Chapter 16 Quadrilateral Area Coordinate Systems, Part Ⅰ—Theory and Formulae

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Abstract This chapter introduces new concepts for developing the quadrilateral finite element models. Firstly, the quadrilateral area coordinate system (QACM-Ⅰ) with four coordinate components, which is a generalization of the triangular area coordinate method, is systematically established in detail. Then, on the basis of the QACM-Ⅰ, another quadrilateral area coordinate system (QACM-Ⅱ) with only two coordinate components is also proposed. These new coordinate systems provide the theoretical bases for the construction of new quadrilateral element models insensitive to mesh distortion, which will be introduced in Chap. 17.

Keywords quadrilateral element, quadrilateral area coordinate system, coordinate components, QACM-Ⅰ, QACM-Ⅱ.

16.1 Introduction

In comparison with the rectangular and triangular elements, the quadrilateral element possesses more flexibility and can be used to model multiform shapes, but its construction procedure is more complicated. The invention of the isoparametric coordinates greatly promotes the development of the quadrilateral elements. At present, the isoparametric coordinate method almost occupies a dominant position in the construction of the quadrilateral elements. But, some disadvantages still exist in this system:

(1) The relation equations (inverse transformations) in which the isoparametric coordinates (ξ, η) are expressed in terms of the Cartesian coordinates (x, y) are too complicated to use.

(2) The stiffness matrix of the isoparametric element generally cannot be evaluated exactly by numerical integration.

(3) When the shape of an element is distorted, the accuracy of the Serendipity isoparametric element will drop obviously.

How to overcome the above shortcomings of the isoparametric element is an interesting topic which attracted many researchers in the finite element method area for a long time, and is also the background of the developments of the quadrilateral area coordinate method and related quadrilateral elements. It can be seen from this chapter and Chap. 17 that, these new methods are effective tools for eliminating the above disadvantages.

Related new developments were proposed in 1997. The first papers about the QACM-Ⅰ are references [1 – 6], and the first papers about the QACM-Ⅱ are references [7,8].

This chapter will introduce the systematic theories of the quadrilateral area coordinates:

(1) Characteristic parameters for the quadrilateral elements are defined and the degeneration conditions under which a quadrilateral degenerates into a parallelogram (including rectangle) or a trapezoid or a triangle are given;

(2) For the QACM-Ⅰ, the area coordinates of any point in a quadrilateral are defined, and transformation relations between the area coordinates and the Cartesian or isoparametric coordinates are presented;

(3) For the QACM-Ⅰ, two identical equations, which the four area coordinate components should satisfy, are given and proved;

(4) Another quadrilateral area coordinate system with only two components (QACM-Ⅱ) is defined and the transformation relations between the QACM-Ⅱ and the Cartesian or isoparametric coordinates are presented;

(5) Related differential and integral formulae are given and proved.

This chapter provides a theoretical basis for the construction of new quadrilateral elements in the next chapter. By the combination of the quadrilateral area and the isoparametric coordinates, excellent elements with curved sides can also be derived.

16.2 The Isoparametric Coordinate Method and the Area Coordinate Method

The isoparametric coordinate method[9,10] has been successfully applied in the construction of the quadrilateral elements. The coordinate transformations between the isoparametric coordinates (ξ, η) and the Cartesian coordinates (x, y) are

$$\left.\begin{aligned} x &= \frac{1}{4}\sum_{i=1}^{4} x_i(1+\xi_i\xi)(1+\eta_i\eta) \\ y &= \frac{1}{4}\sum_{i=1}^{4} y_i(1+\xi_i\xi)(1+\eta_i\eta) \end{aligned}\right\} \tag{16-1}$$

where (x_i, y_i) and (ξ_i, η_i) are the coordinates of node i, respectively. Although the isoparametric coordinates have been broadly applied, there are still some disadvantages which have been mentioned above. Here, some explanations are given:

(1) The inverse transformation of (16-1) is too complicated to use. Reference [11] divided the quadrilateral elements into 6 cases, and derived the corresponding inverse transformation, respectively:

$$\left.\begin{aligned} \xi &= F_1(x, y) \\ \eta &= F_2(x, y) \end{aligned}\right\} \tag{16-2}$$

but $F_1(x, y)$ and $F_2(x, y)$ cannot be expressed by a polynomial in finite terms except for the degenerate case of a parallelogram.

(2) In general, the stiffness matrix of the quadrilateral element constructed by the isoparametric coordinates has to be evaluated by numerical integration instead of exact integration, which leads to the loss of accuracy.

(3) Serendipity isoparametric elements are very sensitive to mesh distortion. Although the shape functions of these elements contain high-order terms of ξ and η, they have only first-order completeness in the Cartesian coordinates x and y. Detailed discussions of this can be found in reference [12], and some convincing numerical examples are also given.

On the other hand, the area coordinate method[13,14] has been successfully applied in the construction of triangular elements. The coordinate transformation between the triangular area coordinates (L_1, L_2, L_3) and Cartesian coordinates (x, y) is

$$L_i = \frac{1}{2A}(a_i + b_i x + c_i y) \quad (i = 1, 2, 3) \tag{16-3}$$

where a_i, b_i, c_i are constants determined by the nodal coordinates (x_i, y_i), e.g.

$$a_1 = x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2 \tag{16-4}$$

There are only two independent coordinates in L_1, L_2, L_3 which should satisfy the identical equation

$$L_1 + L_2 + L_3 = 1 \tag{16-5}$$

The advantages of the triangular area co-ordinate method are:

(1) The area coordinate L_i is natural and invariant, that is, when the Cartesian axes rotate, the area coordinate L_i of the given point is invariant.

(2) The equation of the element boundary line is $L_i = 0$, and therefore the boundary condition is easy to express and be satisfied.

(3) The inverse transformation of (16-3) is

$$x = \sum_{i=1}^{3} L_i x_i, \quad y = \sum_{i=1}^{3} L_i y_i \tag{16-6}$$

which is a linear relation, and vice versa.

(4) The stiffness matrix of the triangular element constructed by the area coordinate method can be easily formulated with exact integration instead of numerical integration.

In this chapter, the traditional area coordinate method, which is an efficient tool in the construction of triangular elements, is generalized to formulate the quadrilateral elements. Two characteristic parameters for quadrilateral elements and the general theory of the area coordinates for quadrilateral elements are presented. Thus, it offers a new way to formulate the quadrilateral elements.

16.3 Two Shape Characteristic Parameters of a Quadrilateral

16.3.1 The Definition of Two Shape Characteristic Parameters of a Quadrilateral

Quadrilaterals have various shapes. In references [1] and [3], two dimensionless parameters g_1 and g_2 are defined as the shape characteristic parameters of a quadrilateral (Fig. 16.1(a),(b)):

$$g_1 = \frac{A(\triangle 124)}{A}, \quad g_2 = \frac{A(\triangle 123)}{A} \tag{16-7}$$

where A is the quadrilateral area, and $A(\triangle 124)$ and $A(\triangle 123)$ are triangular areas of $\triangle 124$ and $\triangle 123$, respectively. If we set $g_3 = 1 - g_1$, $g_4 = 1 - g_2$, then from Fig. 16.1 we see that two triangular areas on both sides of the diagonal 24 are $g_1 A$ and $g_3 A$, respectively; while two triangular areas on both sides of the diagonal 13 are $g_2 A$ and $g_4 A$, respectively.

For a convex quadrilateral, the ranges of parameters g_1 and g_2 are taken as

$$0 < g_1 < 1, \quad 0 < g_2 < 1 \tag{16-8}$$

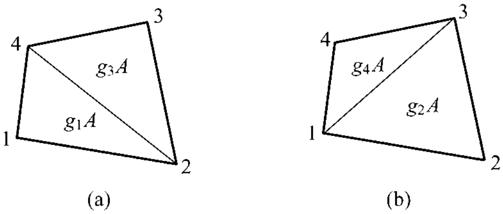

Figure 16.1 Definition of g_1, g_2, g_3 and g_4

from which we have

$$0 < g_3 < 1, \quad 0 < g_4 < 1$$

16.3.2 The Characteristic Conditions for a Quadrilateral to Degenerate into a Parallelogram or a Trapezoid or a Triangle

Special case 1 The characteristic condition under which a quadrilateral degenerates into a parallelogram (including rectangle) is

$$g_1 = g_2 = \frac{1}{2} \tag{16-9}$$

from which we have

$$g_3 = g_4 = \frac{1}{2}$$

Special case 2 The characteristic condition under which a quadrilateral degenerates into a trapezoid is

$$(g_1 - g_2)(g_2 - g_3) = 0 \tag{16-10}$$

which can be resolved into two sub-conditions as follows:

$$g_1 - g_2 = 0 \tag{16-11a}$$

or

$$g_2 - g_3 = 0 \tag{16-11b}$$

Only one of these two sub-conditions needs to be satisfied. They are in correspondence with two degeneration cases respectively as follows:

Condition (16-11a) corresponds to a trapezoid of kind A ($\overline{12} // \overline{34}$, Fig. 16.2(a)).

Condition (16-11b) corresponds to a trapezoid of kind B ($\overline{23} // \overline{41}$, Fig. 16.2(b)).

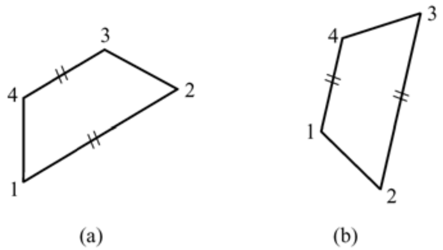

Figure 16.2 Two kinds of trapezoids

(a) Trapezoid of kind $A(\overline{12}//\overline{34})\ g_1-g_2=g_4-g_3=0$; (b) Trapezoid of kind $B(\overline{23}//\overline{41})\ g_2-g_3=g_1-g_4=0$

If both sub-conditions (16-11a,b) are satisfied simultaneously, then the condition (16-9) should be satisfied, which means it degenerates into a parallelogram.

Special case 3 The characteristic condition under which a quadrilateral degenerates into a triangle is

$$g_1 g_2 g_3 g_4 = 0 \tag{16-12}$$

which can be resolved into four sub-conditions as follows:

$$g_1 = 0 \tag{16-13a}$$

or

$$g_2 = 0 \tag{16-13b}$$

or

$$g_3 = 0 \tag{16-13c}$$

or

$$g_4 = 0 \tag{16-13d}$$

If any of the sub-conditions (16-13a,b,c,d) is satisfied, then some three adjacent nodes of the quadrilateral are in line with each other, so the quadrilateral degenerates into one of the four kinds of triangles in Fig. 16.3.

