# Chapter 1
# Basics

**Abstract.** This chapter is the first part of the theoretical survey on the joint spectral radius. We first present precise definitions of the main concepts. We then show that these definitions are well posed, and we present some basic properties on the joint spectral radius. In the last section, we show that these notions are "useful", in the sense that they actually characterize the maximal and minimal growth rates of a switched dynamical system.

This chapter is meant to be a quick survey on the basic behavior of the joint spectral radius. Some of the results presented in this chapter require rather involved proofs. For this reason this chapter is not self-contained, and some proofs are postponed to the next one.

In this introductory chapter, we compare all results for the joint spectral radius to its minimum-growth counterpart: the joint spectral subradius.

A switched linear system in discrete time is characterized by the equation

$$\begin{aligned} x_{t+1} &= A_t x_t : \quad A_t \in \Sigma, \\ x_0 &\in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $\Sigma$ is a set of real $n \times n$ matrices. We would like to estimate the evolution of the vector $x$, and more particularly (if it exists) the asymptotic growth rate of its norm:

$$\lambda = \lim_{t \to \infty} ||x_t||^{1/t}.$$

Clearly, one cannot expect that this limit would exist in general. Indeed, even in dimension one, it is easy to design a dynamical system and a trajectory such that the limit above does not exist. Thus a typical relevant question for such a system is the extremal rate of growth: given a set of matrices $\Sigma$, what is the maximal value for $\lambda$, over all initial vectors $x_0$ and all sequences of matrices $A_t$? In the case of dynamical systems for instance, such an analysis makes a lot of sense. Indeed, by computing

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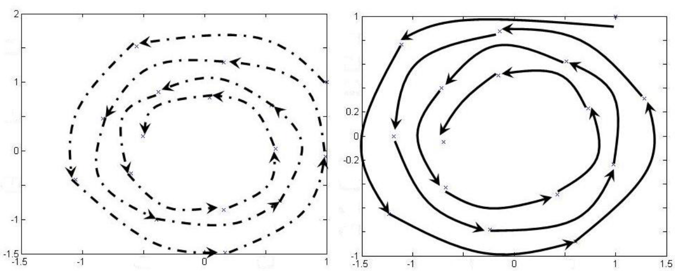
the maximal growth rate one can ensure the stability of the system, provided that this growth rate is less than one. We will see that the quantity characterizing this maximal rate of growth of a switched linear discrete time system is the *joint spectral radius,* introduced in 1960 by Rota and Strang [104]. Thanks to its interpretation in terms of dynamical systems, and for many other reasons that we will present later on, it has been widely studied during the last decades.

When the set of matrices consists in a single matrix $A$, the problem is simple: the maximal growth rate is the largest magnitude of the eigenvalues of $A$. As a consequence, a matrix is stable if and only if the magnitudes of its eigenvalues are less than one. However, if the set of matrices consists in more than just one matrix, the problem is far more complex: the matrices could well all be stable, while the system itself could be unstable! This phenomenon, which motivates the study of the joint spectral radius, is illustrated by the next example. Consider the set of matrices

$$\Sigma = \left\{ A_0 = \frac{2}{3}\begin{pmatrix} \cos 1.5 & \sin 1.5 \\ -2\sin 1.5 & 2\cos 1.5 \end{pmatrix}, A_1 = \frac{2}{3}\begin{pmatrix} 2\cos 1.5 & 2\sin 1.5 \\ -\sin 1.5 & \cos 1.5 \end{pmatrix} \right\}.$$

The dynamics of these matrices are illustrated in Figure 1.1(a) and (b), with the initial point $x_0 = (1,1)$. Since both matrices are stable ($\rho(A_0) = \rho(A_1) = 0.9428$, where $\rho(A)$, the *spectral radius* of $A$, is the largest magnitude of its eigenvalues) the trajectories go to the origin. But if one combines the action of $A_0$ and $A_1$ alternatively, a diverging behavior occurs (Figure 1.2). The explanation is straightforward: the spectral radius of $A_0A_1$ is equal to $1.751 > 1$.




(a) (b)

**Fig. 1.1** Trajectories of two stable matrices

In practical applications, some other quantities can be of importance, as for instance the *minimal* rate of growth. This concept corresponds to the notion of *joint spectral subradius*. In this introductory chapter, we give definitions for these concepts, as well as some basic results. For the sake of conciseness, and to save time for the reader, we decided not to recall too many basic facts or definitions from linear algebra. We instead refer the reader to classical reference books [45, 72].

