}M_y + 2\frac{\partial^2 w}{\partial x \partial y}M_{xy}\right) - \tilde{V}(\boldsymbol{M}) - qw\right]\mathrm{d}x\mathrm{d}y \\
& - \int_{C_{1a}+C_{2a}}\left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)(w - \bar{w})\mathrm{d}s - \int_{C_{3a}}\bar{V}_n w\mathrm{d}s + \int_{C_{1a}} M_n\left(\frac{\partial w}{\partial n} - \bar{\psi}_n\right)\mathrm{d}s \\
& + \int_{C_{2a}+C_{3a}}\bar{M}_n\frac{\partial w}{\partial n}\mathrm{d}s - \sum_{A_{1a}}\Delta M_{ns}(w - \bar{w}) - \sum_{A_{2a}}\bar{R}w
\end{aligned} \tag{2-64}$$

$$\begin{aligned}
\Pi_{2\mathrm{c}}^{(a)} = & \iint_{\Omega_a}\left[\tilde{V}(\boldsymbol{M}) + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q\right)w\right]\mathrm{d}x\mathrm{d}y \\
& - \int_{C_{1a}+C_{2a}}\left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)\bar{w}\mathrm{d}s - \int_{C_{3a}}\left(Q_n + \frac{\partial M_{ns}}{\partial s} - \bar{V}_n\right)w\mathrm{d}s + \int_{C_{1a}} M_n\bar{\psi}_n\mathrm{d}s \\
& + \int_{C_{2a}+C_{3a}}(M_n - \bar{M}_n)\frac{\partial w}{\partial n}\mathrm{d}s - \sum_{A_{1a}}\Delta M_{ns}\bar{w} - \sum_{A_{2a}}(\Delta M_{ns} - \bar{R})w
\end{aligned} \tag{2-65}$$

From Eqs. (2-38), (2-55) and (2-59), the functional expressions of the sub-region two-field generalized mixed energy, potential energy and complementary energy principle can be obtained:

$$\Pi_2 = \Pi_{2\mathrm{p}}^{(a)} - \Pi_{2\mathrm{c}}^{(b)} + H_{\mathrm{pc}} + G_{1\mathrm{pc}} + G_{2\mathrm{pc}} \tag{2-66}$$

$$\Pi_2 = \Pi_{2p}^{(a)} + \Pi_{2p}^{(b)} + H_{pp} + G_{1pp} + G_{2pp} \tag{2-67}$$

$$\Pi_2 = -\Pi_{2c}^{(a)} - \Pi_{2c}^{(b)} + H_{cc} + G_{1cc} + G_{2cc} \tag{2-68}$$

where H_{pc}, H_{pp} and H_{cc} are still given by Eqs. (2-42), (2-56) and (2-60), respectively; G_{1pc}, G_{2pc}, G_{1pp}, G_{2pp}, G_{1cc} and G_{2cc} are still given by Eqs. (2-43), (2-44), (2-57), (2-58), (2-61) and (2-62), respectively.

2. The sub-region single-field variational principle

Now we consider the case where each sub-region is a single-field region. If the region a is a potential energy region, only the displacement w will be taken as the field variable. Thus, the $\Pi_{3p}^{(a)}$ in Eq. (2-39) or the $\Pi_{2p}^{(a)}$ in Eq. (2-64) will transform to the single-field potential energy $\Pi_{1p}^{(a)}$ of the region a:

$$\begin{aligned}\Pi_{1p}^{(a)} = & \iint_{\Omega_a}[\tilde{U}(w) - qw]\mathrm{d}x\mathrm{d}y - \int_{C_{1a}+C_{2a}}\left(\frac{\partial M_{ns}}{\partial s} + Q_n\right)(w - \bar{w})\mathrm{d}s \\ & - \int_{C_{3a}}\bar{V}_n w\mathrm{d}s + \int_{C_{1a}} M_n\left(\frac{\partial w}{\partial n} - \bar{\psi}_n\right)\mathrm{d}s + \int_{C_{2a}+C_{3a}}\bar{M}_n\frac{\partial w}{\partial n}\mathrm{d}s \\ & - \sum_{A_{1a}}\Delta M_{ns}(w - \bar{w}) - \sum_{A_{2a}}\bar{R}w\end{aligned} \tag{2-69a}$$

where $Q_n + \dfrac{\partial M_{ns}}{\partial s}$, M_n and ΔM_{ns} can be expressed as the functions of the displacement w, or looked upon as the Lagrange multipliers on the boundaries and their corner points; $\tilde{U}(w)$ is the strain energy density in terms of the displacement w:

$$\tilde{U}(w) = \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}$$

If the displacement w satisfies the geometrical boundary and corner point conditions in advance, then

$$\Pi_{1p}^{(a)} = \iint_{\Omega_a}[\tilde{U}(w) - qw]\mathrm{d}x\mathrm{d}y - \int_{C_{3a}}\bar{V}_n w\mathrm{d}s + \int_{C_{2a}+C_{3a}}\bar{M}_n\frac{\partial w}{\partial n}\mathrm{d}s - \sum_{A_{2a}}\bar{R}w \tag{2-69b}$$

If the sub-region a is a complementary energy region, only the internal moment $\boldsymbol{M}$ will be taken as the field variable, and $\boldsymbol{M}$ should satisfy the equilibrium differential equation in advance.

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + q = 0$$

Thus, the $\Pi_{3c}^{(a)}$ in Eq. (2-41) or the $\Pi_{2c}^{(a)}$ in Eq. (2-65) will transform to the single-field complementary energy $\Pi_{1c}^{(a)}$:

$$\Pi_{1c}^{(a)} = \iint_{\Omega_a} \tilde{V}(\boldsymbol{M})\mathrm{d}x\mathrm{d}y - \int_{C_{1a}+C_{2a}} \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \overline{w}\mathrm{d}s - \int_{C_{3a}} \left(Q_n + \frac{\partial M_{ns}}{\partial s} - \overline{V}_n \right) w\mathrm{d}s$$
$$+ \int_{C_{1a}} M_n \overline{\psi}_n \mathrm{d}s + \int_{C_{2a}+C_{3a}} (M_n - \overline{M}_n) \frac{\partial w}{\partial n} \mathrm{d}s - \sum_{A_{1a}} \Delta M_{ns} \overline{w} - \sum_{A_{2a}} (\Delta M_{ns} - \overline{R}) w \tag{2-70a}$$

where w and $\frac{\partial w}{\partial n}$ can be looked upon as the Lagrange multipliers on the boundaries and their corner points.

If $\boldsymbol{M}$ satisfies the traction boundary and corner point conditions in advance, then

$$\Pi_{1c}^{(a)} = \iint_{\Omega_a} \tilde{V}(\boldsymbol{M})\mathrm{d}x\mathrm{d}y - \int_{C_{1a}+C_{2a}} \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \overline{w}\mathrm{d}s + \int_{C_{1a}} M_n \overline{\psi}_n \mathrm{d}s - \sum_{A_{1a}} \Delta M_{ns} \overline{w} \tag{2-70b}$$

From Eqs. (2-38), (2-55), (2-59), or (2-66), (2-67), (2-68), the functional expressions of the sub-region single-field mixed energy principle, potential energy principle and complementary energy principle can be obtained:

$$\Pi_1 = \Pi_{1p}^{(a)} - \Pi_{1c}^{(b)} + H_{pc} + G_{1pc} + G_{2pc} \tag{2-71}$$

$$\Pi_1 = \Pi_{1p}^{(a)} + \Pi_{1p}^{(b)} + H_{pp} + G_{1pp} + G_{2pp} \tag{2-72}$$

$$\Pi_1 = -\Pi_{1c}^{(a)} - \Pi_{1c}^{(b)} + H_{cc} + G_{1cc} + G_{2cc} \tag{2-73}$$

where H_{pc}, G_{1pc}, G_{2pc}, H_{pp}, G_{1pp}, G_{2pp}, H_{cc}, G_{1cc} and G_{2cc} are the same as those in Eqs. (2-66) to (2-68).

2.3.4 The General Form of the Sub-Region Generalized Variational Principle for Thin Plate

From the above discussions, a general form of the sub-region generalized variational principle can be obtained.

Let an elastic thin plate be divided into several sub-regions. Each sub-region can be arbitrarily appointed as single-field, two-field and three-field potential energy regions (such as the regions Ω_{p1}, Ω_{p2}, Ω_{p3} in Fig. 2.4) or complementary energy regions (such as the regions Ω_{c1}, Ω_{c2}, Ω_{c3} in Fig. 2.4).

The interfaces between two adjacent sub-regions are of three types, C_{pc}, C_{pp} and C_{cc}: ① one side of C_{pc} is the potential energy region, while the other side is the complementary one; ② both sides of C_{pp} are potential energy regions; and ③ both sides of C_{cc} are complementary energy regions.

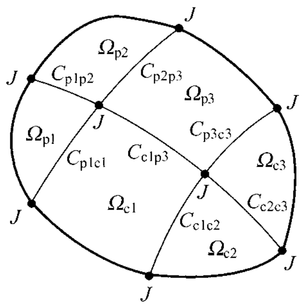

Figure 2.4 A thin plate divided into multi-regions

The node J of the adjacent sub-regions are of two types, J_1 and J_2: J_1 is the node where the displacement $\overline{w}$ is specified; J_2 is the node where the concentrated force $\overline{R}$ is specified. There are r_p potential energy elements e_p and r_c complementary energy element e_c around the node J.

The general form for the functional of the sub-region variational principle can be written as

$$\Pi = \sum_{\Omega_p} \Pi_p - \sum_{\Omega_c} \Pi_c + \sum_{C_{pc}} H_{pc} + \sum_{C_{pp}} H_{pp} + \sum_{C_{cc}} H_{cc} + \sum_{J_1} G_1 + \sum_{J_2} G_2 \tag{2-74}$$

The meanings of the terms on the right-side of this equation are as follows:

The first term denotes the sum of the potential (or generalized potential) energy Π_p of each potential energy region Ω_p, where Π_p can be Π_{1p} or Π_{2p} or Π_{3p}, which is given by Eqs. (2-69), (2-64) and (2-39), respectively;

The second term denotes the sum of the complementary (or generalized complementary) energy Π_c of each complementary energy region Ω_c, where Π_c can be Π_{1c} or Π_{2c} or Π_{3c}, which is given by Eqs. (2-70), (2-65) and (2-41), respectively;

The third term denotes the sum of the additional mixed energy term H_{pc} on the interface C_{pc}, in which H_{pc} is given by Eq. (2-42);

The fourth term denotes the sum of the additional potential energy term H_{pp} on the interface C_{pp}, in which H_{pp} is given by Eq. (2-56);

The fifth term denotes the sum of the additional complementary energy term H_{cc} on the interface C_{cc}, in which H_{cc} is given by Eq. (2-60).

The sixth term denotes the sum of the additional energy term G_1 at the node J_1 (where the displacement is specified) of the adjacent sub-regions, in which

$$G_1 = -\sum_{e_p} (\Delta M_{ns})^{(e_p)} (w^{(e_p)} - \overline{w}) + \sum_{e_c} (\Delta M_{ns})^{(e_c)} \overline{w} \tag{2-75}$$

The first term on the right side of the above equation means the sum of all the potential elements e_{p} around the node; the second term means the sum of all the complementary energy elements e_{c} around the node.