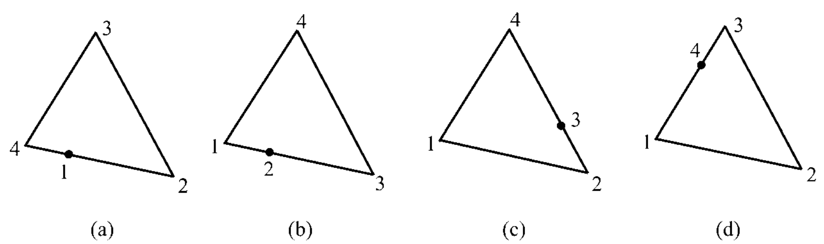

Figure 16.3 Degeneration into four kinds of triangles (some three nodes in line).

(a) $g_1=0$ (4,1,2 in line); (b) $g_2=0$ (1,2,3 in line); (c) $g_3=0$ (2,3,4 in line);

(d) $g_4=0$ (3,4,1 in line)

If any two adjacent sub-conditions (16-13a,b,c,d) are satisfied simultaneously, then some two adjacent nodes are in coincidence with each other, so the quadrilateral degenerates into one of the four kinds of triangles in Fig. 16.4.

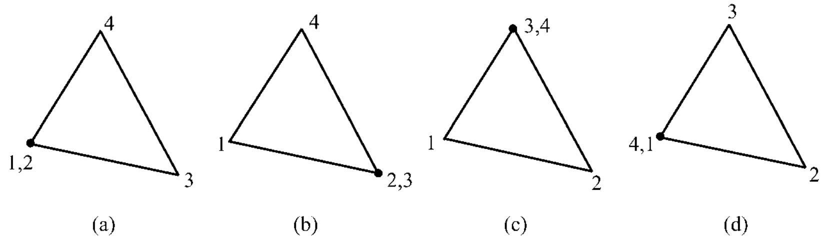

Figure 16.4 Degeneration into four kinds of triangles (some two adjacent nodes in coincidence).

(a) $g_1 = g_2 = 0$ (1,2 in coincidence); (b) $g_2 = g_3 = 0$ (2,3 in coincidence);

(c) $g_3 = g_4 = 0$ (3,4 in coincidence); (d) $g_4 = g_1 = 0$ (4,1 in coincidence)

16.3.3 Two Identical Relations among Nodal Cartesian Coordinates and Parameters g_1 and g_2

The Cartesian coordinates of node i of a quadrilateral are denoted by (x_i, y_i), $i = 1$, 2, 3, 4. There are two identical relations among (x_i, y_i) and (g_1, g_2) as follows:

$$(1-g_1)\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} - (1-g_2)\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + g_1\begin{Bmatrix} x_3 \\ y_3 \end{Bmatrix} - g_2\begin{Bmatrix} x_4 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{16-14}$$

which can also be written as

$$g_3\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} - g_4\begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + g_1\begin{Bmatrix} x_3 \\ y_3 \end{Bmatrix} - g_2\begin{Bmatrix} x_4 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Proof Point 5, for which the coordinates are assumed to be (x_5, y_5), is the intersection point of two quadrilateral diagonals 13 and 24 (Fig. 16.5).

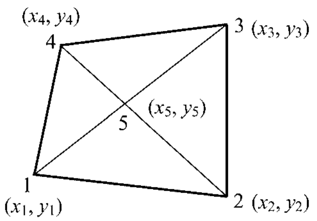

Figure 16.5 Intersection point 5 of diagonals

Firstly, from diagonal 153, we obtain

$$\begin{Bmatrix} x_5 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} + g_1 \begin{Bmatrix} x_3 - x_1 \\ y_3 - y_1 \end{Bmatrix} \tag{16-15}$$

Secondly, from diagonal 254, we obtain

$$\begin{Bmatrix} x_5 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + g_2 \begin{Bmatrix} x_4 - x_2 \\ y_4 - y_2 \end{Bmatrix} \tag{16-16}$$

From the above equations, Eq. (16-14) can be obtained. □

16.3.4 The Quadrilateral Determined by Its Base Triangle and Parameters g_1 and g_2

For convenience, we only discuss the case of a convex quadrilateral.

Take the triangle △123 as the base triangle of the quadrilateral (Fig. 16.1(b)). If the base triangle △123 is given, then the coordinate (x_4, y_4) of the fourth node can be determined by g_1 and g_2 as follows:

$$g_2 \begin{Bmatrix} x_4 \\ y_4 \end{Bmatrix} = (1 - g_1) \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} - (1 - g_2) \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} + g_1 \begin{Bmatrix} x_3 \\ y_3 \end{Bmatrix} \tag{16-17}$$

In fact, Eq. (16-17) can be obtained from Eq. (16-14).

Thus, we can draw a conclusion that a quadrilateral can be determined by the parameters g_1 and g_2 and its base triangle. In other words, if the base triangle is given, the corresponding quadrilateral varies with g_1 and g_2. Otherwise, if g_1 and g_2 are given, it varies with the base triangle.

It can be proved that the above conclusion still holds if the quadrilateral degenerates into a triangle.

16.4 The Definition of Quadrilateral Area Coordinates (QACM-Ⅰ)

16.4.1 Definition

In a quadrilateral, the area coordinates (L_1, L_2, L_3, L_4) of any point P are defined as

$$L_i = \frac{A_i}{A} \quad (i = 1, 2, 3, 4) \tag{16-18}$$

where A_1, A_2, A_3, A_4 are the areas of the four triangles formed by point P and two adjacent vertices in the quadrilateral element, respectively (Fig. 16.6).

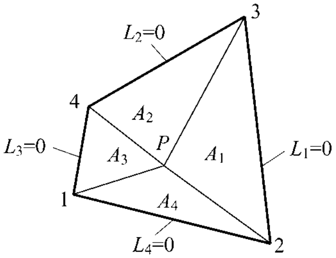

Figure 16.6 The definition of quadrilateral area coordinates

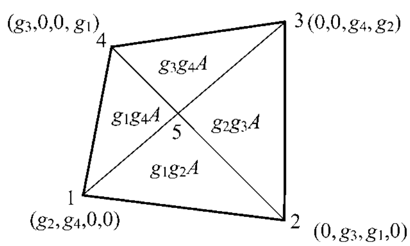

Figure 16.7 Nodal coordinates

Obviously, the equation of every side in a quadrilateral element is

$$L_i = 0 \quad (i = 1,2,3,4) \tag{16-19}$$

The area co-ordinates of the four nodes are (Fig. 16.7):

$$\left.\begin{array}{ll} \text{node 1} & (g_2, g_4, 0, 0) \\ \text{node 2} & (0, g_3, g_1, 0) \\ \text{node 3} & (0, 0, g_4, g_2) \\ \text{node 4} & (g_3, 0, 0, g_1) \end{array}\right\} \tag{16-20}$$

The area coordinates of the intersection point 5 of diagonal lines are (Fig. 16.7)

$$(g_2g_3, \quad g_3g_4, \quad g_4g_1, \quad g_1g_2) \tag{16-21}$$

16.4.2 Area Coordinates Expressed by Cartesian Coordinates

The triangle area A_1, A_2, A_3, A_4 in Fig. 16.6 can be obtained by the determinants:

$$\left.\begin{array}{ll} A_1 = \dfrac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, & A_2 = \dfrac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix} \\ A_3 = \dfrac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_4 & y_4 \\ 1 & x_1 & y_1 \end{vmatrix}, & A_4 = \dfrac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} \end{array}\right\} \tag{16-22}$$

Substituting Eq. (16-22) into Eq. (16-18), we obtain the transformation formula from the Cartesian coordinates to the area coordinates:

$$L_i = \frac{1}{2A}(a_i + b_i x + c_i y) \quad (i = 1,2,3,4) \tag{16-23}$$

in which

$$\begin{gathered} a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j \\ (i = \overleftrightarrow{1,2,3,4}; \quad j = \overleftrightarrow{2,3,4,1}; \quad k = \overleftrightarrow{3,4,1,2}) \end{gathered} \tag{16-24}$$

The transformation formula (16-23) is linear and similar to formula (16-3).

16.4.3 Two Sets of Equalities about a_i, b_i, c_i

About a_i, b_i and c_i the following two sets of equalities can be obtained:

$$\begin{Bmatrix} \sum_{i=1}^{4} a_i \\ \sum_{i=1}^{4} b_i \\ \sum_{i=1}^{4} c_i \end{Bmatrix} = \begin{Bmatrix} 2A \\ 0 \\ 0 \end{Bmatrix} \tag{16-25}$$

$$g_4 g_1 \begin{Bmatrix} a_1 \\ b_1 \\ c_1 \end{Bmatrix} - g_1 g_2 \begin{Bmatrix} a_2 \\ b_2 \\ c_2 \end{Bmatrix} + g_2 g_3 \begin{Bmatrix} a_3 \\ b_3 \\ c_3 \end{Bmatrix} - g_3 g_4 \begin{Bmatrix} a_4 \\ b_4 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16-26}$$

The proof of formula (16-25) is much easier. The first and second formulas of Eq. (16-26) are proved as follows.

The proof of the second formula of Eq. (16-26):

$$\begin{aligned} \text{LHS} &= g_4 g_1 b_1 - g_1 g_2 b_2 + g_2 g_3 b_3 - g_3 g_4 b_4 \\ &= g_4 g_1 (y_2 - y_3) - g_1 g_2 (y_3 - y_4) + g_2 g_3 (y_4 - y_1) - g_3 g_4 (y_1 - y_2) \\ &= -y_1 g_3 + y_2 g_4 - y_3 g_1 + y_4 g_2 \\ &= 0 \end{aligned}$$

□

Equation (16-14) is quoted in the last step of the above proof, and the proof of the third formula of Eq. (16-26) is similar to this.

The proof of the first formula of Eq. (16-26):

$$\begin{aligned}\text{LHS} &= g_4 g_1 a_1 - g_1 g_2 a_2 + g_2 g_3 a_3 - g_3 g_4 a_4 \\ &= g_4 g_1 (x_2 y_3 - x_3 y_2) - g_1 g_2 (x_3 y_4 - x_4 y_3) + g_2 g_3 (x_4 y_1 - x_1 y_4) - g_3 g_4 (x_1 y_2 - x_2 y_1) \\ &= -x_1 g_3 (g_2 y_4 + g_4 y_2) + x_2 g_4 (g_3 y_1 + g_1 y_3) - x_3 g_1 (g_4 y_2 + g_2 y_4) + x_4 g_2 (g_1 y_3 + g_3 y_1) \\ &= (g_2 y_4 + g_4 y_2)(-x_1 g_3 + x_2 g_4 - x_3 g_1 + x_4 g_2) \\ &= 0 \qquad \square\end{aligned}$$

Equation (16-14) is quoted twice in the last two steps of the above proof.