In this chapter we first present precise definitions of the main concepts (Section 1.1). In Section 1.2 we show that these definitions are well posed, and we present

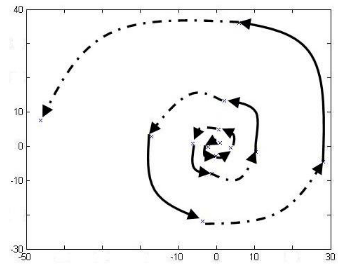



**Fig. 1.2** Unstable behavior by combining two stable matrices

some basic properties on the joint spectral radius and the joint spectral subradius. In the last section, we show that these notions are "useful", in the sense that they actually characterize the maximal and minimal growth rates of a switched dynamical system of the type (1.1). As the reader will discover, this is not so obvious.

Some of the results presented in this chapter require rather involved proofs. For this reason this chapter is not self-contained, and some proofs are postponed to Chapter 2. Nevertheless we had the feeling that a small chapter with all the basic results could be useful for the reader in order to summarize the basic properties of the joint spectral radius and the joint spectral subradius.

## 1.1 Definitions

The joint spectral radius characterizes the maximal asymptotic growth rate of the norms of long products of matrices taken in a set $\Sigma$. By a *norm*, we mean a function that to any matrix $A \in \mathbb{R}^{n \times n}$ associates a real number $||A||$ such that

- $||A|| \geq 0$, $||A|| = 0 \Leftrightarrow A = 0$,
- $\forall k \in \mathbb{R} : ||kA|| = |k|\, ||A||$,
- $||A+B|| \leq ||A|| + ||B||$,
- $||AB|| \leq ||A||\, ||B||$.

The latter condition, called *submultiplicativity* is not required in classical definitions of a norm, but in this monograph we will restrict our attention to them, so that all results involving norms have to be understood in terms of submultiplicative norms. Many norms are submultiplicative, and it is for instance the case of any norm induced by a vector norm. So, let $||\cdot||$ be a matrix norm, and $A \in \mathbb{R}^{n \times n}$ be a real matrix. It is well known that the spectral radius of $A$, that is, the maximal modulus of its eigenvalues, represents the asymptotic growth rate of the norm of the successive powers of $A$:

$$\rho(A) = \lim_{t \to \infty} ||A^t||^{1/t}. \tag{1.2}$$

This quantity does provably not depend on the norm used, and one can see that it characterizes the maximal rate of growth for the norm of a point $x_t$ subject to a Linear Time Invariant dynamical system. In order to generalize this notion to a set of matrices $\Sigma$, let us introduce the following notation:

$$\Sigma^t \triangleq \{A_1 \dots A_t : A_i \in \Sigma\}.$$

Also, it is a common practice to denote by $A^T$ the transpose of $A$. It will always be clear from the context whether $A^T$ denotes the transpose of $A$ of the classical exponentiation.

We define the two following quantities that are good candidates to quantify the "maximal size" of products of length $t$:

$$\hat{\rho}_t(\Sigma, \|\cdot\|) \triangleq \sup\{\|A\|^{1/t} : A \in \Sigma^t\},$$
$$\rho_t(\Sigma) \triangleq \sup\{\rho(A)^{1/t} : A \in \Sigma^t\}.$$

For a matrix $A \in \Sigma^t$, we call $||A||^{1/t}$ and $\rho(A)^{1/t}$ respectively the *averaged norm* and the *averaged spectral radius* of the matrix, in the sense that it is averaged with respect to the length of the product. We also abbreviate $\hat{\rho}_t(\Sigma, \|\cdot\|)$ into $\hat{\rho}_t(\Sigma)$ or even $\hat{\rho}_t$ if this is clear enough with the context. Rota and Strang introduced the *joint spectral radius* as the limit [104]:

$$\hat{\rho}(\Sigma) \triangleq \lim_{t\to\infty} \hat{\rho}_t(\Sigma, \|\cdot\|). \tag{1.3}$$

This definition is independent of the norm used by the equivalence of the norms in $\mathbb{R}^n$. Daubechies and Lagarias introduced the generalized spectral radius as [33]:

$$\rho(\Sigma) \triangleq \limsup_{t\to\infty} \rho_t(\Sigma).$$

We will see in the next chapter that for bounded sets of matrices these two quantities are equal. Based on this equivalence, we use the following definition:

**Definition 1.1.** *The* joint spectral radius *of a bounded set of matrices $\Sigma$ is defined by:*

$$\rho(\Sigma) = \limsup_{t\to\infty} \rho_t(\Sigma) = \lim_{t\to\infty} \hat{\rho}_t(\Sigma).$$

*Example 1.1.* Let us consider the following set of matrices:

$$\Sigma = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

The spectral radius of both matrices is one. However, by multiplying them, one can obtain the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

whose spectral radius is equal to two. Hence, $\rho(\Sigma) \geq \sqrt{2}$, since

$$\lim_{t\to\infty} \hat{\rho}_t(\Sigma) \geq \lim ||A^{t/2}||^{1/t} = \sqrt{2}.$$

Now, $\hat{\rho}_2 = \sqrt{2}$ (where we have chosen the maximum column-sum for the norm) and, as we will see below, $\hat{\rho}_t$ is an upper bound on $\rho$ for any $t$. So we get $\rho(\Sigma) = \sqrt{2}$.

As the reader will see, the proof of the equivalence between the joint spectral radius and the generalized spectral radius necessitates some preliminary work so as to be presented in a natural way. Before to reach this proof, we continue to make the distinction between the joint spectral radius $\hat{\rho}(\Sigma)$ and the generalized spectral radius $\rho(\Sigma)$.

Let us now interest ourself to the minimal rate of growth. We can still define similar quantities, describing the minimal rate of growth of the spectral radius and of the norms of products in $\Sigma^t$. These notions were introduced later than the joint spectral radius ([52], see also [17]).

$$\check{\rho}_t(\Sigma, \|\cdot\|) \triangleq \inf\{\|A\|^{1/t} : A \in \Sigma^t\},$$
$$\underline{\rho}_t(\Sigma) \triangleq \inf\{\rho(A)^{1/t} : A \in \Sigma^t\}.$$

Then, the *joint spectral subradius* is defined as the limit:

$$\check{\rho}(\Sigma) \triangleq \lim_{t\to\infty} \check{\rho}_t(\Sigma, \|\cdot\|), \tag{1.4}$$

Which is still independent of the norm used by equivalence of the norms in $\mathbb{R}^n$. We define the *generalized spectral subradius* as

$$\underline{\rho}(\Sigma) \triangleq \lim_{t\to\infty} \underline{\rho}_t.$$

Again, we will see that for bounded sets of matrices these two quantities are equal, and we use the following definition:

**Definition 1.2.** *The* joint spectral subradius *of a set of matrices $\Sigma$ is defined by:*

$$\check{\rho}(\Sigma) = \lim_{t\to\infty} \check{\rho}_t = \lim_{t\to\infty} \underline{\rho}_t.$$

*Example 1.2.* Let us consider the following set of matrices:

$$\Sigma = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \right\}.$$

The spectral radius of both matrices is greater than one. However, by multiplying them, one can obtain the zero matrix, and then the joint spectral subradius is zero.

The above examples are simple but, as the reader will see, the situation is sometimes much more complex.

## 1.2 Basic Results

In this section, we review basic results on the joint spectral characteristics, that allow to understand what they are and what they are not. We first present the fundamental theorems, proving the equality for the joint and generalized spectral radii (resp. subradii). We then present basic properties of the joint spectral characteristics, some of which had to our knowledge not yet been formalized.

### 1.2.1 Fundamental Theorems

#### The joint spectral radius

First, recall that we defined $\hat{\rho}$ as a limit, and not as a limsup. This is due to a classical result, known as *Fekete's Lemma*:

**Lemma 1.1.** *[43] Let* $\{a_n\} : n \geq 1$ *be a sequence of real numbers such that*

$$a_{m+n} \leq a_m + a_n.$$

*Then the limit*

$$\lim_{n\to\infty} \frac{a_n}{n}$$

*exists and is equal to* $\inf\{\frac{a_n}{n}\}$.