The seventh term denotes the sum of the additional energy term G_2 at the node J_2 (where the concentrated force is specified) of the adjacent sub-regions, in which

$$G_2 = [\sum_{e} (\Delta M_{ns})^{(e)} - \overline{R}] w^{(a)} - \sum_{e_{\mathrm{p}}} (\Delta M_{ns})^{(e_{\mathrm{p}})} w^{(e_{\mathrm{p}})} \tag{2-76}$$

The $\sum_{e}$ in the first term on the right side of the above equation denotes the sum of all the elements e (including all e_{p} and e_{c}) around the node; $w^{(a)}$ is the displacement of any element a around the node; The $\sum_{e_{\mathrm{p}}}$ in the second term on the right side of the above equation denotes the sum of all the potential elements e_{p} around the node.

$G_{1\mathrm{pc}}$ in (2-43), $G_{1\mathrm{pp}}$ in (2-57), and $G_{1\mathrm{cc}}$ in (2-61) are all special cases of G_1 in (2-75). $G_{2\mathrm{pc}}$ in (2-44), $G_{2\mathrm{pp}}$ in (2-58), and $G_{2\mathrm{cc}}$ in (2-62) are all special cases of G_2 in (2-76).

It can be shown that the stationary condition

$$\delta \Pi = 0 \tag{2-77}$$

of the functional Π in Eq. (2-74) is equivalent to all field equations, boundary conditions, interface conditions, corner point and node conditions of the thin plate system with multi-regions.

The procedure for deriving the node conditions of the node J from the stationary condition (2-77) is given in Appendix B.

If all the sub-regions are potential energy regions, the functional of the sub-region potential (or generalized potential) energy principle can be obtained from Eq. (2-74):

$$\Pi = \sum_{\Omega_{\mathrm{p}}} \Pi_{\mathrm{p}} + \sum_{C_{\mathrm{pp}}} H_{\mathrm{pp}} + \sum_{J_1} G_{1\mathrm{pp}} + \sum_{J_2} G_{2\mathrm{pp}} \tag{2-78}$$

where $G_{1\mathrm{pp}}$ and $G_{2\mathrm{pp}}$ can be obtained from Eqs. (2-75) and (2-76):

$$G_{1\mathrm{pp}} = -\sum_{e_{\mathrm{p}}} (\Delta M_{ns})^{(e_{\mathrm{p}})} (w^{(e_{\mathrm{p}})} - \overline{w}) \tag{2-79}$$

$$\begin{aligned} G_{2\mathrm{pp}} &= [\sum_{e_{\mathrm{p}}} (\Delta M_{ns})^{(e_{\mathrm{p}})} - \overline{R}] w^{(a)} - \sum_{e_{\mathrm{p}}} (\Delta M_{ns})^{(e_{\mathrm{p}})} w^{(e_{\mathrm{p}})} \\ &= -\sum_{e_{\mathrm{p}}} (\Delta M_{ns})^{(e_{\mathrm{p}})} (w^{(e_{\mathrm{p}})} - w^{(a)}) - \overline{R} w^{(a)} \end{aligned} \tag{2-80}$$

Equations (2-55), (2-67) and (2-72) are all the special cases of (2-78). One of the special cases of the sub-region potential energy principle is that each sub-region is appointed as a single-field potential energy region. At this time, Π_{p} in Eq. (2-78) will be replaced by $\Pi_{1\mathrm{p}}$ in Eq. (2-69a):

$$\Pi = \sum_{\Omega_{\mathrm{p}}} \Pi_{1\mathrm{p}} + \sum_{C_{\mathrm{pp}}} H_{\mathrm{pp}} + \sum_{J_1} G_{1\mathrm{pp}} + \sum_{J_2} G_{2\mathrm{pp}} \tag{2-81}$$

If all the sub-regions are complementary energy regions, the functional of the sub-region complementary (or generalized complementary) energy principle can be obtained from Eq. (2-74):

$$\Pi = -\sum_{\Omega_{\mathrm{c}}} \Pi_{\mathrm{c}} + \sum_{C_{\mathrm{cc}}} H_{\mathrm{cc}} + \sum_{J_1} G_{1\mathrm{cc}} + \sum_{J_2} G_{2\mathrm{cc}} \tag{2-82}$$

where

$$G_{1\mathrm{cc}} = \sum_{e_{\mathrm{c}}} (\Delta M_{ns})^{(e_{\mathrm{c}})} \,\overline{w} \tag{2-83}$$

$$G_{2\mathrm{cc}} = [\sum_{e_{\mathrm{c}}} (\Delta M_{ns})^{(e_{\mathrm{c}})} - \overline{R}] w^{(a)} \tag{2-84}$$

Equations (2-59), (2-68) and (2-73) are all the special cases of (2-82). One of the special cases of sub-region complementary energy principle is that each sub-region is appointed as a two-field complementary energy region. At this time, Π_{c} in Eq. (2-82) will be replaced by $\Pi_{2\mathrm{c}}$ in Eq. (2-65):

$$\Pi = -\sum_{\Omega_{\mathrm{c}}} \Pi_{2\mathrm{c}} + \sum_{C_{\mathrm{cc}}} H_{\mathrm{cc}} + \sum_{J_1} G_{1\mathrm{cc}} + \sum_{J_2} G_{2\mathrm{cc}} \tag{2-85}$$

2.4 The Sub-Region Variational Principle for Elastic Thick Plate

In the previous section, the sub-region variational principle for thin plate is discussed. This section will consider the thick plate case.

Compared with the thin plate theory, the characteristics of the thick plate theory are as follows: the influences of the transverse shear strain γ_x and γ_y (abbreviations of γ_{xz} and γ_{yz}) are considered; the rotations ψ_x and ψ_y are not dependent on the deflection w, thereby, w, ψ_x and ψ_y are three independent displacements; Besides the normal load $\overline{q}$, there still are couple loads $\overline{m}_x$ and $\overline{m}_y$; the transverse shear forces Q_x and Q_y are not dependent on the bending and twisting moments M_x, M_y and M_{xy}.

The sub-region variational principle for elastic thick plate was proposed in [13].

For comparison, the arrangement of this section is the same as that of the previous one, which can make it easy to understand the similarities and differences of the two principles.

2.4.1 The Sub-Region Three-Field Generalized Mixed Variational Principle for Thick Plate

Here we consider an elastic plate with moderate thickness, i.e. an elastic thick plate. A Cartesian co-ordinate system is established on the mid-surface of the plate (see Fig. 2.5), and the positive direction of the z-axis is downward. n and s denote the directions of the outer normal and the tangent along the boundary, respectively. And, the positive direction of s is shown in Fig. 2.5.

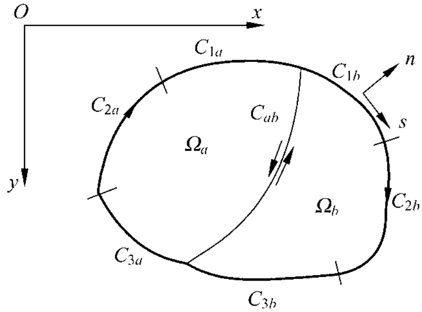

Figure 2.5 A thick plate divided into two sub-regions

Let a thick plate be divided into two sub-regions a and b, and Ω_a and Ω_b represent the domains of the regions a and b, respectively. The outer boundaries C_a and C_b of the regions a and b are both composed of three parts:

$$C_a = C_{1a} + C_{2a} + C_{3a}, \quad C_b = C_{1b} + C_{2b} + C_{3b}$$

where C_{1a} and C_{1b} are the fixed boundaries (the deflection w, the normal rotation ψ_n and the tangent rotation ψ_s of the mid-surface normal line are specified by $\overline{w}$, $\overline{\psi}_n$ and $\overline{\psi}_s$, respectively); C_{2a} and C_{2b} are the simply-supported boundaries (the deflection w, the tangent rotation ψ_s of the mid-surface normal line and the normal bending moment M_n are specified by $\overline{w}$, $\overline{\psi}_s$ and $\overline{M}_n$, respectively); C_{3a} and C_{3b} are the free boundaries (the normal bending moment M_n, the twisting moment M_{ns} and the transverse shear force Q_n are specified by $\overline{M}_n$, $\overline{M}_{ns}$ and $\overline{Q}_n$, respectively). The interface of the two regions is denoted by C_{ab}. The positive deflection w is downward; the positive normal rotation ψ_n rotates from n to z; the positive tangent rotation ψ_s rotates from s to z; the normal bending moment M_n is positive when the bottom of the plate is under tension; the twisting moment M_{ns}

is positive when it produces positive shear stress τ_{ns} along the positive direction of s at the bottom of the plate; and the positive transverse shear force Q_n is also downward.

The key points of the sub-region generalized mixed variational principle can be listed as follows.

1. The field variables

Both regions a and b possess three field variables:

Displacements $\boldsymbol{d}^{(a)} = [w \quad \psi_x \quad \psi_y]^{\mathrm{T}(a)}$, $\boldsymbol{d}^{(b)} = [w \quad \psi_x \quad \psi_y]^{\mathrm{T}(b)}$

Internal forces $\boldsymbol{S}^{(a)} = [M_x \quad M_y \quad M_{xy} \quad Q_x \quad Q_y]^{\mathrm{T}(a)}$

$\boldsymbol{S}^{(b)} = [M_x \quad M_y \quad M_{xy} \quad Q_x \quad Q_y]^{\mathrm{T}(b)}$

Strain $\boldsymbol{E}^{(a)} = [\kappa_x \quad \kappa_y \quad 2\kappa_{xy} \quad \gamma_x \quad \gamma_y]^{\mathrm{T}(a)}$, $\boldsymbol{E}^{(b)} = [\kappa_x \quad \kappa_y \quad 2\kappa_{xy} \quad \gamma_x \quad \gamma_y]^{\mathrm{T}(b)}$

The positive rotations ψ_x and ψ_y of the normal line rotate from x to z and from y to z, respectively; the bending moment M_x and M_y are positive when the bottom of the plate is under tension; the twisting moment M_{xy} is positive when it produces positive shear stress τ_{xy} at the bottom of the plate; the positive shear forces Q_x and Q_y on the positive surfaces are all downward. The positive curvatures κ_x, κ_y and κ_{xy}, shear strains γ_x (γ_{xz}) and γ_y (γ_{yz}) are all corresponding to the deformations caused by the positive M_x, M_y, M_{xy}, Q_x and Q_y, respectively. The above three-field variables are not required to satisfy any conditions in advance within the domain and on the boundaries and interfaces.