16.4.4 g_1 and g_2 Expressed by b_i and c_i

From Fig. 16.1(a), we have

$$2Ag_1 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 0 & c_4 & -b_4 \\ 0 & -c_3 & b_3 \end{vmatrix} = b_3 c_4 - b_4 c_3$$

With cyclic permutation of $g_1 \to g_2 \to g_3 \to g_4 \to g_1$, we obtain the following formulae:

$$\left.\begin{aligned} 2Ag_1 &= b_3 c_4 - b_4 c_3 \\ 2Ag_2 &= b_4 c_1 - b_1 c_4 \\ 2Ag_3 &= b_1 c_2 - b_2 c_1 \\ 2Ag_4 &= b_2 c_3 - b_3 c_2 \end{aligned}\right\} \tag{16-27}$$

which can be used to obtain

$$\left.\begin{aligned} 2A(g_3 - g_2) &= b_3 c_1 - b_1 c_3 \\ 2A(g_2 - g_1) &= b_2 c_4 - b_4 c_2 \\ 2A &= (b_3 c_4 - b_4 c_3) + (b_1 c_2 - b_2 c_1) = (b_4 c_1 - b_1 c_4) + (b_2 c_3 - b_3 c_2) \\ &= \frac{1}{2}[(b_1 - b_3)(c_2 - c_4) - (b_2 - b_4)(c_1 - c_3)] \end{aligned}\right\} \tag{16-28}$$

16.5 Two Identical Relations Among Area Coordinates (QACM-Ⅰ)

16.5.1 Two Identical Relations Satisfied by the Area Co-Ordinates (QACM-Ⅰ)

An arbitrary point in a quadrilateral has two DOFs. Obviously, in the four area co-ordinates, only two are independent. Namely, there are two identical relations which the four area co-ordinates should satisfy. They are firstly presented and proved in reference [1]:

$$L_1 + L_2 + L_3 + L_4 = 1 \tag{16-29}$$

$$g_4 g_1 L_1 - g_1 g_2 L_2 + g_2 g_3 L_3 - g_3 g_4 L_4 = 0 \tag{16-30}$$

These two identical relations (16-29) and (16-30), are the fundamental part in the complete definition of the quadrilateral area coordinates. In other words, Eq. (16-18), together with the relations (16-29) and (16-30), constitutes a complete definition of the quadrilateral area coordinate system.

Equation (16-29) holds, obviously, and Eq. (16-30) can be proved as follows.

The proof of Eq. (16-30):

By quoting Eqs. (16-23) and (16-26), Eq. (16-30) can be proved as follows:

$$\begin{aligned}
\text{LHS} &= g_4 g_1 L_1 - g_1 g_2 L_2 + g_2 g_3 L_3 - g_3 g_4 L_4 \\
&= \frac{1}{2A}[g_4 g_1 (a_1 + b_1 x + c_1 y) - g_1 g_2 (a_2 + b_2 x + c_2 y) + g_2 g_3 (a_3 + b_3 x + c_3 y) \\
&\quad - g_3 g_4 (a_4 + b_4 x + c_4 y)] \\
&= \frac{1}{2A}\{[g_4 g_1 a_1 - g_1 g_2 a_2 + g_2 g_3 a_3 - g_3 g_4 a_4] + x[g_4 g_1 b_1 - g_1 g_2 b_2 \\
&\quad + g_2 g_3 b_3 - g_3 g_4 b_4] + y[g_4 g_1 c_1 - g_1 g_2 c_2 + g_2 g_3 c_3 - g_3 g_4 c_4]\} \\
&= 0
\end{aligned}$$

□

Though a definition of the quadrilateral area coordinates similar to Eq. (16-18) has been suggested for constructing several simple contact functions in reference [15], only the establishment of the above identical Eq. (16-30) indicates that the QACM-Ⅰ can be treated as a complete system.

16.5.2 Use Independent Area Co-Ordinates to Express Others

There are only two independent area coordinates in (L_1, L_2, L_3, L_4). Independent coordinates can be taken in many ways, but they must be adjacent coordinates, e.g.

$$(L_1, L_2), \quad (L_2, L_3), \quad (L_3, L_4), \quad (L_4, L_1)$$

Otherwise, two opposite coordinates, e.g. (L_1, L_3) or (L_2, L_4), cannot be taken as independent coordinates. From the rectangle case (a special case of quadrilateral) we can easily understand this conclusion.

Take (L_1, L_2) as independent coordinates:

$$\left.\begin{aligned} L_3 &= g_4 - \frac{g_4}{g_3}L_1 + \frac{g_1 - g_4}{g_3}L_2 \\ L_4 &= g_2 + \frac{g_1 - g_2}{g_3}L_1 - \frac{g_2}{g_3}L_2 \end{aligned}\right\} \tag{16-31}$$

Take (L_2, L_3) as independent coordinates:

$$\left.\begin{aligned} L_4 &= g_1 - \frac{g_1}{g_4}L_2 + \frac{g_2 - g_1}{g_4}L_3 \\ L_1 &= g_3 + \frac{g_2 - g_3}{g_4}L_2 - \frac{g_3}{g_4}L_3 \end{aligned}\right\} \tag{16-32}$$

Take (L_3, L_4) as independent coordinates:

$$\left.\begin{aligned} L_1 &= g_2 - \frac{g_2}{g_1}L_3 + \frac{g_3 - g_2}{g_1}L_4 \\ L_2 &= g_4 + \frac{g_3 - g_4}{g_1}L_3 - \frac{g_4}{g_1}L_4 \end{aligned}\right\} \tag{16-33}$$

Take (L_4, L_1) as independent coordinates:

$$\left.\begin{aligned} L_2 &= g_3 - \frac{g_3}{g_2}L_4 + \frac{g_4 - g_3}{g_2}L_1 \\ L_3 &= g_1 + \frac{g_4 - g_1}{g_2}L_4 - \frac{g_1}{g_2}L_1 \end{aligned}\right\} \tag{16-34}$$

16.6 Transformation Relations Between the Area Coordinate System (QACM-Ⅰ) and the Cartesian or Isoparametric Coordinate System

16.6.1 Cartesian Coordinates Expressed by the Area Coordinates (QACM-Ⅰ)

Equation (16-23) is the coordinate transformation formula from Cartesian to area co-ordinates. Now, we derive its inverse formula with which the Cartesian coordinates are expressed by the area coordinates.

Since there are four different ways to take the independent area coordinates: (L_1, L_2), (L_2, L_3), (L_3, L_4), (L_4, L_1), four inverse transformation formulae can be obtained.

Take (L_1, L_2) as independent coordinates:

$$\left.\begin{aligned} x &= \frac{1}{g_3}(c_2L_1 - c_1L_2) + x_3 \\ y &= \frac{1}{g_3}(-b_2L_1 + b_1L_2) + y_3 \end{aligned}\right\} \tag{16-35}$$

Take (L_2, L_3) as independent coordinates:

$$\left.\begin{aligned} x &= \frac{1}{g_4}(c_3L_2 - c_2L_3) + x_4 \\ y &= \frac{1}{g_4}(-b_3L_2 + b_2L_3) + y_4 \end{aligned}\right\} \tag{16-36}$$

Take (L_3, L_4) as independent coordinates:

$$\left.\begin{aligned} x &= \frac{1}{g_1}(c_4L_3 - c_3L_4) + x_1 \\ y &= \frac{1}{g_1}(-b_4L_3 + b_3L_4) + y_1 \end{aligned}\right\} \tag{16-37}$$

Take (L_4, L_1) as independent coordinates:

$$\left.\begin{aligned} x &= \frac{1}{g_2}(c_1L_4 - c_4L_1) + x_2 \\ y &= \frac{1}{g_2}(-b_1L_4 + b_4L_1) + y_2 \end{aligned}\right\} \tag{16-38}$$

16.6.2 Quadrilateral Area Coordinates (QACM-Ⅰ) Expressed by the Isoparametric Coordinates

Area coordinates (L_1, L_2, L_3, L_4) can be expressed by the quadrilateral isoparametric coordinates (ξ, η) as follows:

$$\left.\begin{aligned} L_1 &= \frac{1}{4}(1-\xi)[g_2(1-\eta) + g_3(1+\eta)] \\ L_2 &= \frac{1}{4}(1-\eta)[g_3(1+\xi) + g_4(1-\xi)] \\ L_3 &= \frac{1}{4}(1+\xi)[g_4(1+\eta) + g_1(1-\eta)] \\ L_4 &= \frac{1}{4}(1+\eta)[g_1(1-\xi) + g_2(1+\xi)] \end{aligned}\right\} \tag{16-39}$$

It is known that area co-ordinates L_i ($i=1,2,3,4$) are the linear functions of the Cartesian coordinates (x, y). Assume that Z stands for any arbitrary linear function of (x, y); then it can be expressed by the quadrilateral isoparametric coordinate (ξ, η) as follows:

$$Z=\frac{1}{4}\sum_{i=1}^{4}Z_i(1+\xi_i\xi)(1+\eta_i\eta) \tag{16-40}$$

where Z_i, ξ_i, η_i are the values of Z, ξ, η at node i. According to Eq. (16-40) we can obtain Eq. (16-39). Taking the first formula of Eq. (16-39) as an example, Z stands for L_1. From Eq. (16-20), we get

$$[Z_1 \quad Z_2 \quad Z_3 \quad Z_4]=[g_2 \quad 0 \quad 0 \quad g_3] \tag{16-41}$$

Substituting this into Eq. (16-40), the first formula of Eq. (16-39) can be obtained.