2. Definition of the functional

Let the region a be the potential energy region, and the region b be the complementary energy region. Then, the definition of the functional is

$$\Pi_3 = \Pi_{3\mathrm{p}}^{(a)} - \Pi_{3\mathrm{c}}^{(b)} + H_{\mathrm{pc}} \tag{2-86}$$

where $\Pi_{3\mathrm{p}}^{(a)}$ is the three-field generalized potential energy of the region a (excluding the interface C_{ab}):

$$\begin{aligned}\Pi_{3\mathrm{p}}^{(a)} = & \iint_{\Omega_a} [\tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa}) + \tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma}) - M_x\left(\kappa_x + \frac{\partial \psi_x}{\partial x}\right) - M_y\left(\kappa_y + \frac{\partial \psi_y}{\partial y}\right) \\ & - M_{xy}\left(2\kappa_{xy} + \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) - Q_x\left(\gamma_x - \frac{\partial w}{\partial x} + \psi_x\right) - Q_y\left(\gamma_y - \frac{\partial w}{\partial y} + \psi_y\right) \\ & - \bar{m}_x\psi_x - \bar{m}_y\psi_y - \bar{q}w]\mathrm{d}x\mathrm{d}y + \int_{C_{1a}+C_{2a}} [M_{ns}(\psi_s - \bar{\psi}_s) - Q_n(w - \bar{w})]\mathrm{d}s \\ & + \int_{C_{3a}} (\bar{M}_{ns}\psi_s - \bar{Q}_n w)\mathrm{d}s + \int_{C_{1a}} (\psi_n - \bar{\psi}_n)M_n\mathrm{d}s + \int_{C_{2a}+C_{3a}} \bar{M}_n\psi_n\mathrm{d}s\end{aligned} \tag{2-87}$$

Here, $\bar{q}$ is the load density, and its positive direction is downward. $\bar{m}_x$ and $\bar{m}_y$ are the couple load densities, and their positive directions are the same as those of ψ_x and ψ_y, respectively. $\tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa})$ and $\tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma})$ are the densities of bending and shear strain energies, respectively:

$$\tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa}) = \frac{D}{2}[\kappa_x^2 + \kappa_y^2 + 2\mu\kappa_x\kappa_y + 2(1-\mu)\kappa_{xy}^2] \tag{2-88}$$

$$\tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma}) = \frac{C}{2}(\gamma_x^2 + \gamma_y^2) \tag{2-89}$$

where $D = \dfrac{Eh^3}{12(1-\mu^2)}$ and $C = \dfrac{Eh}{2(1+\mu)k}$ are the bending and shear stiffness, respectively; μ is the Poisson's ratio; and coefficient $k = 1.2$.

$\Pi_{3\mathrm{c}}^{(b)}$ is the three-field generalized complementary energy of the region b (excluding the interface C_{ab}):

$$\begin{aligned}\Pi_{3\mathrm{c}}^{(b)} = & \iint_{\Omega_b}[-\tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa}) - \tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma}) + M_x\kappa_x + M_y\kappa_y + 2M_{xy}\kappa_{xy} + Q_x\gamma_x + Q_y\gamma_y \\ & -\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x - \overline{m}_x\right)\psi_x - \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y - \overline{m}_y\right)\psi_y \\ & + \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \overline{q}\right)w]\mathrm{d}x\mathrm{d}y + \int_{C_{1b}+C_{2b}}(\overline{\psi}_s M_{ns} - \overline{w}Q_n)]\mathrm{d}s \\ & + \int_{C_{3b}}[(M_{ns} - \overline{M}_{ns})\psi_s - (Q_n - \overline{Q}_n)w]\mathrm{d}s + \int_{C_{1b}}\overline{\psi}_n M_n \mathrm{d}s \\ & + \int_{C_{2b}+C_{3b}}(M_n - \overline{M}_n)\psi_n \mathrm{d}s \end{aligned} \tag{2-90}$$

H_{pc} is the additional energy term on the interface C_{ab}:

$$H_{\mathrm{pc}} = \int_{C_{ab}}(M_n^{(b)}\psi_n^{(a)} + M_{ns}^{(b)}\psi_s^{(a)} + Q_n^{(b)}w^{(a)})\mathrm{d}s \tag{2-91}$$

3. Stationary condition

The stationary condition of the functional is

$$\delta\Pi_3 = \delta\Pi_{3\mathrm{p}}^{(a)} - \delta\Pi_{3\mathrm{c}}^{(b)} + \delta H_{\mathrm{pc}} = 0 \tag{2-92}$$

which is equivalent to all field equations, boundary conditions and interface conditions of the thick plate sub-region system , including:

The constitutive, geometrical and equilibrium equations within Ω_a and Ω_b:

$$M_x = D(\kappa_x + \mu\kappa_y),\quad M_y = D(\kappa_y + \mu\kappa_x),\quad M_{xy} = D(1-\mu)\kappa_{xy} \tag{2-93}$$

$$\kappa_x = -\frac{\partial\psi_x}{\partial x},\qquad \kappa_y = -\frac{\partial\psi_y}{\partial y},\qquad 2\kappa_{xy} = -\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) \tag{2-94}$$

$$\left.\begin{aligned}&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x-\overline{m}_x=0\\&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y-\overline{m}_y=0\\&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\overline{q}=0\end{aligned}\right\}\tag{2-95}$$

The geometrical and force boundary conditions:

$$\left.\begin{aligned}&\psi_s=\overline{\psi}_s,\quad w=\overline{w}&&(\text{on } C_{1a}+C_{2a}+C_{1b}+C_{2b})\\&\psi_n=\overline{\psi}_n&&(\text{on } C_{1a}+C_{1b})\end{aligned}\right\}\tag{2-96}$$

$$\left.\begin{aligned}&M_{ns}=\overline{M}_{ns},\quad Q_n=\overline{Q}_n&&(\text{on } C_{3a}+C_{3b})\\&M_n=\overline{M}_n&&(\text{on } C_{2a}+C_{3a}+C_{2b}+C_{3b})\end{aligned}\right\}\tag{2-97}$$

The interface conditions on C_{ab}:

$$Q_n^{(a)}=-Q_n^{(b)},\quad M_n^{(a)}=M_n^{(b)},\quad M_{ns}^{(a)}=M_{ns}^{(b)}\tag{2-98}$$

$$w^{(a)}=w^{(b)},\quad \psi_n^{(a)}=-\psi_n^{(b)},\quad \psi_s^{(a)}=-\psi_s^{(b)}\tag{2-99}$$

2.4.2 The Sub-Region Three-Field Generalized Potential and Complementary Energy Principles for Thick Plate

1. The transformation relation between $\varPi_{3\mathrm{p}}^{(a)}$ and $\varPi_{3\mathrm{c}}^{(a)}$

The three-field generalized potential energy $\varPi_{3\mathrm{p}}^{(a)}$ and generalized complementary energy $\varPi_{3\mathrm{c}}^{(a)}$ of the region a (excluding the interface C_{ab}) have the following transformation relation:

$$\varPi_{3\mathrm{p}}^{(a)}+\varPi_{3\mathrm{c}}^{(a)}=\int_{C_{ab}}[Q_n^{(a)}w^{(a)}-M_n^{(a)}\psi_n^{(a)}-M_{ns}^{(a)}\psi_s^{(a)}]\mathrm{d}s\tag{2-100}$$

Proof From Eqs. (2-87) and (2-90) (replace b by a in Eq. (2-90)), we have

$$\begin{aligned}\varPi_{3\mathrm{p}}^{(a)}+\varPi_{3\mathrm{c}}^{(a)}=&\iint_{\Omega_a}\left[-M_x\frac{\partial\psi_x}{\partial x}-M_y\frac{\partial\psi_y}{\partial y}-M_{xy}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+Q_x\left(\frac{\partial w}{\partial x}-\psi_x\right)\right.\\&+Q_y\left(\frac{\partial w}{\partial y}-\psi_y\right)-\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x\right)\psi_x-\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y\right)\psi_y\\&\left.+\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)w\right]\mathrm{d}x\mathrm{d}y-\int_{C_{1a}+C_{2a}+C_{3a}}(Q_nw-M_n\psi_n-M_{ns}\psi_s)\mathrm{d}s\end{aligned}\tag{2-101}$$

By using integration by parts, the following identical relation can be obtained:

$$\iint_{\Omega_a}\left[-M_x\frac{\partial\psi_x}{\partial x}-M_y\frac{\partial\psi_y}{\partial y}-M_{xy}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+Q_x\left(\frac{\partial w}{\partial x}-\psi_x\right)+Q_y\left(\frac{\partial w}{\partial y}-\psi_y\right)\right]\mathrm{d}x\mathrm{d}y$$
$$=\iint_{\Omega_a}\left[\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x\right)\psi_x+\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y\right)\psi_y-\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)w\right]\mathrm{d}x\mathrm{d}y$$
$$+\int_{C_{1a}+C_{2a}+C_{3a}+C_{ab}}(Q_n w-M_n\psi_n-M_{ns}\psi_s)\mathrm{d}s \tag{2-102}$$

Substitution of Eq. (2-102) into Eq. (2-101) yields Eq. (2-100). □

If the whole domain is not divided into sub-regions, C_{ab} will no longer exist, so we have

$$\Pi_{3\mathrm{p}}^{(a)}+\Pi_{3\mathrm{c}}^{(a)}=0 \tag{2-103}$$

2. The sub-region three-field generalized potential energy principle

In the functional expression (2-86) of the sub-region three-field generalized mixed variational principle, the region a represents the generalized potential energy region, and the region b represents the generalized complementary region. Now, if the region b is changed to represent the generalized potential region, then from Eq. (2-100), we have

$$\Pi_{3\mathrm{c}}^{(b)}=-\Pi_{3\mathrm{p}}^{(b)}+\int_{C_{ab}}[Q_n^{(b)}w^{(b)}-M_n^{(b)}\psi_n^{(b)}-M_{ns}^{(b)}\psi_s^{(b)}]\mathrm{d}s$$

Substitution of this equation into (2-86) yields

$$\Pi_3=\Pi_{3\mathrm{p}}^{(a)}+\Pi_{3\mathrm{p}}^{(b)}+H_{\mathrm{pp}} \tag{2-104}$$

where H_{pp} is the additional potential energy term on the interface C_{ab}:

$$H_{\mathrm{pp}}=\int_{C_{ab}}[Q_n^{(b)}(w^{(a)}-w^{(b)})+M_n^{(b)}(\psi_n^{(a)}+\psi_n^{(b)})+M_{ns}^{(b)}(\psi_s^{(a)}+\psi_s^{(b)})]\mathrm{d}s \tag{2-105a}$$

Equations (2-104) and (2-105a) are the functional expressions of the sub-region three-field generalized potential energy principle. It can be shown that the stationary conditions of this functional is equivalent to all field equations, boundary conditions and interface conditions of the thick plate with sub-regions.

Another expression of H_{pp} can also be obtained by interchanging a and b in Eq. (2-105a):

$$H_{\mathrm{pp}}=\int_{C_{ab}}[Q_n^{(a)}(w^{(b)}-w^{(a)})+M_n^{(a)}(\psi_n^{(a)}+\psi_n^{(b)})+M_{ns}^{(a)}(\psi_s^{(a)}+\psi_s^{(b)})]\mathrm{d}s \tag{2-105b}$$

If the displacement continuous conditions (2-99) on the interface C_{ab} are satisfied in advance, then from Eqs. (2-105a) and (2-105b), we can obtain

$$H_{\mathrm{pp}} = 0 \tag{2-106}$$

3. The sub-region three-field generalized complementary energy principle

In Eq. (2-86), if the sub-region a is changed to represent the generalized complementary energy region, substitution of (2-100) into (2-86) will yield

$$\varPi_3 = -\varPi_{3\mathrm{c}}^{(a)} - \varPi_{3\mathrm{c}}^{(b)} + H_{\mathrm{cc}} \tag{2-107}$$

where H_{cc} is the additional complementary energy term on the interface C_{ab}:

$$H_{\mathrm{cc}} = \int_{C_{ab}} [(Q_n^{(a)} + Q_n^{(b)}) w^{(a)} + (M_n^{(b)} - M_n^{(a)}) \psi_n^{(a)} + (M_{ns}^{(b)} - M_{ns}^{(a)}) \psi_s^{(a)}] \mathrm{d}s \tag{2-108a}$$

Equations (2-107) and (2-108a) are the functional expressions of the sub-region three-field generalized complementary energy principle. Another expression of H_{cc} can also be obtained by interchanging a and b in equation (2-108a):

$$H_{\mathrm{cc}} = \int_{C_{ab}} [(Q_n^{(a)} + Q_n^{(b)}) w^{(b)} + (M_n^{(a)} - M_n^{(b)}) \psi_n^{(b)} + (M_{ns}^{(a)} - M_{ns}^{(b)}) \psi_s^{(b)}] \mathrm{d}s \tag{2-108b}$$

If the traction conditions (2-98) on the interface C_{ab} are satisfied in advance, then from Eqs. (2-108), we can obtain:

$$H_{\mathrm{cc}} = 0 \tag{2-109}$$

2.4.3 The Sub-Region Two-Field and Single-Field Variational Principle for Thick Plate

1. The sub-region two-field generalized variational principle

By using the following relations between the strain energy density, $\tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa})$ and $\tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma})$, and the strain complementary energy density, $\tilde{V}_{\mathrm{b}}(\boldsymbol{M})$ and $\tilde{V}_{\mathrm{s}}(\boldsymbol{Q})$:

$$\left.\begin{aligned} \tilde{V}_{\mathrm{b}}(\boldsymbol{M}) &= M_x\kappa_x + M_y\kappa_y + 2M_{xy}\kappa_{xy} - \tilde{U}_{\mathrm{b}}(\boldsymbol{\kappa}) \\ \tilde{V}_{\mathrm{s}}(\boldsymbol{Q}) &= Q_x\gamma_x + Q_y\gamma_y - \tilde{U}_{\mathrm{s}}(\boldsymbol{\gamma}) \end{aligned}\right\} \tag{2-110}$$

the strain $\boldsymbol{E}$ in the three-field generalized potential energy $\varPi_{3\mathrm{p}}^{(a)}$ and the three-field generalized complementary energy $\varPi_{3\mathrm{c}}^{(a)}$ of the region a (excluding the interface C_{ab}) can be eliminated. Thereby, the two-field (displacement $\boldsymbol{d}$ and internal force $\boldsymbol{S}$) generalized potential energy $\varPi_{2\mathrm{p}}^{(a)}$ and the two-field generalized complementary energy $\varPi_{2\mathrm{c}}^{(a)}$ can be written as follows:

$$\begin{aligned}\varPi_{2\mathrm{p}}^{(a)} = &\iint_{\Omega_a}\left[-M_x\frac{\partial\psi_x}{\partial x}-M_y\frac{\partial\psi_y}{\partial y}-M_{xy}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)+Q_x\left(\frac{\partial w}{\partial x}-\psi_x\right)\right.\\&\left.+Q_y\left(\frac{\partial w}{\partial y}-\psi_y\right)-\tilde{V}_{\mathrm{b}}(\boldsymbol{M})-\tilde{V}_{\mathrm{s}}(\boldsymbol{Q})-\bar{m}_x\psi_x-\bar{m}_y\psi_y-\bar{q}w\right]\mathrm{d}x\mathrm{d}y\\&+\int_{C_{3a}}(\bar{M}_{ns}\psi_s-\bar{Q}_n w)\mathrm{d}s+\int_{C_{1a}+C_{2a}}[(\psi_s-\bar{\psi}_s)M_{ns}-(w-\bar{w})Q_n]\mathrm{d}s\\&+\int_{C_{1a}}(\psi_n-\bar{\psi}_n)M_n\mathrm{d}s+\int_{C_{2a}+C_{3a}}\bar{M}_n\psi_n\mathrm{d}s\end{aligned}\tag{2-111}$$

$$\begin{aligned}\varPi_{2\mathrm{c}}^{(a)} = &\iint_{\Omega_a}\left[\tilde{V}_{\mathrm{b}}(\boldsymbol{M})+\tilde{V}_{\mathrm{s}}(\boldsymbol{Q})-\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x-\bar{m}_x\right)\psi_x\right.\\&\left.-\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y-\bar{m}_y\right)\psi_y+\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\bar{q}\right)w\right]\mathrm{d}x\mathrm{d}y\\&+\int_{C_{1a}+C_{2a}}(\bar{\psi}_s M_{ns}-\bar{w}Q_n)\mathrm{d}s+\int_{C_{3a}}[(M_{ns}-\bar{M}_{ns})\psi_s-(Q_n-\bar{Q}_n)w]\mathrm{d}s\\&+\int_{C_{1a}}\bar{\psi}_n M_n\mathrm{d}s+\int_{C_{2a}+C_{3a}}(M_n-\bar{M}_n)\psi_n\mathrm{d}s\end{aligned}\tag{2-112}$$

From equations (2-86), (2-104) and (2-107), the functional expressions of the sub-region two-field generalized mixed energy, potential energy and complementary energy principles can be obtained:

$$\varPi_2=\varPi_{2\mathrm{p}}^{(a)}-\varPi_{2\mathrm{c}}^{(b)}+H_{\mathrm{pc}}\tag{2-113}$$

$$\varPi_2=\varPi_{2\mathrm{p}}^{(a)}+\varPi_{2\mathrm{p}}^{(b)}+H_{\mathrm{pp}}\tag{2-114}$$

$$\varPi_2=-\varPi_{2\mathrm{c}}^{(a)}-\varPi_{2\mathrm{c}}^{(b)}+H_{\mathrm{cc}}\tag{2-115}$$

where H_{pc}, H_{pp} and H_{cc} are still given by Eqs. (2-91), (2-105) and (2-108), respectively.

2. The sub-region single-field variational principle

Now we consider the case where each sub-region is a single-field region.

If the region a is a potential energy region, only the displacement $\boldsymbol{d}$ will be taken as the field variable. Thus, the $\varPi_{3\mathrm{p}}^{(a)}$ in Eq. (2-87) or the $\varPi_{2\mathrm{p}}^{(a)}$ in Eq. (2-111) will transform to the single-field potential energy $\varPi_{1\mathrm{p}}^{(a)}$ of the region a:

$$\begin{aligned}\Pi_{1\text{p}}^{(a)} = & \iint_{\Omega_a}[\tilde{U}_{\text{b}}(\boldsymbol{d}) + \tilde{U}_{\text{s}}(\boldsymbol{d}) - \overline{m}_x\psi_x - \overline{m}_y\psi_y - \overline{q}w]\text{d}x\text{d}y \\ & + \int_{C_{1a}+C_{2a}}[(\psi_s - \overline{\psi}_s)\hat{M}_{ns} - (w-\overline{w})\hat{Q}_n]\text{d}s + \int_{C_{3a}}(\overline{M}_{ns}\psi_s - \overline{Q}_n w)\text{d}s \\ & + \int_{C_{1a}}(\psi_n - \overline{\psi}_n)\hat{M}_n\text{d}s + \int_{C_{2a}+C_{3a}}\overline{M}_n\psi_n\text{d}s \end{aligned} \tag{2-116a}$$

where $\hat{Q}_n$, $\hat{M}_n$ and $\hat{M}_{ns}$ are the boundary force variables, and can also be expressed by the functions of the displacement $\boldsymbol{d}$; $\tilde{U}_{\text{b}}(\boldsymbol{d})$ and $\tilde{U}_{\text{s}}(\boldsymbol{d})$ are the strain energy densities expressed by the displacement $\boldsymbol{d}$:

$$\tilde{U}_{\text{b}}(\boldsymbol{d}) = \frac{D}{2}\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2 + \left(\frac{\partial\psi_y}{\partial y}\right)^2 + 2\mu\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right)^2\right] \tag{2-117}$$

$$\tilde{U}_{\text{s}}(\boldsymbol{d}) = \frac{C}{2}\left[\left(\frac{\partial w}{\partial x} - \psi_x\right)^2 + \left(\frac{\partial w}{\partial y} - \psi_y\right)^2\right] \tag{2-118}$$

If the displacement $\boldsymbol{d}$ satisfies the geometrical boundary conditions in advance, then

$$\begin{aligned}\Pi_{1\text{p}}^{(a)} = & \iint_{\Omega_a}[\tilde{U}_{\text{b}}(\boldsymbol{d}) + \tilde{U}_{\text{s}}(\boldsymbol{d}) - \overline{m}_x\psi_x - \overline{m}_y\psi_y - \overline{q}w]\text{d}x\text{d}y \\ & + \int_{C_{3a}}(\overline{M}_{ns}\psi_s - \overline{Q}_n w)\text{d}s + \int_{C_{2a}+C_{3a}}\overline{M}_n\psi_n\text{d}s \end{aligned} \tag{2-116b}$$

If the sub-region a is a complementary energy region, only the internal force $\boldsymbol{S}$ will be taken as the field variable, and $\boldsymbol{S}$ should satisfy the equilibrium differential Eq. (2-95) in advance. Thus, the $\Pi_{3\text{c}}^{(a)}$ in Eq. (2-90) or the $\Pi_{2\text{c}}^{(a)}$ in Eq. (2-112) will transform to the single-field complementary energy $\Pi_{1\text{c}}^{(a)}$:

$$\begin{aligned}\Pi_{1\text{c}}^{(a)} = & \iint_{\Omega_a}[\tilde{V}_{\text{b}}(\boldsymbol{M}) + \tilde{V}_{\text{s}}(\boldsymbol{Q})]\,\text{d}x\text{d}y + \int_{C_{1a}+C_{2a}}[(\overline{\psi}_s M_{ns} - \overline{w}Q_n)]\text{d}s \\ & + \int_{C_{3a}}[(M_{ns} - \overline{M}_{ns})\hat{\psi}_s - (Q_n - \overline{Q}_n)\hat{w})]\text{d}s + \int_{C_{1a}}\overline{\psi}_n M_n\text{d}s \\ & + \int_{C_{2a}+C_{3a}}(M_n - \overline{M}_n)\hat{\psi}_n\text{d}s \end{aligned} \tag{2-119a}$$

where $\hat{w}$, $\hat{\psi}_n$ and $\hat{\psi}_s$ are the boundary displacement variables.