16.7 Differential Formulae (QACM-Ⅰ)

16.7.1 Transformations of Derivatives (QACM-Ⅰ)

In a quadrilateral element, any point P has four area coordinate components (L_1, L_2, L_3, L_4) to which the transformation from the Cartesian coordinates (x, y) is given by Eq. (16-23), i.e.,

$$L_i=\frac{1}{2A}(a_i+b_ix+c_iy) \quad (i=1,2,3,4)$$

From the above equation, the transformation of derivatives of the first and second order in both coordinate systems is presented as follows:

(1) The transformation of the derivatives of the first order:

$$\begin{Bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\end{Bmatrix}=\frac{1}{2A}\begin{bmatrix}b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\end{bmatrix}\partial \tag{16-42}$$

where

$$\partial=\begin{bmatrix}\dfrac{\partial}{\partial L_1} & \dfrac{\partial}{\partial L_2} & \dfrac{\partial}{\partial L_3} & \dfrac{\partial}{\partial L_4}\end{bmatrix}^{\mathrm{T}} \tag{16-43}$$

(2) The transformation of the derivatives of the second order:

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ 2\dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \boldsymbol{T}\partial^2 \tag{16-44}$$

where

$$\partial^2 = \begin{bmatrix} \dfrac{\partial^2}{\partial L_1^2} & \dfrac{\partial^2}{\partial L_2^2} & \dfrac{\partial^2}{\partial L_3^2} & \dfrac{\partial^2}{\partial L_4^2} & \dfrac{\partial^2}{\partial L_1 \partial L_2} & \dfrac{\partial^2}{\partial L_2 \partial L_3} & \dfrac{\partial^2}{\partial L_3 \partial L_4} & \dfrac{\partial^2}{\partial L_4 \partial L_1} & \dfrac{\partial^2}{\partial L_1 \partial L_3} & \dfrac{\partial^2}{\partial L_2 \partial L_4} \end{bmatrix}^{\mathrm{T}} \tag{16-45}$$

$$\boldsymbol{T} = \frac{1}{4A^2}\begin{bmatrix} b_1^2 & b_2^2 & b_3^2 & b_4^2 & 2b_1b_2 & 2b_2b_3 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 & 2c_1c_2 & 2c_2c_3 \\ 2b_1c_1 & 2b_2c_2 & 2b_3c_3 & 2b_4c_4 & 2(b_1c_2+b_2c_1) & 2(b_2c_3+b_3c_2) \end{bmatrix.$$
$$\begin{matrix} 2b_3b_4 & 2b_4b_1 & 2b_1b_3 & 2b_2b_4 \\ 2c_3c_4 & 2c_4c_1 & 2c_1c_3 & 2c_2c_4 \\ 2(b_3c_4+b_4c_3) & 2(b_4c_1+b_1c_4) & 2(b_1c_3+b_3c_1) & 2(b_2c_4+b_4c_2) \end{matrix}\right] \tag{16-46}$$

16.7.2 Normal and Tangential Derivatives (QACM-Ⅰ)

Assume that n_i and s_i stand for unit vectors oriented in the normal and the tangential direction respectively of the side i in a quadrilateral element (Fig. 16.8). The length of each side is

$$d_i = \sqrt{b_i^2 + c_i^2} \quad (i = 1, 2, 3, 4) \tag{16-47}$$

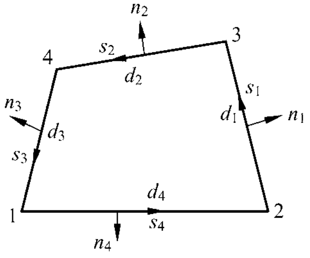

Figure 16.8 Normal and tangential directions of quadrilateral element sides

The direction cosines of the normal vector of each side are:

$$\cos(n_i, x) = -\frac{b_i}{d_i}, \quad \cos(n_i, y) = -\frac{c_i}{d_i} \tag{16-48}$$

The direction cosines of the tangential vector of each side are:

$$\cos(s_i, x) = \frac{c_i}{d_i}, \quad \cos(s_i, y) = -\frac{b_i}{d_i} \tag{16-49}$$

The normal derivative of each side is:

$$\begin{aligned}\frac{\partial}{\partial n_i} &= -\frac{1}{d_i}\left(b_i \frac{\partial}{\partial x} + c_i \frac{\partial}{\partial y}\right) = -\frac{1}{d_i}[b_i \quad c_i]\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \\ &= -\frac{1}{2Ad_i}[b_i \quad c_i]\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}\partial \end{aligned} \tag{16-50}$$

The tangential derivative of each side is

$$\begin{aligned}\frac{\partial}{\partial s_i} &= \frac{1}{d_i}\left(c_i \frac{\partial}{\partial x} - b_i \frac{\partial}{\partial y}\right) = \frac{1}{d_i}[c_i \quad -b_i]\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \\ &= \frac{1}{2Ad_i}[c_i \quad -b_i]\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}\partial \end{aligned} \tag{16-51}$$

16.8 Integral Formulae (QACM-Ⅰ)

16.8.1 The Basic Formulae for Area Integrals (QACM-Ⅰ)

In a quadrilateral element, two equivalent basic integral formulae, which can be applied to evaluate the area integrals for the arbitrary power function $L_1^m L_2^n L_3^p L_4^q$ of the area coordinates, are given as follows:

$$\begin{aligned}\iint_A L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = &\frac{m!n!p!q!}{(m+n+p+q+2)!}2A \\ &\times\left[g_3^{m+n+1}\sum_{k=0}^{q}\sum_{j=0}^{p} \mathrm{C}_{m+q-k}^{m}\mathrm{C}_{n+p-j}^{n}\mathrm{C}_{k+j}^{k} g_2^k g_4^j g_1^{p+q-k-j} \right. \\ &\left. + g_1^{p+q+1}\sum_{k=0}^{m}\sum_{j=0}^{n} \mathrm{C}_{m+q-k}^{q}\mathrm{C}_{n+p-j}^{p}\mathrm{C}_{k+j}^{k} g_2^k g_4^j g_3^{m+n-k-j} \right] \end{aligned} \tag{A}$$

$$\iint_A L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = \frac{m!n!p!q!}{(m+n+p+q+2)!} 2A$$
$$\times \left[g_4^{n+p+1} \sum_{k=0}^{m} \sum_{j=0}^{q} \mathrm{C}_{n+m-k}^{n} \mathrm{C}_{p+q-j}^{p} \mathrm{C}_{k+j}^{k} g_3^k g_1^j g_2^{m+q-k-j} \right.$$
$$\left. + g_2^{m+q+1} \sum_{k=0}^{n} \sum_{j=0}^{p} \mathrm{C}_{m+n-k}^{m} \mathrm{C}_{q+p-j}^{q} \mathrm{C}_{k+j}^{k} g_3^k g_1^j g_4^{n+p-k-j} \right] \tag{B}$$

where
$$\mathrm{C}_k^i = \frac{k!}{(k-i)!i!} \tag{16-52}$$

In fact, with the cyclic permutation that we change (m, n, p, q) to (n, p, q, m) and (g_1, g_2, g_3, g_4) to (g_2, g_3, g_4, g_1) in Eq. (A), Eq. (B) can be obtained.

If a quadrilateral element degenerates into a triangular element, e.g. when $g_1 = g_2 = 0$, nodes 1 and 2 become coincident; thus $L_4 = 0$, $q = 0$, and we substitute them into Eq. (A) or Eq. (B), so that the following famous integral formula for the arbitrary power function of the area coordinates over the triangular element is obtained[14]:

$$\iint_A L_1^m L_2^n L_3^p \mathrm{d}A = \frac{m!n!p!}{(m+n+p+2)!} 2A \tag{16-53}$$

16.8.2 Area Integral Formulae for Lower Power Functions (QACM-Ⅰ)

For convenience in application, we list the area integral formulae for lower power functions (from first order to third order) as follows according to the basic formulae (A) or (B).

(1) The first power terms (four in one group)

$$\iint_A \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \mathrm{d}A = \frac{A}{3} \begin{Bmatrix} 1 - g_4 g_1 \\ 1 - g_1 g_2 \\ 1 - g_2 g_3 \\ 1 - g_3 g_4 \end{Bmatrix} \tag{16-54}$$

(2) The second power terms (ten in three groups)

$$\iint_A \begin{Bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_4^2 \end{Bmatrix} \mathrm{d}A = \frac{A}{6} \begin{Bmatrix} g_3^2 + g_1 g_2 (g_2 + g_3) \\ g_4^2 + g_2 g_3 (g_3 + g_4) \\ g_1^2 + g_3 g_4 (g_4 + g_1) \\ g_2^2 + g_4 g_1 (g_1 + g_2) \end{Bmatrix} \tag{16-55}$$

$$\iint_A \begin{Bmatrix} L_1L_2 \\ L_2L_3 \\ L_3L_4 \\ L_4L_1 \end{Bmatrix} \mathrm{d}A = \frac{A}{12} \begin{Bmatrix} g_3 + 2g_4g_1g_2 \\ g_4 + 2g_1g_2g_3 \\ g_1 + 2g_2g_3g_4 \\ g_2 + 2g_3g_4g_1 \end{Bmatrix} \tag{16-56}$$

$$\iint_A \begin{Bmatrix} L_1L_3 \\ L_2L_4 \end{Bmatrix} \mathrm{d}A = \frac{A}{12} \begin{Bmatrix} g_4 - g_1 + 2g_1g_2 \\ g_1 - g_2 + 2g_2g_3 \end{Bmatrix} \tag{16-57}$$

(3) The third power terms (20 in five groups)

Considering the above integral formulae (16-54) – (16-57), we find that if the formula in the first line of any group is known, then the others in this group can easily be obtained with the cyclic permutation of ($L_1 \to L_2 \to L_3 \to L_4 \to L_1$ and $g_1 \to g_2 \to g_3 \to g_4 \to g_1$). Thus, only the first formula in each group needs to be listed, for brevity:

$$\iint_A L_1^3 \mathrm{d}A = \frac{A}{10}[g_3^3 + g_1g_2(g_2^2 + g_2g_3 + g_3^2)] \tag{16-58}$$

$$\iint_A L_1^2 L_2 \mathrm{d}A = \frac{A}{30}[g_3^2 + 2g_1g_2g_3 + g_2^2g_1(2+g_1) - 3g_2^3g_1] \tag{16-59}$$

$$\iint_A L_1^2 L_3 \mathrm{d}A = \frac{A}{30}[g_3^2 + g_2g_3(1-2g_3) + g_2^2g_1^2] \tag{16-60}$$

$$\iint_A L_1^2 L_4 \mathrm{d}A = \frac{A}{30}[3g_1g_3^2 + g_2g_3(g_3^2 + 2g_1^2) + g_1^2g_2^2] \tag{16-61}$$

$$\iint_A L_1L_2L_3 \mathrm{d}A = \frac{A}{60}[g_4 - g_1 + 3g_1g_2 - 2g_1^2g_2^2] \tag{16-62}$$

16.8.3 The Basic Formulae for Line Integrals (QACM-Ⅰ)

In a quadrilateral element, the following basic formulae can be used to evaluate the line integral for the arbitrary power function of the area coordinates along each side $L_i = 0$ ($i = 1,2,3,4$):

Along side $\overline{12}(L_4 = 0)$

$$\int_0^1 L_1^m L_2^n L_3^p \mathrm{d}\overline{s} = \frac{m!n!p!}{(m+n+p+1)!} g_2^m g_1^p \sum_{k=0}^{n} g_3^{n-k} g_4^k \mathrm{C}_{p+n-k}^{p} \mathrm{C}_{m+k}^{m} \tag{C1}$$

Along side $\overline{23}(L_1 = 0)$

$$\int_0^1 L_2^n L_3^p L_4^q \mathrm{d}\overline{s} = \frac{n!p!q!}{(n+p+q+1)!} g_3^n g_2^q \sum_{k=0}^{p} g_4^{p-k} g_1^k \mathrm{C}_{q+p-k}^q \mathrm{C}_{n+k}^n \tag{C2}$$

Along side $\overline{34}(L_2 = 0)$

$$\int_0^1 L_3^p L_4^q L_1^m \mathrm{d}\overline{s} = \frac{p!q!m!}{(p+q+m+1)!} g_4^p g_3^m \sum_{k=0}^{q} g_1^{q-k} g_2^k \mathrm{C}_{m+q-k}^m \mathrm{C}_{p+k}^p \tag{C3}$$

Along side $\overline{41}(L_3 = 0)$

$$\int_0^1 L_4^q L_1^m L_2^n \mathrm{d}\overline{s} = \frac{q!m!n!}{(q+m+n+1)!} g_1^q g_4^n \sum_{k=0}^{m} g_2^{m-k} g_3^k \mathrm{C}_{n+m-k}^n \mathrm{C}_{q+k}^q \tag{C4}$$

where $\overline{s}$ is a dimensionless coordinate along side $\overline{jk}$, and it is 0 at node j and 1 at node k.