If $\boldsymbol{S}$ satisfies the force boundary conditions in advance, then

$$\Pi_{1\text{c}}^{(a)} = \iint_{\Omega_a}[\tilde{V}_{\text{b}}(\boldsymbol{M}) + \tilde{V}_{\text{s}}(\boldsymbol{Q})]\text{d}x\text{d}y + \int_{C_{1a}+C_{2a}}[(\overline{\psi}_s M_{ns} - \overline{w}Q_n)]\text{d}s + \int_{C_{1a}}\overline{\psi}_n M_n\text{d}s \tag{2-119b}$$

From Eqs. (2-86), (2-104), (2-107), or (2-113), (2-114), (2-115), the functional expressions of the sub-region single-field mixed energy principle, potential energy principle and complementary energy principle can be obtained:

$$\Pi_1 = \Pi_{1\mathrm{p}}^{(a)} - \Pi_{1\mathrm{c}}^{(b)} + H_{\mathrm{pc}} \tag{2-120}$$

$$\Pi_1 = \Pi_{1\mathrm{p}}^{(a)} + \Pi_{1\mathrm{p}}^{(b)} + H_{\mathrm{pp}} \tag{2-121}$$

$$\Pi_1 = -\Pi_{1\mathrm{c}}^{(a)} - \Pi_{1\mathrm{c}}^{(b)} + H_{\mathrm{cc}} \tag{2-122}$$

where H_{pc}, H_{pp} and H_{cc} are the same as those in Eq. (2-113) to Eq. (2-115).

2.4.4 The General Form of the Sub-Region Generalized Variational Principle for Thick Plate

From the above discussions, a general form of the sub-region generalized variational principle for elastic thick plate can be obtained. Let an elastic thick plate be divided into several sub-regions. Each sub-region can be arbitrarily appointed as single-field, two-field and three-field potential energy regions (such as the regions Ω_{p1}, Ω_{p2}, Ω_{p3} in Fig. 2.6) or complementary energy regions (such as the regions Ω_{c1}, Ω_{c2}, Ω_{c3} in Fig. 2.6). The interfaces between two adjacent sub-regions are of three types, C_{pc}, C_{pp} and C_{cc}: ① one side of C_{pc} is the potential energy region, while the other side is the complementary one; ② both sides of C_{pp} are potential energy regions; and ③ both sides of C_{cc} are complementary energy regions.

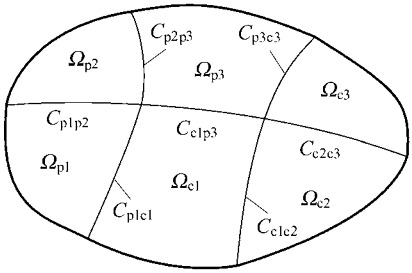

Figure 2.6 A thick plate divided into multi-regions

The general form for the functional of the sub-region variational principle can be written as

$$\Pi = \sum_{\Omega_{\mathrm{p}}} \Pi_{\mathrm{p}} - \sum_{\Omega_{\mathrm{c}}} \Pi_{\mathrm{c}} + \sum_{C_{\mathrm{pc}}} H_{\mathrm{pc}} + \sum_{C_{\mathrm{pp}}} H_{\mathrm{pp}} + \sum_{C_{\mathrm{cc}}} H_{\mathrm{cc}} \tag{2-123}$$

The meanings of the terms on the right-side of this equation are as follows:

The first term denotes the sum of the potential (or generalized potential) energy Π_p of each potential energy region Ω_p, where Π_p can be Π_{1p} or Π_{2p} or Π_{3p}, which is given by Eqs. (2-116), (2-111) and (2-87), respectively;

The second term denotes the sum of the complementary (or generalized complementary) energy Π_c of each complementary energy region Ω_c, where Π_c can be Π_{1c} or Π_{2c} or Π_{3c}, which is given by Eqs. (2-119), (2-112) and (2-90), respectively;

The third term denotes the sum of the additional mixed energy term H_{pc} on the interface C_{pc}, in which H_{pc} is given by Eq. (2-91);

The fourth term denotes the sum of the additional potential energy term H_{pp} on the interface C_{pp}, in which H_{pp} is given by Eq. (2-105);

The fifth term denotes the sum of the additional complementary energy term H_{cc} on the interface C_{cc}, in which H_{cc} is given by Eq. (2-108).

It can be shown that the stationary condition

$$\delta \Pi = 0 \tag{2-124}$$

of the functional Π in Eq. (2-123) is equivalent to all field equations, boundary conditions and interface conditions of the thick plate system with multi-regions.

If all the sub-regions are potential energy regions, the functional of the sub-region potential (or generalized potential) energy principle can be obtained from Eq. (2-123):

$$\Pi = \sum_{\Omega_p} \Pi_p + \sum_{C_{pp}} H_{pp} \tag{2-125}$$

Equations (2-104), (2-114) and (2-121) are all the special cases of (2-125).

If all the sub-regions are complementary energy regions, the functional of the sub-region complementary (or generalized complementary) energy principle can be obtained from Eq. (2-123):

$$\Pi = -\sum_{\Omega_c} \Pi_c + \sum_{C_{cc}} H_{cc} \tag{2-126}$$

Equations (2-107), (2-115) and (2-122) are all the special cases of (2-126).

The functional expression (2-123) of the sub-region generalized variational principle for elastic thick plate is the most general functional form of the variational principle for thick plate, and builds a bridge linking various special functional forms of the variational principle.

By using the relation (2-100), the direct transformation between the different functional forms of the variational principle for thick plate can be performed conveniently.

The sub-region mixed variational principle for thick plate and its functional expression (2-86) provide the fundamentals for the applications of the sub-region mixed finite element method in thick plate problems.

2.5 The Sub-Region Variational Principle for Elastic Shallow Shell

This section will discuss the sub-region variational principle for elastic shallow shell[14]. The fundamental equations and the variational principles for shallow shell were systematically introduced in [3].

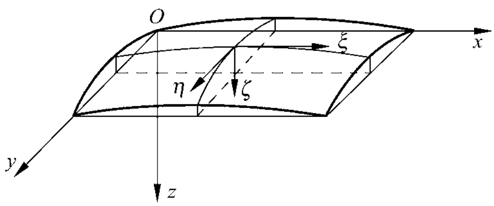

Figure 2.7 The shallow shell

Let the bottom plane of the shallow shell be the xOy plane, and the z-axis be normal to the bottom plane (Fig. 2.7). Then, the mid-surface equation of the shallow shell is

$$z = z(x, y)$$

The initial curvatures of the mid-surface are

$$\kappa_x = -\frac{\partial^2 z}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 z}{\partial y^2}, \quad \kappa_{xy} = -\frac{\partial^2 z}{\partial x \partial y}$$

And, another movable co-ordinate system (ξ, η, ζ) is also adopted where ζ-axis is the normal of the mid-surface, and ξ-axis and η-axis are the tangents of the mid-surface within xz-plane and yz-plane, respectively.

The load components along ξ, η and ζ directions of an arbitrary point within the mid-surface are p_x, p_y and p_z; and the displacement components are u, v and w. There are three membrane internal force components N_x, N_y and N_{xy} in shallow shell structures, and their corresponding strains are ε_x, ε_y and γ_{xy}. There are also three independent internal moment components M_x, M_y and M_{xy}, and their corresponding generalized strains are the curvature variety values κ_x, κ_y and κ_{xy}. Furthermore, the transverse shear forces Q_x and Q_y are dependent internal force components, and can be determined by M_x, M_y and M_{xy}. In thin shells, the transverse shear strain γ_{xz} and γ_{yz} are both zero.

On the boundary line C of the shallow shell, let n and s be the outer normal and the tangent directions. The displacement components along n, s and ζ directions of an arbitrary point on the boundary line are u_n, v_s and w, and the corresponding boundary forces are normal tension N_n, tangent shear force N_{ns}, and equivalent transverse shear force $V_n = Q_n + \frac{\partial M_{ns}}{\partial s}$ which is synthesized by the transverse shear force Q_n and the twisting moment M_{ns}. The rotation on the boundary within $n\zeta$ plane is $\psi_n = \dfrac{\partial w}{\partial n}$, and the corresponding boundary moment is the normal bending moment M_n.

The boundary line C of the shallow shell contains different line segments:

$$C = C_{u_n} + C_{N_n} = C_{v_s} + C_{N_{ns}} = C_w + C_{V_n} = C_{\psi_n} + C_{M_n}$$

where $C_{u_n}, C_{N_n}, C_{v_s}, C_{N_{ns}}, C_w, C_{V_n}, C_{\psi_n}$ and C_{M_n} denote the boundary segments on which u_n, N_n, v_s, N_{ns}, w, V_n, ψ_n and M_n are specified, respectively.

A denotes the corner point on the boundary line, and is generally composed of two types:

$$A = A_w + A_R$$

where A_w and A_R are the corner points where the deflection $\bar{w}$ and transverse concentrated force $\bar{R}$ are specified, respectively. The twisting moment increment of the two sides of corner point A is $(\Delta M_{ns})_A$.

In the sub-region generalized variational principle for shallow shell, the mid and the bottom surfaces of the shallow shell are divided into several sub-regions. Each sub-region can be arbitrarily appointed as single-field, two-field and three-field potential energy regions (such as Ω_{p1}, Ω_{p2} and Ω_{p3} in Fig. 2.8), or the complementary energy regions (such as Ω_{c1}, Ω_{c2} and Ω_{c3} in Fig. 2.8). The interfaces between two adjacent sub-regions are of three types, C_{pc}, C_{pp} and C_{cc} (Fig. 2.8): ① one side of C_{pc} is the potential energy region, while the other side is the complementary one; ② both sides of C_{pp} are potential energy regions; and ③ both sides of C_{cc} are complementary energy regions. The node J of the adjacent sub-regions generally is also classified into two types, J_w and J_R: J_w and J_R are the nodes where the displacement $\bar{w}$ and the transverse concentrated force $\bar{R}$ are specified, respectively. r_p and r_c denote the numbers of the elements e_p in the potential energy regions and the elements e_c in the complementary energy regions around the node J, respectively.

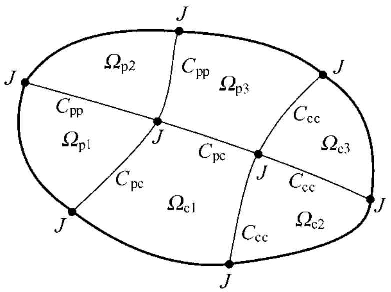

Figure 2.8 A shallow shell divided into multi-regions

The functional of the sub-region generalized variational principle for elastic shallow shell can be written as:

$$\Pi = \sum_{\Omega_{\rm p}} \Pi_{\rm p} - \sum_{\Omega_{\rm c}} \Pi_{\rm c} + \sum_{C_{\rm pc}} H_{\rm pc} + \sum_{C_{\rm pp}} H_{\rm pp} + \sum_{C_{\rm cc}} H_{\rm cc} + \sum_{J_w} G_w + \sum_{J_R} G_R \tag{2-127}$$

There are seven terms on the right side of the above equation, where the first two terms are the energy of all the sub-regions; the middle three terms are the energy on the interfaces; and the last two terms are the energy at the nodes. Now, the expressions and their meanings of all the terms are given as follows.

The first term on the right side of Eq. (2-127) denotes the sum of the potential (or generalized potential) energy $\Pi_{\rm p}$ of each potential energy region $\Omega_{\rm p}$; the second term denotes the sum of the complementary (or generalized complementary) energy $\Pi_{\rm c}$ of each complementary energy region $\Omega_{\rm c}$. If the sub-region e is a three-field region, then, $\Pi_{\rm p}$ and $\Pi_{\rm c}$ are the following $\Pi_{3\rm p}$ and $\Pi_{3\rm c}$, respectively.

$$\Pi_{3\rm p}^{(e)} = \Pi_{3\rm p}'^{(e)} + \Pi_{3\rm p}''^{(e)} + I^{(e)} \tag{2-128}$$

$$\Pi_{3\rm c}^{(e)} = \Pi_{3\rm c}'^{(e)} + \Pi_{3\rm c}''^{(e)} - I^{(e)} \tag{2-129}$$

where

$$\begin{aligned}\Pi_{3\rm p}'^{(e)} = & \iint_{\Omega_e} \left[\tilde{U}'(\boldsymbol{\varepsilon}) - p_x u - p_y v - \left(\varepsilon_x - \frac{\partial u}{\partial x} \right) N_x - \left(\varepsilon_y - \frac{\partial v}{\partial y} \right) N_y \right. \\ & \left. - \left(\gamma_{xy} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) N_{xy} \right] {\rm d}x{\rm d}y \\ & - \int_{C_{N_n}^e} \overline{N}_n u_n {\rm d}s - \int_{C_{N_{ns}}^e} \overline{N}_{ns} v_s {\rm d}s - \int_{C_{u_n}^e} (u_n - \overline{u}_n) N_n {\rm d}s \\ & - \int_{C_{v_s}^e} (v_s - \overline{v}_s) N_{ns} {\rm d}s \end{aligned} \tag{2-130}$$

$$\begin{aligned}\Pi_{3\rm p}''^{(e)} = & \iint_{\Omega_e} \left[\tilde{U}''(\boldsymbol{\kappa}) - p_z w - \left(\kappa_x + \frac{\partial^2 w}{\partial x^2} \right) M_x - \left(\kappa_y + \frac{\partial^2 w}{\partial y^2} \right) M_y \right. \\ & \left. - 2\left(\kappa_{xy} + \frac{\partial^2 w}{\partial x \partial y} \right) M_{xy} \right] {\rm d}x{\rm d}y \\ & - \int_{C_{V_n}^e} \overline{V}_n w {\rm d}s + \int_{C_{M_n}^e} \overline{M}_n \frac{\partial w}{\partial n} {\rm d}s - \sum_{A_{R^e}} \overline{R} w - \int_{C_{w^e}} (w - \overline{w}) V_n {\rm d}s \\ & + \int_{C_{\psi_n}^e} \left(\frac{\partial w}{\partial n} - \overline{\psi}_n \right) M_n {\rm d}s - \sum_{A_{w^e}} (w - \overline{w}) \Delta M_{ns} \end{aligned} \tag{2-131}$$

$$I^e = \iint_{\Omega_e} (\kappa_x N_x + \kappa_y N_y + 2\kappa_{xy} N_{xy}) w {\rm d}x{\rm d}y \tag{2-132}$$

$$\Pi_{3c}'^{(e)} = \iint_{\Omega_e} \left[-\tilde{U}'(\boldsymbol{\varepsilon}) + \varepsilon_x N_x + \varepsilon_y N_y + \gamma_{xy} N_{xy} + \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x \right) u + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y \right) v \right] \mathrm{d}x\mathrm{d}y - \int_{C_{N_n^e}} (N_n - \bar{N}_n) u_n \mathrm{d}s - \int_{C_{N_{ns}^e}} (N_{ns} - \bar{N}_{ns}) v_s \mathrm{d}s - \int_{C_{u_n^e}} \bar{u}_n N_n \mathrm{d}s - \int_{C_{v_s^e}} \bar{v}_s N_{ns} \mathrm{d}s \tag{2-133}$$

$$\Pi_{3c}''^{(e)} = \iint_{\Omega_e} \left[-\tilde{U}''(\boldsymbol{\kappa}) + \kappa_x M_x + \kappa_y M_y + 2\kappa_{xy} M_{xy} + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + p_z \right) w \right] \mathrm{d}x\mathrm{d}y - \int_{C_{V_n^e}} (V_n - \bar{V}_n) w \mathrm{d}s + \int_{C_{M_n^e}} (M_n - \bar{M}_n) \frac{\partial w}{\partial n} \mathrm{d}s - \sum_{A_{R^e}} (\Delta M_{ns} - \bar{R}) w - \int_{C_{w^e}} \bar{w} V_n \mathrm{d}s + \int_{C_{\psi_n^e}} \bar{\psi}_n M_n \mathrm{d}s - \sum_{A_{w^e}} \bar{w} \Delta M_{ns} \tag{2-134}$$

where $\tilde{U}'(\boldsymbol{\varepsilon})$ and $\tilde{U}''(\boldsymbol{\kappa})$ are the strain energy density of the in-plane action and the thin plate bending, respectively:

$$\tilde{U}'(\boldsymbol{\varepsilon}) = \frac{Eh}{2(1-\mu^2)} \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\mu\varepsilon_x\varepsilon_y + \frac{1-\mu}{2}\gamma_{xy}^2 \right) \tag{2-135}$$

$$\tilde{U}''(\boldsymbol{\kappa}) = \frac{Eh^3}{24(1-\mu^2)} (\kappa_x^2 + \kappa_y^2 + 2\mu\kappa_x\kappa_y + 2(1-\mu)\kappa_{xy}^2) \tag{2-136}$$

E and μ are the Young's modulus and Poisson's ratio, respectively; h is the thickness of the thin shell.

If the sub-region e is a two-field region, then, Π_p and Π_c are the following Π_{2p} and Π_{2c}, respectively:

$$\Pi_{2p}^{(e)} = \Pi_{2p}'^{(e)} + \Pi_{2p}''^{(e)} + I^{(e)} \tag{2-137}$$

$$\Pi_{2c}^{(e)} = \Pi_{2c}'^{(e)} + \Pi_{2c}''^{(e)} - I^{(e)} \tag{2-138}$$

where

$$\Pi_{2p}'^{(e)} = \iint_{\Omega_e} \left[-\tilde{V}'(\boldsymbol{N}) - p_x u - p_y v + N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \mathrm{d}x\mathrm{d}y - \int_{C_{N_n^e}} \bar{N}_n u_n \mathrm{d}s - \int_{C_{N_{ns}^e}} \bar{N}_{ns} v_s \mathrm{d}s - \int_{C_{u_n^e}} (u_n - \bar{u}_n) N_n \mathrm{d}s - \int_{C_{v_s^e}} (v_s - \bar{v}_s) N_{ns} \mathrm{d}s \tag{2-139}$$

$$\Pi''^{(e)}_{2\mathrm{p}} = \iint_{\Omega_e}\left[-\tilde{V}''(\boldsymbol{M}) - p_z w - M_x\frac{\partial^2 w}{\partial x^2} - M_y\frac{\partial^2 w}{\partial y^2} - 2M_{xy}\frac{\partial^2 w}{\partial x\partial y}\right]\mathrm{d}x\mathrm{d}y$$
$$-\int_{C_{V_n}^e}\bar{V}_n w\mathrm{d}s + \int_{C_{M_n}^e}\bar{M}_n\frac{\partial w}{\partial n}\mathrm{d}s - \sum_{A_{R^e}}\bar{R}w - \int_{C_{w^e}}(w-\bar{w})V_N\mathrm{d}s$$
$$+\int_{C_{\psi_n}^e}\left(\frac{\partial w}{\partial n} - \bar{\psi}_n\right)M_n\mathrm{d}s - \sum_{A_{w^e}}(w-\bar{w})\Delta M_{ns} \quad (2\text{-}140)$$

$$\Pi'^{(e)}_{2\mathrm{c}} = \iint_{\Omega_e}\left[\tilde{V}'(\boldsymbol{N}) + \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x\right)u + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y\right)v\right]\mathrm{d}x\mathrm{d}y$$
$$-\int_{C_{N_n}^e}(N_n - \bar{N}_n)u_n\mathrm{d}s - \int_{C_{N_{ns}}^e}(N_{ns} - \bar{N}_{ns})v_s\mathrm{d}s - \int_{C_{u_n}^e}\bar{u}_n N_n\mathrm{d}s - \int_{C_{v_s}^e}\bar{v}_s N_{ns}\mathrm{d}s$$
$$(2\text{-}141)$$

$$\Pi''^{(e)}_{2\mathrm{c}} = \iint_{\Omega_e}\left[\tilde{V}''(\boldsymbol{M}) + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + p_z\right)w\right]\mathrm{d}x\mathrm{d}y$$
$$-\int_{C_{V_n}^e}(V_n - \bar{V}_n)w\mathrm{d}s + \int_{C_{M_n}^e}(M_n - \bar{M}_n)\frac{\partial w}{\partial n}\mathrm{d}s - \sum_{A_{R^e}}(\Delta M_{ns} - \bar{R})w$$
$$-\int_{C_{w^e}}\bar{w}V_n\mathrm{d}s + \int_{C_{\psi_n}^e}\bar{\psi}_n M_n\mathrm{d}s - \sum_{A_{w^e}}\bar{w}\Delta M_{ns} \quad (2\text{-}142)$$

$I^{(e)}$ is still given by Eq. (2-132); $\tilde{V}'(\boldsymbol{N})$ and $\tilde{V}''(\boldsymbol{M})$ are the strain complementary energy density of the in-plane action and the thin plate bending, respectively:

$$\tilde{V}'(\boldsymbol{N}) = \frac{1}{2}\cdot\frac{1}{Eh}[N_x^2 + N_y^2 - 2\mu N_x N_y + 2(1+\mu)N_{xy}^2] \quad (2\text{-}143)$$

$$\tilde{V}''(\boldsymbol{M}) = \frac{1}{2}\cdot\frac{12}{Eh^3}[M_x^2 + M_y^2 - 2\mu M_x M_y + 2(1+\mu)M_{xy}^2] \quad (2\text{-}144)$$

If the sub-region e is a single-field region, then, Π_{p} and Π_{c} are the following $\Pi_{1\mathrm{p}}$ and $\Pi_{1\mathrm{c}}$, respectively:

$$\Pi^{(e)}_{1\mathrm{p}} = \Pi'^{(e)}_{1\mathrm{p}} + \Pi''^{(e)}_{1\mathrm{p}} \quad (2\text{-}145)$$

where

$$\Pi'^{(e)}_{1\mathrm{p}} = \iint_{\Omega_e}[\tilde{U}'(u,v,w) - p_x u - p_y v]\mathrm{d}x\mathrm{d}y - \int_{C_{N_n}^e}\bar{N}_n u_n\mathrm{d}s - \int_{C_{N_{ns}}^e}\bar{N}_{ns}v_s\mathrm{d}s$$
$$-\int_{C_{u_n}^e}(u_n - \bar{u}_n)N_n\mathrm{d}s - \int_{C_{v_s}^e}(v_s - \bar{v}_s)N_{ns}\mathrm{d}s \quad (2\text{-}146)$$

$$\Pi_{1p}''^{(e)} = \iint_{\Omega_e} [\tilde{U}''(w) - p_z w] \mathrm{d}x\mathrm{d}y - \int_{C_{V_n^e}} \bar{V}_n w \mathrm{d}s + \int_{C_{M_n^e}} \bar{M}_n \frac{\partial w}{\partial n} \mathrm{d}s - \sum_{A_{R^e}} \bar{R} w$$
$$- \int_{C_{w^e}} (w - \bar{w}) V_n \mathrm{d}s + \int_{C_{\psi_n^e}} \left(\frac{\partial w}{\partial n} - \bar{\psi}_n \right) M_n \mathrm{d}s - \sum_{A_{w^e}} (w - \bar{w}) \Delta M_{ns} \quad (2\text{-}147)$$

in which only a single field, i.e. displacement field (u, v, w), exists within the sub-region e; and N_n, N_{ns}, V_n, M_n and ΔM_{ns} are only the boundary variables or corner point variables defined on the element boundaries and corner points. $\tilde{U}'(u,v,w)$ and $\tilde{U}''(w)$ are the strain energy densities, expressed by the displacement, of the in-plane strain and thin plate bending, respectively.