In fact, if the cyclic permutation is inserted in Eq. (C1), then Eqs. (C2), (C3) and (C4) can be obtained.

If quadrilateral elements degenerate into triangular elements, e.g. when $g_1 = g_2 = 0$, nodes 1 and 2 become coincident; thus $L_4 = 0$, $q = 0$, we substitute them into Eqs. (C2), (C3) and (C4), so that the following line integral formulae for the arbitrary power function over the triangular element sides are obtained:

$$\left.\begin{aligned} &\int_0^1 L_2^n L_3^p \mathrm{d}\overline{s} = \frac{n!p!}{(n+p+1)!} \qquad (\text{along } L_1 = 0) \\ &\int_0^1 L_3^p L_1^m \mathrm{d}\overline{s} = \frac{p!m!}{(p+m+1)!} \qquad (\text{along } L_2 = 0) \\ &\int_0^1 L_1^m L_2^n \mathrm{d}\overline{s} = \frac{m!n!}{(m+n+1)!} \qquad (\text{along } L_3 = 0) \end{aligned}\right\} \tag{16-63}$$

16.9 The Proof of the Basic Formulae (A) and (B) (QACM-Ⅰ)

16.9.1 Preparative Formula (D)

$$\int_0^1 (1-t)^i t^j \mathrm{d}t = \frac{i!j!}{(i+j+1)!} \tag{D}$$

Proof Using integration by parts

$$\int_0^1 u\mathrm{d}v = uv\Big|_0^1 - \int_0^1 v\mathrm{d}u \tag{16-64}$$

we have

$$\text{LHS} = \frac{1}{j+1}\int_0^1 (1-t)^i \, \mathrm{d}t^{j+1} = \frac{i}{j+1}\int_0^1 (1-t)^{i-1} t^{j+1} \mathrm{d}t$$

$$= \frac{i!j!}{(i+j)!}\int_0^1 t^{j+i} \mathrm{d}t = \frac{i!j!}{(i+j+1)!} = \text{RHS} \qquad \square$$

16.9.2 Preparative Formula (E)

$$\int_0^1 t^m (1-t)^n [\alpha t + \beta(1-t)]^p \, \mathrm{d}t = \frac{m!n!p!}{(m+n+p+1)!}\sum_{k=0}^{p} \alpha^{p-k}\beta^k \mathrm{C}_{m+p-k}^m \mathrm{C}_{n+k}^n \qquad \text{(E)}$$

Proof From formula (D), we can write that

$$\text{LHS} = \int_0^1 t^m (1-t)^n \left[\sum_{k=0}^{p} \frac{p!}{(p-k)!k!} \alpha^{p-k}\beta^k t^{p-k}(1-t)^k\right] \mathrm{d}t$$

$$\overset{\text{(D)}}{=\!=} \sum_{k=0}^{p} \frac{p!}{(p-k)!k!} \alpha^{p-k}\beta^k \frac{(m+p-k)!(n+k)!}{(m+n+p+1)!}$$

$$= \frac{m!n!p!}{(m+n+p+1)!}\sum_{k=0}^{p} \alpha^{p-k}\beta^k \mathrm{C}_{m+p-k}^m \mathrm{C}_{n+k}^n = \text{RHS} \qquad \square$$

16.9.3 Subdivision of the Quadrilateral

In order to obtain the basic formula (A), the quadrilateral 1234 is subdivided into two triangles: $\triangle 234$ and $\triangle 124$ (Fig. 16.9). Assume that A, A' and A'' represent the areas of the quadrilateral 1234, $\triangle 234$ and $\triangle 124$, respectively, thus

$$A' = (1-g_1)A = g_3 A \qquad (16\text{-}65)$$

$$A'' = g_1 A \qquad (16\text{-}66)$$

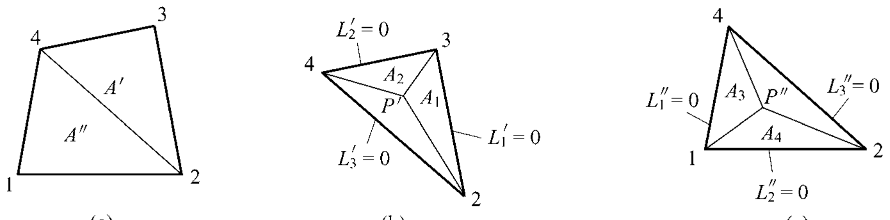

Figure 16.9 Subdivision Ⅰ of a quadrilateral

Since the integral in the basic formula (A) is the integral over the quadrilateral area A (denoted by I), it can be expressed by the sum of the integrals over triangles A' and A'' (denoted by I_1 and I_2).

$$I = \iint_A L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = \iint_{A'} L_1^m L_2^n L_3^p L_4^q \mathrm{d}A + \iint_{A''} L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = I_1 + I_2 \tag{16-67}$$

Consider an arbitrary point P' in △234 (Fig. 16.9(b)). We use two coordinate systems to describe point P': quadrilateral area coordinates (L_1, L_2, L_3, L_4) and triangular area coordinates (L_1', L_2', L_3'). The relation between these two coordinate systems is:

$$L_1 = g_3 L_1' \tag{16-68a}$$

$$L_2 = g_3 L_2' \tag{16-68b}$$

$$L_3 = g_1 L_2' + g_4 L_3' \tag{16-68c}$$

$$L_4 = g_1 L_1' + g_2 L_3' \tag{16-68d}$$

In fact, the first two formulae (16-68a,b) can be obtained from Eq. (16-65):

$$L_1 = \frac{A_1}{A} = A_1\left(\frac{g_3}{A'}\right) = g_3 L_1'$$

$$L_2 = \frac{A_2}{A} = A_2\left(\frac{g_3}{A'}\right) = g_3 L_2'$$

The last two formulae (16-68c,d) can be obtained from (16-31):

$$L_3 = g_4 - \frac{g_4}{g_3} L_1 + \frac{g_1 - g_4}{g_3} L_2 = g_4 - g_4 L_1' + (g_1 - g_4) L_2' = g_1 L_2' + g_4 L_3'$$

$$L_4 = g_2 + \frac{g_1 - g_2}{g_3} L_1 - \frac{g_2}{g_3} L_2 = g_2 + (g_1 - g_2) L_1' - g_2 L_2' = g_1 L_1' + g_2 L_3'$$

Similarly, the relations between these two coordinate systems of any point P'' in △124 are

$$L_3 = g_1 L_1'' \tag{16-69a}$$

$$L_4 = g_1 L_2'' \tag{16-69b}$$

$$L_1 = g_3 L_2'' + g_2 L_3'' \tag{16-69c}$$

$$L_2 = g_3 L_1'' + g_4 L_3'' \tag{16-69d}$$

16.9.4 Area Integral I_1

From the preparative formula (E) and the coordinate transformation (16-68), the area integral I_1 can be evaluated:

$$I_1 = \iint_{A'} L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = \frac{m!n!p!q!}{(m+n+p+q+2)!} 2A$$
$$\times g_3^{m+n+1} \sum_{k=0}^{q} \sum_{j=0}^{p} \mathrm{C}_{m+q-k}^{m} \mathrm{C}_{n+p-j}^{n} \mathrm{C}_{k+j}^{k} g_2^k g_4^j g_1^{p+q-k-j} \qquad (16\text{-}70)$$

Now, we prove it as follows.

First, applying the coordinate transformation (16-68) and the differential area formula $\mathrm{d}A = 2Ag_3 \mathrm{d}L_1' \mathrm{d}L_2'$, we have

$$I_1 = \iint_{A'} g_3^{m+n} L_1'^m L_2'^n (g_1 L_2' + g_4 L_3')^p (g_1 L_1' + g_2 L_3')^q \cdot 2Ag_3 \mathrm{d}L_1' \mathrm{d}L_2' \qquad (16\text{-}71)$$

Second, introducing a new variable $t = \dfrac{L_2'}{1 - L_1'}$, thus

$$I_1 = 2Ag_3^{m+n+1} \int_0^1 \{L_1'^m (1 - L_1')^{n+p+1} \int_0^1 t^n [g_1 t + g_4 (1-t)]^p$$
$$\times [g_1 L_1' + g_2 (1 - L_1')(1-t)]^q \mathrm{d}t\} \mathrm{d}L_1' \qquad (16\text{-}72)$$

Since

$$[g_1 L_1' + g_2 (1 - L_1')(1-t)]^q = \sum_{k=0}^{q} \mathrm{C}_q^k g_1^{q-k} g_2^k L_1'^{q-k} (1 - L_1')^k (1-t)^k \qquad (16\text{-}73)$$

substituting it into (16-72), we obtain

$$I_1 = 2Ag_3^{m+n+1} \sum_{k=0}^{q} \mathrm{C}_q^k g_1^{q-k} g_2^k \left(\int_0^1 L_1'^{m+q-k} (1 - L_1')^{m+p+k+1} \mathrm{d}L_1'\right)\left(\int_0^1 t^n (1-t)^k [g_1 t + g_4 (1-t)]^p \mathrm{d}t\right)$$

in which the two integrals can be evaluated by the preparative formulae (D) and (E), and finally Eq. (16-70) is obtained.

16.9.5 Area Integral I_2 and the Derivation of Formula (A)

Similarly, using the coordinate transformation (16-69), we obtain the area integral I_2 as follows:

$$I_2 = \iint_{A''} L_1^m L_2^n L_3^p L_4^q \mathrm{d}A = \frac{m!n!p!q!}{(m+n+p+q+2)!} 2A$$
$$\times g_1^{p+q+1} \sum_{k=0}^{m} \sum_{j=0}^{n} \mathrm{C}_{m+q-k}^{q} \mathrm{C}_{n+p-j}^{p} \mathrm{C}_{k+j}^{k} g_2^k g_4^j g_1^{m+n-k-j} \qquad (16\text{-}74)$$

Superpose I_1 and I_2 obtained from (16-70) and (16-74), thus we can get the expression of area integral I, that is, the basic formula (A).

16.9.6 The Derivation of Formula (B)

The quadrilateral 1234 can be divided into two triangles: △123 and △134 in the way shown in Fig. 16.10. With similar steps, the basic formula (B) for the area integral can be derived.