$$\Pi_{1c}^{e} = \Pi_{1c}'^{e} + \Pi_{1c}''^{e} \quad (2\text{-}148)$$

where

$$\Pi_{1c}'^{(e)} = \int_{\Omega_e} \tilde{V}'(\boldsymbol{N}) \mathrm{d}x\mathrm{d}y - \int_{C_{N_n^e}} (N_n - \bar{N}_n) u_n \mathrm{d}s - \int_{C_{N_{ns}^e}} (N_{ns} - \bar{N}_{ns}) v_s \mathrm{d}s$$
$$- \int_{C_{u_n^e}} \bar{u}_n N_n \mathrm{d}s - \int_{C_{v_s^e}} \bar{v}_s N_{ns} \mathrm{d}s \quad (2\text{-}149)$$

$$\Pi_{1c}''^{(e)} = \iint_{\Omega_e} \tilde{V}''(\boldsymbol{M}) \mathrm{d}x\mathrm{d}y - \int_{C_{V_n^e}} (V_n - \bar{V}_n) w \mathrm{d}s + \int_{C_{M_n^e}} (M_n - \bar{M}_n) \frac{\partial w}{\partial n} \mathrm{d}s$$
$$- \sum_{A_{R^e}} (\Delta M_{ns} - \bar{R}) w - \int_{C_{w^e}} \bar{w} V_n \mathrm{d}s + \int_{C_{\psi_n^e}} \bar{\psi}_n M_n \mathrm{d}s - \sum_{A_{w^e}} \bar{w} \Delta M_{ns} \quad (2\text{-}150)$$

Here, only a single-field, i.e. internal force field (N_x, N_y, N_{xy}, M_x, M_y, M_{xy}), exists within the sub-region e, and these internal forces in advance satisfy the equilibrium differential equation of the shallow shell:

$$\left.\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x = 0 \\ &\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y = 0 \\ &\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - k_x N_x - k_y N_y - 2k_{xy} N_{xy} + p_z = 0 \end{aligned}\right\} \quad (2\text{-}151)$$

u_n, v_s, w and $\dfrac{\partial w}{\partial n}$ are only the boundary or corner point variables defined on the element boundaries or corner points.

The third, fourth and fifth terms on the right side of Eq. (2-127) are the sum of the additional energy H_{pc}, H_{pp} and H_{cc} on the interfaces C_{pc}, C_{pp} and C_{cc} between the adjacent sub-regions e and e', respectively, where

$$H_{\text{pc}} = \int_{C_{ee'}} \left[-N_n^{e'} u_n^e - N_{ns}^{e'} v_s^e + M_n^{e'} \left(\frac{\partial w}{\partial n} \right)^e + V_n^{e'} w^e \right] \mathrm{d}s \tag{2-152}$$

(e is the potential energy region; e' is the complementary energy region)

$$\begin{aligned} H_{\text{pp}} &= \int_{C_{ee'}} \left[-N_n^{e'} (u_n^e + u_n^{e'}) - N_{ns}^{e'} (v_s^e + v_s^{e'}) + M_n^{e'} \left(\left(\frac{\partial w}{\partial n} \right)^e + \left(\frac{\partial w}{\partial n} \right)^{e'} \right) + V_n^{e'} (w^e - w^{e'}) \right] \mathrm{d}s \\ &= \int_{C_{ee'}} \left[-N_n^e (u_n^e + u_n^{e'}) - N_{ns}^e (v_s^e + v_s^{e'}) + M_n^e \left(\left(\frac{\partial w}{\partial n} \right)^e + \left(\frac{\partial w}{\partial n} \right)^{e'} \right) + V_n^e (w^{e'} - w^e) \right] \mathrm{d}s \end{aligned} \tag{2-153}$$

$$\begin{aligned} H_{\text{cc}} &= \int_{C_{ee'}} \left[(N_n^e - N_n^{e'}) u_n^e + (N_{ns}^e - N_{ns}^{e'}) v_s^e - (M_n^e - M_n^{e'}) \left(\frac{\partial w}{\partial n} \right)^e + (V_n^e + V_n^{e'}) w^e \right] \mathrm{d}s \\ &= \int_{C_{ee'}} \left[(N_n^{e'} - N_n^e) u_n^{e'} + (N_{ns}^{e'} - N_{ns}^e) v_s^{e'} - (M_n^{e'} - M_n^e) \left(\frac{\partial w}{\partial n} \right)^{e'} + (V_n^{e'} + V_n^e) w^{e'} \right] \mathrm{d}s \end{aligned} \tag{2-154}$$

The last two terms on the right side of Eq. (2-127) are the sum of additional energy G_w and G_R at the nodes J_w and J_R, respectively, where

$$G_w = -\sum_{e_{\text{p}}} (\Delta M_{ns})^{(e_{\text{p}})} (w^{(e_{\text{p}})} - \overline{w}) + \sum_{e_{\text{c}}} (\Delta M_{ns})^{(e_{\text{c}})} \overline{w} \tag{2-155}$$

$$G_R = \left[\sum_e (\Delta M_{ns})^{(e)} - \overline{R} \right] w^{(a)} - \sum_{e_{\text{p}}} (\Delta M_{ns})^{(e_{\text{p}})} w^{(e_{\text{p}})} \tag{2-156}$$

where $\sum_{e_{\text{p}}}$, $\sum_{e_{\text{c}}}$ and $\sum_e$ denote the sum of all the potential energy elements e_{p}, all the complementary energy elements e_{c} and all the elements e around the nodes, respectively; $w^{(a)}$ is the displacement of any element a around the nodes.

It can be shown that the stationary condition

$$\delta \Pi = 0 \tag{2-157}$$

of the functional Π in Eq. (2-127) is equivalent to all field equations, boundary conditions, interface conditions, corner point and node conditions of the shallow shell system with multi-regions.

As a special case, if each sub-region is appointed as a potential energy region (or complementary energy region), then, the functional of the sub-region generalized potential (or complementary) energy principle can be obtained from Eq. (2-127).

2.6 The Sub-Region Mixed Energy Partial Derivative Theorem

This section will discuss the sub-region mixed energy partial derivative theorem and its extensions[15].

Castigliano first and second theorems are two famous energy partial derivative theorems in history, and both of them are the special cases of the sub-region mixed energy partial derivative theorem.

2.6.1 The Sub-Region Mixed Energy Partial Derivative Theorem and Its Proof

1. The definition of the sub-region mixed energy

Let a structure be divided into two regions: complementary energy region (region a) and potential energy region (region b). The complementary energy region has n_1 independent force variables $X_1, X_2, \cdots, X_{n_1}$, and its complementary energy $(\Pi_{\mathrm{c}})_a$ is expressed as a function of these force variables. The potential energy region has n_2 displacements at the supports (or constrained displacements) $\Delta_{n_1+1}, \Delta_{n_1+2}, \cdots, \Delta_{n_1+n_2}$ as independent displacement variables, and its potential energy $(\Pi_{\mathrm{p}})_b$ is expressed as a function of these displacement variables. Furthermore, the additional energy Π_J at the interface J between the regions a and b equals to the work done by the constrained force $(\hat{F}_J)_a$ of the region a along the displacement $(\hat{D}_J)_b$ of the region b:

$$\Pi_J = \sum_J (\hat{F}_J)_a (\hat{D}_J)_b$$

The sub-region mixed energy Π_m is defined as:

$$\Pi_m = (\Pi_{\mathrm{p}})_b - (\Pi_{\mathrm{c}})_a + \Pi_J = (\Pi_{\mathrm{p}})_b - (\Pi_{\mathrm{c}})_a + \sum_J (\hat{F}_J)_a (\hat{D}_J)_b \tag{2-158}$$

As an example, consider a frame shown in Fig. 2.9(a). The left side of the interface J is the complementary energy region (region a), and the right side is the potential energy region (region b). There is force variable X_1 operating in the region a. Let $(M)_a$ be the bending moment of the region a, then the complementary energy $(\Pi_{\mathrm{c}})_a$ of the region a is

$$(\Pi_{\mathrm{c}})_a = \sum_a \int \frac{1}{2EI} (M)_a^2 \mathrm{d}s \tag{2-159}$$

There is a displacement variable Δ_2 (the nodal rotation) in the region b.

Furthermore, the structure is also under a constant load P. Let $(M)_b$ be the bending moment of the region b, D be the corresponding displacement of load P, then the potential energy $(\Pi_\text{p})_b$ of the region b is

$$(\Pi_\text{p})_b = \sum_b \int \frac{1}{2EI}(M)_b^2 \text{ds} - \sum_b (PD)_b \tag{2-160}$$

At the interface J, the displacement $(\hat{D}_J)_b$ of the region b is the nodal rotation Δ_2, the constrained force $(\hat{F}_J)_a$ of the region a is the bending moment $(M_J)_a$ of cross section J. The additional energy Π_J on the interface is

$$\Pi_J = (\hat{F}_J)_a(\hat{D}_J)_b = (M_J)_a \Delta_2 \tag{2-161}$$

Substitution of Eqs. (2-159), (2-160) and (2-161) into (2-158) yields

$$\Pi_\text{m} = \sum_b \int \frac{1}{2EI}(M)_b^2 \text{d}s - \sum_b (PD)_b - \sum_a \int \frac{1}{2EI}(M)_a^2 \text{ds} + (M_J)_a \Delta_2 \tag{2-162}$$

2. The description of the sub-region mixed energy partial derivative theorem

If the sub-region mixed energy Π_m of the structure is defined by Eq. (2-158), the partial derivative of Π_m with respect to force variable X_i of the complementary energy region will be equal to a minus value of the displacement D_i which corresponds to X_i, and the partial derivative of Π_m with respect to displacement variable Δ_j of the potential energy region will be equal to the constrained force F_j which corresponds to Δ_j, i.e.,

$$\left.\begin{aligned} D_i &= -\frac{\partial \Pi_\text{m}}{\partial X_i} \qquad (i = 1, 2, \cdots, n_1) \\ F_j &= \frac{\partial \Pi_\text{m}}{\partial \Delta_j} \qquad (j = n_1 + 1, n_1 + 2, \cdots, n_1 + n_2) \end{aligned}\right\} \tag{2-163}$$

3. The proof for the sub-region mixed energy partial derivative theorem

Consider the frame shown in Fig. 2.9(a), the partial derivative formulae (2-163) can be rewritten as

$$D_1 = -\frac{\partial \Pi_\text{m}}{\partial X_1}, \quad F_2 = \frac{\partial \Pi_\text{m}}{\partial \Delta_2} \tag{2-164}$$

These two expressions can be derived by the virtual force equation and the virtual displacement equation, respectively.

Firstly, we will deduce the first expression of Eq. (2-164). As shown in Fig. 2.9(b), in order to solve the displacement D_1, a virtual force system is established: a virtual force increment δX_1 is assumed at point A, then the bending moment increment of the region a is $(\delta M)_a = \frac{\partial(M)_a}{\partial X_1}\delta X_1$, and the constrained moment increment at interface J is $(\delta M_J)_a = \frac{\partial(M_J)_a}{\partial X_1}\delta X_1$. Let the virtual force system of the region a in Fig. 2.9(b) do virtual work on the deformation state in Fig. 2.9(a), the virtual force equation is

$$(\delta X_1)D_1 + (\delta M_J)_a \varDelta_2 = \sum_a \int (\delta M)_a \frac{(M)_a}{EI}\mathrm{d}s \tag{2-165}$$

Then we have

$$D_1 = \sum_a \int \frac{1}{EI}(M)_a \frac{\partial(M)_a}{\partial X_1}\mathrm{d}s - \frac{\partial(M_J)_a}{\partial X_1}\varDelta_2 \tag{2-166}$$

By using Eq. (2-162), the above equation can be rewritten as

$$D_1 = -\frac{\partial \varPi_{\mathrm{m}}}{\partial X_1}$$

Thereby, the first expression of Eq. (2-164) has been derived.

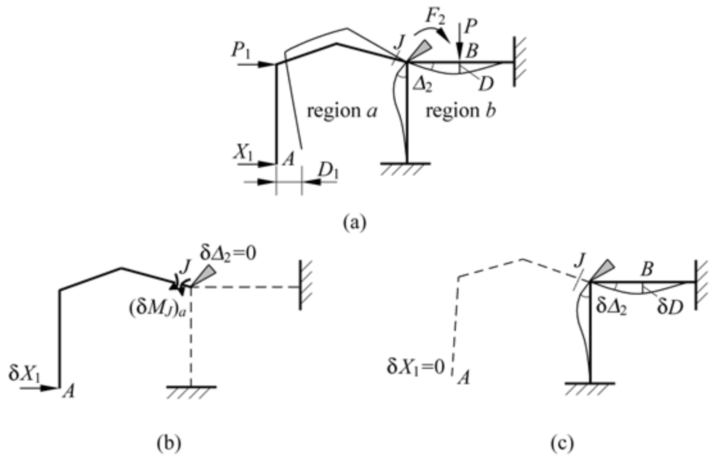

Figure 2.9 A frame divided into two regions

Secondly, we will deduce the second expression of Eq. (2-164). As shown in Fig. 2.9(c), in order to solve the constrained moment F_2, a virtual displacement system is established: a virtual displacement increment $\delta\varDelta_2$ is assumed at point J, then the displacement increment at the point B where the load P acts is $\delta D = \frac{\partial D}{\partial \varDelta_2}\delta\varDelta_2$, the moment increment of the region b is $\delta(M)_b = \frac{\partial(M)_b}{\partial \varDelta_2}\delta\varDelta_2$. Let the force system of the region b (including the interface J) in Fig. 2.9(a) do the virtual work on the virtual displacements of the region b in Fig. 2.9(c), the virtual displacement equation is

$$[F_2 - (M_J)_a]\delta\Delta_2 + \sum_b (P\delta D)_b = \sum_b \int (M)_b \frac{(\delta M)_b}{EI} \mathrm{d}s \tag{2-167}$$

Then we have

$$F_2 = \sum_b \int \frac{1}{EI}(M)_b \frac{\partial (M)_b}{\partial \Delta_2} \mathrm{d}s - \sum_b P \frac{\partial (D)_b}{\partial \Delta_2} + (M_J)_a \tag{2-168}$$

By using Eq. (2-162), the above equation can be rewritten as

$$F_2 = \frac{\partial \Pi_{\mathrm{m}}}{\partial \Delta_2}$$

Thereby, the second expression of Eq. (2-164) has also been derived.

2.6.2 Three Deductions of the Sub-Region Mixed Energy Partial Derivative Theorem

1. The sub-region mixed energy stationary principle

Let us analyze the frame plotted in Fig. 2.10 by using the sub-region mixed energy method. Node J is the interface, and the region on the left side of the node J is the complementary energy region. Then, according to the force method, the reaction force X_1 along the horizontal bar at point A is taken as the fundamental unknown variable. The region on the right side of the node J is the potential energy region. Then, according to the displacement method, the angular displacement Δ_2 at the node J is taken as the fundamental unknown variable.

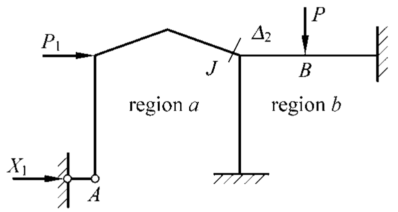

Figure 2.10 A frame

The fundamental system is shown in Fig. 2.9(a): in region a, the horizontal bar at point A is eliminated and replaced by the force variable X_1; and in region b, an additional constraint is added at the node J, and the node rotation is made as the displacement variable Δ_2. The sub-region mixed energy Π_m of the fundamental system is given by Eq. (2-162), and the displacement D_1 corresponding to X_1 and the constrained moment F_2 corresponding to Δ_2 are given by Eq. (2-164).

The original structure in Fig. 2.10 should satisfy the following fundamental equation

$$D_1 = 0, \quad F_2 = 0 \tag{2-169}$$

Substitution of the above equation into Eq. (2-164) yields

$$\frac{\partial \Pi_m}{\partial X_1} = 0, \quad \frac{\partial \Pi_m}{\partial \Delta_2} = 0 \tag{2-170}$$

The above equation is the stationary conditions of the sub-region mixed energy Π_m. Thereby, the sub-region mixed energy stationary principle can be derived from the sub-region mixed energy partial derivative theorem. Furthermore, the sub-region potential energy principle and the sub-region complementary energy principle are the special cases of the sub-region mixed energy principle.

2. The potential energy partial derivative theorem and related approach, principle and theorem

If the whole structure is looked upon as the potential energy region and no complementary energy region existing, the sub-region mixed energy Π_m will degenerate to the potential energy Π_p of the whole region, and the sub-region mixed energy partial derivative formulae (2-163) will degenerate to the potential energy partial derivative formulae:

$$F_i = \frac{\partial \Pi_p}{\partial \Delta_i} \quad (i = 1, 2, \cdots, n) \tag{2-171}$$

This is the mathematical expression of the potential energy partial derivative theorem. And, the theorem can be stated as follows: A structure has n support displacements $\Delta_i (i = 1, 2, \cdots, n)$ treated as the independent displacement variables, other support displacements and loads are all specified by the given values, and the potential energy Π_p of the structure is expressed as a function of $\Delta_1, \Delta_2, \cdots, \Delta_n$, then the partial derivative of the potential energy Π_p with respect to the displacement variable Δ_i will be equal to the constrained force F_i corresponding to Δ_i.

There are some other deductions which can also be obtained from the potential energy partial derivative theorem.

(1) Both the potential energy partial derivative theorem and the unit support displacement method are the approaches for solving the support reaction force F_i, and they have a close relation. Their differences are as follows: the unit support displacement method possesses a broader application range, and does not involve

physical conditions; the application range of the potential energy partial derivative theorem is relatively narrow, only suitable for elastic structures, but its formulae are quite simple and convenient.

(2) If the constrained force F_i corresponding to the displacement variable Δ_i does not exist, equation (2-171) will degenerate to:

$$\frac{\partial \Pi_{\mathrm{p}}}{\partial \Delta_i} = 0 \quad (i = 1, 2, \cdots, n) \tag{2-172}$$

This is the potential energy stationary condition. So, the potential energy stationary principle can also be derived from the potential energy partial derivative theorem.

(3) If there is no other load in the structure except for the displacement variable Δ_i and its constrained force $F_i (i = 1, 2, \cdots, n)$, the potential energy Π_{p} will be equal to the strain energy U, and Eq. (2-171) will be simplified as:

$$F_i = \frac{\partial U}{\partial \Delta_i} \tag{2-173}$$

This is the Castigliano first theorem.

3. The complementary energy partial derivative theorem and related approach, principle and theorem

If the whole structure is looked upon as the complementary energy region and no potential energy region existing, the sub-region mixed energy Π_{m} will degenerate to the minus value of the complementary energy of the whole region, i.e. $(-\Pi_{\mathrm{c}})$. Eq. (2-163) will degenerate to:

$$D_i = \frac{\partial \Pi_{\mathrm{c}}}{\partial X_i} \quad (i = 1, 2, \cdots, n) \tag{2-174}$$

This is the mathematical expression of the complementary energy partial derivative theorem. And, the theorem can be stated as follows: A structure has n independent variable loads or independent force variables $X_i (i = 1, 2, \cdots, n)$, other loads and support displacements are all specified by the given values, and the complementary energy Π_{c} of the structure is expressed as a function of $X_1, X_2, \cdots, X_n$, then the partial derivative of the complementary energy Π_{c} with respect to the displacement variable X_i will be equal to the displacement D_i corresponding to X_i.

There are some other deductions which can also be obtained from the complementary energy partial derivative theorem.

(1) Both the complementary energy partial derivative theorem and the unit load method are the approaches for solving the displacement D_i, and they have a close relation. Their differences are as follows: the unit load method possesses a broader application range, and does not involve physical conditions; the application range

of the complementary energy partial derivative theorem is relatively narrow, only suitable for elastic structures, but its formulae are quite simple and convenient.

(2) If the force variables $X_1, X_2, \cdots, X_n$ are all redundant constrained forces of the statically indeterminate structure, and their corresponding displacements $D_1, D_2, \cdots, D_n$ are all zero, then Eq. (2-174) will degenerate to:

$$\frac{\partial \Pi_{\mathrm{c}}}{\partial X_i} = 0 \quad (i = 1, 2, \cdots, n) \tag{2-175}$$

This is the complementary energy stationary conditions. So, the complementary energy stationary principle can also be derived from the complementary energy partial derivative theorem.

(3) If the support displacements of the structure are zero, then, the complementary energy Π_{c} will be equal to the strain complementary energy V, and Eq. (2-174) will be simplified as

$$D_i = \frac{\partial V}{\partial X_i} \quad (i = 1, 2, \cdots, n) \tag{2-176}$$

This is the Crotti-Engesser theorem.

(4) If the structure is linear elastic, and has no initial strain, then the strain complementary energy V and the strain energy U are equal to each other, and Eq. (2-176) can be written as

$$D_i = \frac{\partial U}{\partial X_i} \quad (i = 1, 2, \cdots, n) \tag{2-177}$$

This is the Castigliano second theorem.

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