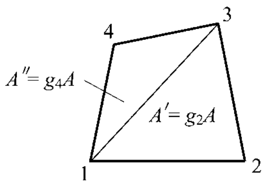

Figure 16.10 Subdivision Ⅱ of a quadrilateral

16.10 The Proof of the Basic Formulae (C) (QACM-Ⅰ)

16.10.1 Variation of Area Co-Ordinates (L_1, L_2, L_3, L_4) along Each Side

Define a dimensionless coordinate $\bar{s}$ along side $\overline{jk}$ in the way that it is 0 at node j and 1 at k. Thus, the area coordinate L_i is a linear function of $\bar{s}$ along each side, which are listed in the following table.

$L_i =$	Along line $\overline{12}$ ($L_4 = 0$)	Along line $\overline{23}$ ($L_1 = 0$)	Along line $\overline{34}$ ($L_2 = 0$)	Along line $\overline{41}$ ($L_3 = 0$)
$L_1 =$	$g_2(1-\bar{s})$	0	$g_3\bar{s}$	$g_2\bar{s}+g_3(1-\bar{s})$
$L_2 =$	$g_3\bar{s}+g_4(1-\bar{s})$	$g_3(1-\bar{s})$	0	$g_4\bar{s}$
$L_3 =$	$g_1\bar{s}$	$g_4\bar{s}+g_1(1-\bar{s})$	$g_4(1-\bar{s})$	0
$L_4 =$	0	$g_2\bar{s}$	$g_1\bar{s}+g_2(1-\bar{s})$	$g_1(1-\bar{s})$

(16-75)

16.10.2 The Proof of Formula (C1)

Along side $\overline{12}(L_4 = 0)$, the area coordinates L_1, L_2, L_3 are linear functions of $\bar{s}$

as listed in the first column of the above table. Substitution into the LHS of (C1) yields

$$\begin{aligned}
\text{LHS} &= g_2^m g_1^p \int_0^1 (1-\overline{s})^m \overline{s}^p [g_3\overline{s} + g_4(1-\overline{s})]^n \mathrm{d}\overline{s} \\
&= g_2^m g_1^p \int_0^1 (1-\overline{s})^m \overline{s}^p [\sum_{k=0}^{n} \mathrm{C}_n^k g_3^{n-k} \overline{s}^{n-k} g_4^k (1-\overline{s})^k]\mathrm{d}\overline{s} \\
&= g_2^m g_1^p \sum_{k=0}^{n} \mathrm{C}_n^k g_3^{n-k} g_4^k \int_0^1 \overline{s}^{p+n-k} (1-\overline{s})^{m+k} \mathrm{d}\overline{s} \\
&\underline{\underline{(\mathrm{D})}} g_2^m g_1^p \sum_{k=0}^{n} \mathrm{C}_n^k g_3^{n-k} g_4^k \frac{(p+n-k)!(m+k)!}{(m+n+p+1)!} = \text{RHS}
\end{aligned}$$

So, the formula (C1) holds. The preparative formula (D) is applied in the above steps. Similarly, the other three formulae of equation (C) can be proved.

16.11 The Quadrilateral Area Coordinate System with Only Two Components (QACM-Ⅱ)

According to the definition of the QACM-Ⅰ, this system contains four area coordinate components (L_1, L_2, L_3, L_4), among which two are independent. The existence of the four components must bring distinct complexity to the construction of an element, for example, users may be confused as to how to formulate a complete high order polynomial. And, most formulations expressed by the QACM-Ⅰ are more complicated than those by the isoparametric coordinates. In view of the shortcomings of the QACM-Ⅰ, another quadrilateral area coordinate method, denoted as QACM-Ⅱ, is systematically established in this section. This new QACM-Ⅱ possesses new physical meanings and contains only two coordinate components. It can not only avoid the defects mentioned above, but also keep the most important advantage of the QACM-Ⅰ, that is, the linear relationship with the Cartesian coordinates.

Recently the third version of quadrilateral area coordinate method (QACM-Ⅲ) was developed in reference [16].

16.11.1 The Definition of the New Quadrilateral Area Coordinates Method (QACM-Ⅱ)

As shown in Fig. 16.11, M_i ($i = 1,2,3,4$) are the mid-side points of element sides $\overline{23}$, $\overline{34}$, $\overline{41}$ and $\overline{12}$, respectively. Thus, the position of an arbitrary point P

within the quadrilateral element $\overline{1234}$ can be uniquely specified by the new two-component area coordinates Z_1 and Z_2 (QACM-Ⅱ), which are defined as:

$$Z_1 = 4\frac{\Omega_1}{A}, \quad Z_2 = 4\frac{\Omega_2}{A} \tag{16-76}$$

where A is still the area of the quadrilateral element; Ω_1 and Ω_2 are the *generalized areas* of $\triangle PM_2M_4$ and $\triangle PM_3M_1$, respectively. It must be noted here that the values of *generalized areas* Ω_1 and Ω_2 can be both positive and negative: for $\triangle PM_2M_4$ (or $\triangle PM_3M_1$), if the permutation order of points P, M_2 and M_4 (or P, M_3 and M_1) is anti-clockwise, a positive Ω_1 (or Ω_2) should be taken; otherwise, Ω_1 (or Ω_2) should be negative.

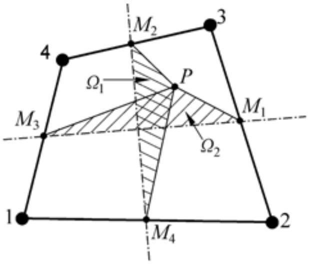

Figure 16.11 Definition of the quadrilateral area coordinates Z_i of QACM-Ⅱ

Though the QACM-Ⅱ (Eq.(16-76)) and the QACM-Ⅰ (Eq. (16-18)) have different physical meanings, it can be proved that they satisfy the following simple linear relations:

$$\left.\begin{aligned} Z_1 &= 2(L_3 - L_1) + (g_2 - g_1) \\ Z_2 &= 2(L_4 - L_2) + (g_3 - g_2) \end{aligned}\right\} \tag{16-77}$$

where g_i (i = 1,2,3,4) are the shape parameters of the quadrangle and given in Sect. 16.2.

Proof Let (x_i, y_i) (i = 1,2,3,4) be the Cartesian coordinates of the four corner nodes 1, 2, 3 and 4, respectively; and (x, y) be the Cartesian coordinates of an arbitrary point P within the element. And, from Eq. (16-28), we have

$$\left.\begin{aligned} 2A(g_2 - g_1) &= -b_4c_2 + b_2c_4 \\ 2A(g_3 - g_2) &= b_3c_1 - b_1c_3 \end{aligned}\right\} \tag{16-78}$$

where b_i and c_i (i = 1,2,3,4) are given by Eq. (16-24). Then, the *generalized areas* of $\triangle PM_2M_4$ and $\triangle PM_3M_1$ in Fig. 16.11 can be obtained as:

$$\Omega_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & \dfrac{x_3 + x_4}{2} & \dfrac{y_3 + y_4}{2} \\ 1 & \dfrac{x_1 + x_2}{2} & \dfrac{y_1 + y_2}{2} \end{vmatrix}, \quad \Omega_2 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & \dfrac{x_4 + x_1}{2} & \dfrac{y_4 + y_1}{2} \\ 1 & \dfrac{x_2 + x_3}{2} & \dfrac{y_2 + y_3}{2} \end{vmatrix} \tag{16-79}$$

Substitution of Eqs. (16-24), (16-78) and (16-79) into Eq. (16-76) yields

$$\begin{aligned} 4\frac{\Omega_1}{A} &= \frac{1}{2}\frac{\left[(x_3 + x_4)(y_1 + y_2) - (x_1 + x_2)(y_3 + y_4)\right]}{A} + \frac{x(y_3 + y_4 - y_1 - y_2)}{A} \\ &\quad + \frac{y(x_1 + x_2 - x_3 - x_4)}{A} \\ &= \frac{\left[(x_4 y_1 - x_1 y_4) + x(y_4 - y_1) + y(x_1 - x_4)\right]}{A} \\ &\quad - \frac{\left[(x_2 y_3 - x_3 y_2) + x(y_2 - y_3) + y(x_3 - x_2)\right]}{A} \\ &\quad + \frac{(x_2 - x_1)(y_3 - y_4) + (x_3 - x_4)(y_1 - y_2)}{2A} \\ &= 2(L_3 - L_1) + \frac{b_2 c_4 - b_4 c_2}{2A} = 2(L_3 - L_1) + (g_2 - g_1) \end{aligned} \tag{16-80a}$$

$$\begin{aligned} 4\frac{\Omega_2}{A} &= \frac{1}{2}\frac{\left[(x_4 + x_1)(y_2 + y_3) - (y_4 + y_1)(x_2 + x_3)\right]}{A} + \frac{x(y_4 + y_1 - y_2 - y_3)}{A} \\ &\quad + \frac{y(x_2 + x_3 - x_4 - x_1)}{A} \\ &= \frac{\left[(x_1 y_2 - x_2 y_1) + x(y_1 - y_2) + y(x_2 - x_1)\right]}{A} \\ &\quad - \frac{\left[(x_3 y_4 - x_4 y_3) + x(y_3 - y_4) + y(x_4 - x_3)\right]}{A} \\ &\quad + \frac{(x_3 - x_2)(y_4 - y_1) + (x_4 - x_1)(y_2 - y_3)}{2A} \\ &= 2(L_4 - L_2) + \frac{b_3 c_1 - b_1 c_3}{2A} = 2(L_4 - L_2) + (g_3 - g_2) \end{aligned} \tag{16-80b}$$

□

So, according to Eqs. (16-23) and (16-77), the new area coordinates Z_1 and Z_2 will also keep the linear relationship with the Cartesian coordinates (x, y).

Here, for convenience, two new shape parameters $\bar{g}_1$ and $\bar{g}_2$ are defined as:

$$\left.\begin{aligned} \bar{g}_1 &= g_2 - g_1 \\ \bar{g}_2 &= g_3 - g_2 \end{aligned}\right\} \tag{16-81}$$

It can be seen that $\bar{g}_1 = \bar{g}_2 = 0$ for the rectangle cases. Thus, Eq. (16-77) can be rewritten as

$$\left.\begin{aligned} Z_1 &= 2(L_3 - L_1) + \bar{g}_1 \\ Z_2 &= 2(L_4 - L_2) + \bar{g}_2 \end{aligned}\right\} \tag{16-82}$$

And, the new local coordinates of the corner nodes and mid-side points can be written as:

$$\begin{array}{llll} \text{node}\,1 & (-1+\bar{g}_2, -1+\bar{g}_1); & \text{node}\,2 & (1-\bar{g}_2, -1-\bar{g}_1) \\ \text{node}\,3 & (1+\bar{g}_2, 1+\bar{g}_1); & \text{node}\,4 & (-1-\bar{g}_2, 1-\bar{g}_1) \\ M_1 & (1, 0); & M_2 & (0, 1) \\ M_3 & (-1, 0); & M_4 & (0, -1) \end{array}$$

It is interesting that the above coordinate values are only small modifications for the isoparametric coordinates.

16.11.2 The Relationship Between the QACM-Ⅱ and the Cartesian Coordinates

Substitution of Eqs. (16-23) and (16-81) into Eq. (16-82) yields

$$\left.\begin{aligned} Z_1 &= \frac{1}{A}[(a_3 - a_1) + (b_3 - b_1)x + (c_3 - c_1)y] + \bar{g}_1 = \frac{1}{A}[\bar{a}_1 + \bar{b}_1 x + \bar{c}_1 y] + \bar{g}_1 \\ Z_2 &= \frac{1}{A}[(a_4 - a_2) + (b_4 - b_2)x + (c_4 - c_2)y] + \bar{g}_2 = \frac{1}{A}[\bar{a}_2 + \bar{b}_2 x + \bar{c}_2 y] + \bar{g}_2 \end{aligned}\right\} \tag{16-83}$$

where

$$\left.\begin{aligned} \bar{a}_1 &= a_3 - a_1, \quad \bar{b}_1 = b_3 - b_1, \quad \bar{c}_1 = c_3 - c_1 \\ \bar{a}_2 &= a_4 - a_2, \quad \bar{b}_2 = b_4 - b_2, \quad \bar{c}_2 = c_4 - c_2 \end{aligned}\right\} \tag{16-84}$$

The linear relationship between the QACM-Ⅱ and the Cartesian coordinates is clearly illustrated.

16.11.3 The Relationship Between the QACM-Ⅱ and the Isoparametric Coordinates

By substituting the relationship (16-39) of the QACM-Ⅰ and the isoparametric coordinates and Eq. (16-81) into Eq. (16-82), we have

$$\left.\begin{aligned} Z_1 &= \xi + \overline{g}_2 \xi\eta \\ Z_2 &= \eta + \overline{g}_1 \xi\eta \end{aligned}\right\} \tag{16-85}$$

From Eq. (16-85), it can be seen that the new area coordinates Z_1 and Z_2 will degenerate to be the isoparametric coordinates ξ and η for the rectangular element cases.

16.11.4 Some Discussions on QACM-II for Various Distortion Modes

Some typical shapes of a quadrilateral element and the corresponding shape parameters are summarized as follows:

Parallelogram

$$\overline{g}_1 = \overline{g}_2 = 0 \quad \text{or} \quad g_1 = g_2 = g_3 = g_4 = \frac{1}{2} \tag{16-86}$$

Trapezoid (refer to Fig. 16.12)

$$\overline{g}_1 = 0 \quad \text{or} \quad g_1 = g_2 \quad (\text{for } \overline{12} // \overline{34}) \tag{16-87a}$$

$$\overline{g}_2 = 0 \quad \text{or} \quad g_2 = g_3 \quad (\text{for } \overline{23} // \overline{41}) \tag{16-87b}$$

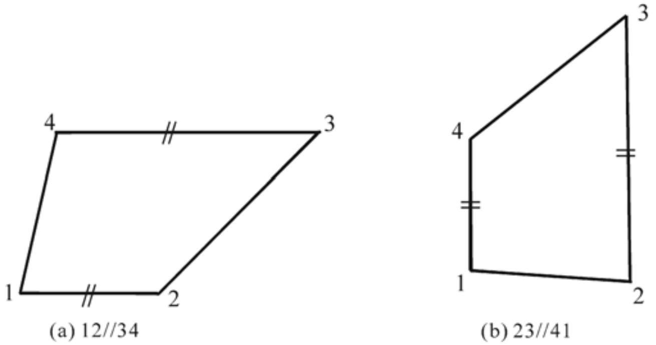

Figure 16.12 Two kinds of trapezoids

Triangular mode I: some three nodes are in line (refer to Fig. 16.13)

$$\begin{cases} \overline{g}_1 = g_2 \\ \overline{g}_2 = g_4 \end{cases} \text{or} \begin{cases} g_1 = 0 \\ g_3 = 1 \end{cases} \text{(Fig. 16.13(a): nodes 4, 1, 2 are in line)} \tag{16-88a}$$

$$\begin{cases} \overline{g}_1 = -g_1 \\ \overline{g}_2 = g_3 \end{cases} \text{or} \begin{cases} g_2 = 0 \\ g_4 = 1 \end{cases} \text{(Fig. 16.13(b): nodes 1, 2, 3 are in line)} \tag{16-88b}$$

$$\begin{cases}\bar{g}_1 = -g_4 \\ \bar{g}_2 = -g_2\end{cases} \text{ or } \begin{cases}g_3 = 0 \\ g_1 = 1\end{cases} \quad \text{(Fig. 16.13(c): nodes 2, 3, 4 are in line)} \qquad (16\text{-}88c)$$

$$\begin{cases}\bar{g}_1 = g_3 \\ \bar{g}_2 = -g_1\end{cases} \text{ or } \begin{cases}g_4 = 0 \\ g_2 = 1\end{cases} \quad \text{(Fig. 16.13(d): nodes 3, 4, 1 are in line)} \qquad (16\text{-}88d)$$

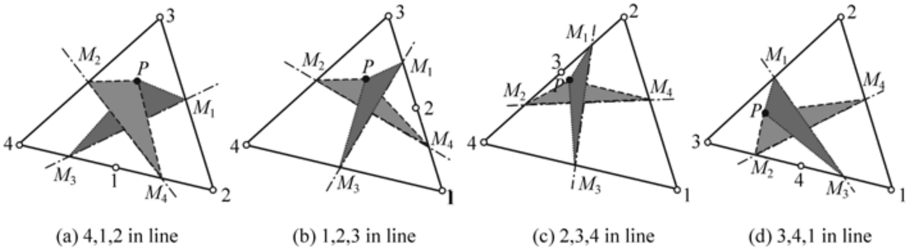

Figure 16.13 Degeneration into triangular mode Ⅰ: some three nodes are in line

Triangular mode Ⅱ: some two nodes are in coincidence (refer to Fig. 16.14)

$$\begin{cases}\bar{g}_1 = 0 \\ \bar{g}_2 = 1\end{cases} \text{ or } \begin{cases}g_1 = g_2 = 0 \\ g_3 = g_4 = 1\end{cases} \quad \text{(Fig. 16.14(a): nodes 1, 2 are in coincidence)}$$

(16-89a)

$$\begin{cases}\bar{g}_1 = -1 \\ \bar{g}_2 = 0\end{cases} \text{ or } \begin{cases}g_2 = g_3 = 0 \\ g_4 = g_1 = 1\end{cases} \quad \text{(Fig. 16.14(b): nodes 2, 3 are in coincidence)}$$

(16-89b)

$$\begin{cases}\bar{g}_1 = 0 \\ \bar{g}_2 = -1\end{cases} \text{ or } \begin{cases}g_3 = g_4 = 0 \\ g_1 = g_2 = 1\end{cases} \quad \text{(Fig. 16.14(c): nodes 3, 4 are in coincidence)}$$

(16-89c)

$$\begin{cases}\bar{g}_1 = 1 \\ \bar{g}_2 = 0\end{cases} \text{ or } \begin{cases}g_4 = g_1 = 0 \\ g_2 = g_3 = 1\end{cases} \quad \text{(Fig. 16.14(d): nodes 4, 1 are in coincidence)}$$

(16-89d)

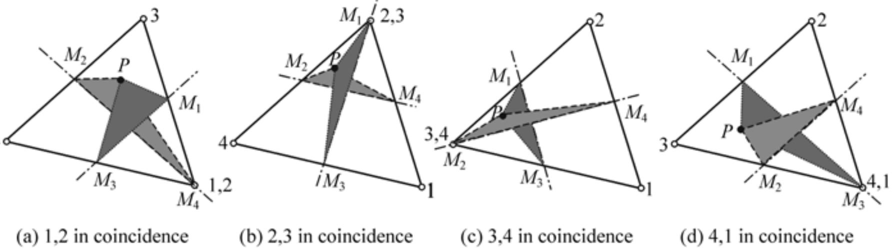

Figure 16.14 Degeneration into triangular mode Ⅱ: some two nodes are in coincidence

16.11.5 Some Basic Differential Formulae of the QACM-Ⅱ

(1) The transformation of the derivatives of the first order:

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \frac{1}{A}\begin{bmatrix} \bar{b}_1 & \bar{b}_2 \\ \bar{c}_1 & \bar{c}_2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial Z_1} \\ \dfrac{\partial}{\partial Z_2} \end{Bmatrix} \tag{16-90}$$

(2) The transformation of the derivatives of the second order:

$$\begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \frac{1}{A^2}\begin{bmatrix} \bar{b}_1^2 & \bar{b}_2^2 & 2\bar{b}_1\bar{b}_2 \\ \bar{c}_1^2 & \bar{c}_2^2 & 2\bar{c}_1\bar{c}_2 \\ \bar{b}_1\bar{c}_1 & \bar{b}_2\bar{c}_2 & \bar{b}_1\bar{c}_2 + \bar{b}_2\bar{c}_1 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2}{\partial Z_1^2} \\ \dfrac{\partial^2}{\partial Z_2^2} \\ \dfrac{\partial^2}{\partial Z_1 \partial Z_2} \end{Bmatrix} \tag{16-91}$$

(3) Normal and tangential derivatives

Assume that n_i and s_i stand for unit vectors oriented in the normal and the tangential direction respectively of the side i in a quadrilateral element (Fig. 16.8). The normal derivative of each side is:

$$\frac{\partial}{\partial n_i} = -\frac{1}{d_i}\begin{bmatrix} b_i & c_i \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = -\frac{1}{Ad_i}\begin{bmatrix} b_i & c_i \end{bmatrix}\begin{bmatrix} \bar{b}_1 & \bar{b}_2 \\ \bar{c}_1 & \bar{c}_2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial Z_1} \\ \dfrac{\partial}{\partial Z_2} \end{Bmatrix} \tag{16-92}$$

The tangential derivative of each side is:

$$\frac{\partial}{\partial s_i} = \frac{1}{d_i}\begin{bmatrix} c_i & -b_i \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \frac{1}{Ad_i}\begin{bmatrix} c_i & -b_i \end{bmatrix}\begin{bmatrix} \bar{b}_1 & \bar{b}_2 \\ \bar{c}_1 & \bar{c}_2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial}{\partial Z_1} \\ \dfrac{\partial}{\partial Z_2} \end{Bmatrix} \tag{16-93}$$

16.11.6 Some Basic Integration Formulae of the QACM-Ⅱ

(1) The area integral formulae within a quadrilateral element

The integration formulae for evaluating the area integrals of the arbitrary power function $Z_1^m Z_2^n$ can be written as:

$$\iint_A Z_1^m Z_2^n \mathrm{d}A = \frac{A}{4}\sum_{i=0}^{m}\sum_{j=0}^{n} \mathrm{C}_m^i \mathrm{C}_n^j \bar{g}_2^i \bar{g}_1^j P \tag{16-94}$$

where *m* and *n* are arbitrary positive integers,

$$P = \frac{[1-(-1)^M][1-(-1)^N]}{MN} + \bar{g}_1 \frac{[1-(-1)^{M+1}][1-(-1)^N]}{(M+1)N} + \bar{g}_2 \frac{[1-(-1)^M][1-(-1)^{N+1}]}{M(N+1)} \tag{16-95}$$

with

$$M = m + j + 1, \quad N = n + i + 1 \tag{16-96}$$

Thus, from Eqs. (16-95) and (16-96), we have

$$\left.\begin{aligned} &P = \frac{4}{MN} && \text{(for } M \text{ and } N \text{ are both odd numbers)} \\ &P = 0 && \text{(for } M \text{ and } N \text{ are both even numbers)} \\ &P = \frac{4}{M(N+1)}\bar{g}_2 && \text{(for } M \text{ is odd, } N \text{ is even)} \\ &P = \frac{4}{(M+1)N}\bar{g}_1 && \text{(for } M \text{ is even, } N \text{ is odd)} \end{aligned}\right\} \tag{16-97}$$

and C_m^n is defined as

$$\mathrm{C}_m^n = \frac{m!}{(m-n)!n!} \tag{16-98}$$

The proof of Eq. (16-94) can be easily performed by using the relationship (16-85) between the QACM-Ⅱ and the isoparametric coordinates.

Proof of Eq. (16-94)

Substitution of Eq. (16-85) into the left side of Eq. (16-94) yields

$$\iint_A Z_1^m Z_2^n \mathrm{d}A = \iint_A \left(\bar{g}_2\xi\eta + \xi\right)^m \left(\bar{g}_1\xi\eta + \eta\right)^n |\boldsymbol{J}| \mathrm{d}\xi \mathrm{d}\eta \tag{16-99}$$

where $|\boldsymbol{J}|$ is the Jacobi determinant; $\boldsymbol{J}$ is the Jacobi matrix, and it is the same as that of the usual 4-node bilinear isoparametric element Q4:

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial N_1^{\mathrm{Q4}}}{\partial \xi} & \dfrac{\partial N_2^{\mathrm{Q4}}}{\partial \xi} & \dfrac{\partial N_3^{\mathrm{Q4}}}{\partial \xi} & \dfrac{\partial N_4^{\mathrm{Q4}}}{\partial \xi} \\ \dfrac{\partial N_1^{\mathrm{Q4}}}{\partial \eta} & \dfrac{\partial N_2^{\mathrm{Q4}}}{\partial \eta} & \dfrac{\partial N_3^{\mathrm{Q4}}}{\partial \eta} & \dfrac{\partial N_4^{\mathrm{Q4}}}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{16-100}$$

with

$$N_i^{\mathrm{Q4}} = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \quad (i=1,2,3,4) \tag{16-101}$$

where ξ_i and η_i are the isoparametric coordinates of the four corner nodes. And, from Eqs. (16-25), (16-78) and (16-81), we can obtain:

$$\sum_{i=1}^{4} a_i = 2A, \quad b_2c_4 - b_4c_2 = 2A\overline{g}_1, \quad b_3c_1 - b_1c_3 = 2A\overline{g}_2 \tag{16-102}$$

Then, the Jacobi determinant $|\boldsymbol{J}|$ can be written as:

$$|\boldsymbol{J}| = \frac{1}{8}\sum_{i=1}^{4} a_i + \frac{1}{8}(b_2c_4 - b_4c_2)\xi + \frac{1}{8}(b_3c_1 - b_1c_3)\eta = \frac{A}{4}[1+\overline{g}_1\xi + \overline{g}_2\eta] \tag{16-103}$$

Substitution of Eq. (16-103) into Eq. (16-99) yields

$$\begin{aligned}
\iint_A Z_1^m Z_2^n \mathrm{d}A &= \frac{A}{4}\iint_A \xi^m\eta^n(1+\overline{g}_2\eta)^m(1+\overline{g}_1\xi)^n(1+\overline{g}_1\xi+\overline{g}_2\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{A}{4}\iint_A \xi^m\eta^n\sum_{i=0}^{m}\mathrm{C}_m^i\overline{g}_2^i\eta^i\sum_{j=0}^{n}\mathrm{C}_n^j\overline{g}_1^j\xi^j(1+\overline{g}_1\xi+\overline{g}_2\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{A}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}\mathrm{C}_m^i\mathrm{C}_n^j\overline{g}_2^i\overline{g}_1^j\iint_A(\xi^{m+j}\eta^{n+i}+\overline{g}_1\xi^{m+j+1}\eta^{n+i}+\overline{g}_2\xi^{m+j}\eta^{n+i+1})\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{A}{4}\sum_{i=0}^{m}\sum_{j=0}^{n}\mathrm{C}_m^i\mathrm{C}_n^j\overline{g}_2^i\overline{g}_1^j P
\end{aligned} \tag{16-104}$$

□

(2) Some area integral formulae for lower power functions

For convenience in application, we list the area integral formulae for lower power functions (from first order to fourth order) as follows according to Eq. (16-94):

The first power terms:

$$\iint_A \begin{Bmatrix} Z_1 \\ Z_2 \end{Bmatrix} \mathrm{d}A = \frac{A}{3}\begin{Bmatrix} \overline{g}_1 \\ \overline{g}_2 \end{Bmatrix} \tag{16-105}$$

The second power terms:

$$\iint_A \begin{Bmatrix} Z_1^2 \\ Z_2^2 \\ Z_1Z_2 \end{Bmatrix} \mathrm{d}A = \frac{A}{3}\begin{Bmatrix} 1+\overline{g}_2^2 \\ 1+\overline{g}_1^2 \\ \overline{g}_1\overline{g}_2 \end{Bmatrix} \tag{16-106}$$

The third power terms:

$$\iint_A \begin{Bmatrix} Z_1^3 \\ Z_1^2 Z_2 \\ Z_1 Z_2^2 \\ Z_2^3 \end{Bmatrix} \mathrm{d}A = \frac{A}{15} \begin{Bmatrix} 3\bar{g}_1(1+\bar{g}_2^2) \\ \bar{g}_2^3 + 2\bar{g}_1^2\bar{g}_2 + 5\bar{g}_2 \\ \bar{g}_1^3 + 2\bar{g}_1\bar{g}_2^2 + 5\bar{g}_1 \\ 3\bar{g}_2(1+\bar{g}_1^2) \end{Bmatrix} \tag{16-107}$$

The fourth power terms:

$$\iint_A \begin{Bmatrix} Z_1^4 \\ Z_1^3 Z_2 \\ Z_1^2 Z_2^2 \\ Z_1 Z_2^3 \\ Z_2^4 \end{Bmatrix} \mathrm{d}A = \frac{A}{5} \begin{Bmatrix} \bar{g}_2^4 + \frac{10}{3}\bar{g}_2^2 + 1 \\ \bar{g}_1\bar{g}_2^3 + \frac{7}{3}\bar{g}_1\bar{g}_2 \\ \bar{g}_1^2\bar{g}_2^2 + \bar{g}_1^2 + \bar{g}_2^2 + \frac{5}{9} \\ \bar{g}_1^3\bar{g}_2 + \frac{7}{3}\bar{g}_1\bar{g}_2 \\ \bar{g}_1^4 + \frac{10}{3}\bar{g}_1^2 + 1 \end{Bmatrix} \tag{16-108}$$

(3) Some basic formulae for line integral

In a quadrilateral element, the following basic formulae can be used to evaluate the line integral for the arbitrary power function of QACM-Ⅱ along each side.

Along side $\overline{23}$

$$\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} = \frac{1}{2}(1+\bar{g}_1)^n \sum_{i=0}^{m} \mathrm{C}_m^i g_2^i \frac{1-(-1)^{n+i+1}}{n+i+1} \tag{16-109}$$

Along side $\overline{34}$

$$\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} = -\frac{1}{2}(1+\bar{g}_2)^m \sum_{i=0}^{n} \mathrm{C}_n^i g_1^i \frac{1-(-1)^{m+i+1}}{m+i+1} \tag{16-110}$$

Along side $\overline{41}$

$$\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} = -\frac{1}{2}(-1)^m(1-\bar{g}_1)^n \sum_{i=0}^{m} \mathrm{C}_m^i g_2^i \frac{1-(-1)^{n+i+1}}{n+i+1} \tag{16-111}$$

Along side $\overline{12}$

$$\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} = -\frac{1}{2}(-1)^n(1-\bar{g}_2)^m \sum_{i=0}^{n} \mathrm{C}_n^i g_1^i \frac{1-(-1)^{m+i+1}}{m+i+1} \tag{16-112}$$

where $\bar{s}$ is a dimensionless coordinate along side $\overline{jk}$ (jk = 23,34,41,12), it is 0 at node j and 1 at node k.

Proof of Eq. (16-111)

By using Eq. (16-85), Z_1 and Z_2 can be replaced by isoparametric coordinates ξ and η, then

$$\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} = \int_0^1 (\xi + \bar{g}_2 \xi\eta)^m (\eta + \bar{g}_1 \xi\eta)^n \mathrm{d}\bar{s} \tag{16-113}$$

Along side $\overline{23}$, we have

$$\xi = 1, \quad -1 \leqslant \eta \leqslant 1, \quad \bar{s} = \frac{1+\eta}{2} \tag{16-114}$$

Thus, Eq. (16-113) can be rewritten as

$$\begin{aligned}\int_0^1 Z_1^m Z_2^n \mathrm{d}\bar{s} &= \frac{1}{2}\int_{-1}^1 (1+\bar{g}_2\eta)^m (\eta + \bar{g}_1\eta)^n \mathrm{d}\eta \\ &= \frac{1}{2}(1+\bar{g}_1)^n \int_{-1}^1 (1+\bar{g}_2\eta)^m \eta^n \mathrm{d}\eta \\ &= \frac{1}{2}(1+\bar{g}_1)^n \int_{-1}^1 \eta^n \sum_{i=0}^m \mathrm{C}_m^i \bar{g}_2^i \eta^i \mathrm{d}\eta \\ &= \frac{1}{2}(1+\bar{g}_1)^n \int_{-1}^1 \sum_{i=0}^m \mathrm{C}_m^i \bar{g}_2^i \eta^{n+i} \mathrm{d}\eta \\ &= \frac{1}{2}(1+\bar{g}_1)^n \sum_{i=0}^m \mathrm{C}_m^i \bar{g}_2^i \frac{1-(-1)^{n+i+1}}{n+i+1}\end{aligned} \tag{16-115}$$

□

The proof procedures of the Eqs. (16-110), (16-111) and (16-112) are similar to the procedure given above.

Thus, a new area coordinate system QACM-II, which only contains two independent components, is successfully established.

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