In the above lemma, the limit can be equal to $-\infty$, but this is not possible in our case since the sequence is nonnegative. We are now in position to prove the convergence:

**Lemma 1.2.** *For any bounded set* $\Sigma \subset \mathbb{R}^{n\times n}$*, the function* $t \to \hat{\rho}_t(\Sigma)$ *converges when* $t \to \infty$*. Moreover,*

$$\lim_{t\to\infty} \hat{\rho}_t(\Sigma) = \inf\{\hat{\rho}_t(\Sigma)\}.$$

*Proof.* Since the norms considered are submultiplicative, the sequence

$$\log\left(\sup\{\|A\| : A \in \Sigma^t\}\right) = \log \hat{\rho}_t^t$$

is subadditive. That is,

$$\log \hat{\rho}_{t+t'}^{t+t'} \leq \log \hat{\rho}_t^t + \log \hat{\rho}_{t'}^{t'}.$$

If for all $t$, $\hat{\rho}_t \neq 0$, then by Fekete's lemma,

$$\frac{1}{t} \log \hat{\rho}_t^t = \log \hat{\rho}_t$$

converges and is equal to $\inf \log \hat{\rho}_t$.

If there is an integer $t$ such that $\hat{\rho}_t = 0$, then clearly, for all $t' \geq t, \hat{\rho}_{t'} = 0$, and the proof is done.

Unlike the maximal norm, the behavior of the maximal spectral radius, $\rho_t$ is not as simple, and in general the limsup in the definition of $\rho(\Sigma)$ cannot be replaced by a simple limit. In the following simple example, $\limsup \rho_t = 1$, but $\lim \rho_t$ does not exist:

$$\Sigma = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Indeed, for any $t$, $\rho_{2t}(\Sigma) = 1$, but $\rho_{2t+1}(\Sigma) = 0$.

**The Joint Spectral Radius Theorem**

It is well known that the spectral radius of a matrix satisfies $\rho(A^k) = \rho(A)^k$, $\rho(A) = \lim ||A^t||^{1/t}$. One would like to generalize these relations to "inhomogeneous" products of matrices, that is, products where factors are not all equal to a same matrix $A$. This is possible, as has been proved in 1992 by Berger and Wang [5] in the so-called *Joint Spectral Radius Theorem*:

*For bounded sets, the values* $\hat{\rho}(\Sigma)$ *and* $\rho(\Sigma)$ *are equal.*

No elementary proof is known for this theorem. Elsner [41] provides a self-contained proof that is somewhat simpler than (though inspired by) the original one from [5]. Since both proofs use rather involved results on the joint spectral radius, we postpone an exposition of the proof to the next chapter. The reader can check that the elementary facts presented in the remainder of this chapter do not make use of this result.

Observe that the joint spectral radius theorem cannot be generalized to unbounded sets of matrices, as can be seen on the following example:

$$\Sigma = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \dots \right\}.$$

Indeed for this set we have $\rho(\Sigma) = 1$, while $\hat{\rho}(\Sigma) = \infty$.

### The joint spectral subradius

Let us now consider the joint spectral subradius. It appears that now both $\underline{\rho}_t$ and $\check{\rho}_t$ converge:

**Proposition 1.1.** *For any set* $\Sigma \subset \mathbb{R}^{n\times n}$, *the function* $t \to \check{\rho}_t(\Sigma)$ *converges when* $t \to \infty$, *and*

$$\lim \check{\rho}_t(\Sigma) = \inf_{t>0} \check{\rho}_t(\Sigma).$$

*Moreover, the function* $t \to \underline{\rho}_t(\Sigma)$ *converges when* $t \to \infty$, *and*

$$\lim \underline{\rho}_t(\Sigma) = \inf_{t>0} \underline{\rho}_t(\Sigma).$$

*Proof.* Again the sequence $\log(\check{\rho}_n)$ is subadditive, which proves the first part. Let us now prove the second assertion. We define $\underline{\rho} = \liminf \underline{\rho}_t$ and we will show that the limit actually exists. Fix an $\varepsilon > 0$. For any sufficiently long $t$, we will construct a product of matrices $B \in \Sigma^t$ such that $||B||^{1/t} \leq \underline{\rho} + \varepsilon$, and thus $\rho(B)^{1/t} \leq \underline{\rho} + \varepsilon$. Indeed, for any norm $||\cdot||$ and any matrix $A$, the relation

$$\rho(A) \leq ||A||$$

always holds. In order to do that, we will pick a matrix $A$ whose spectral radius is small enough, and we define $B = A^k C$, where $C$ is a short product that does not perturb too much the norm of $B$.

By the definition of $\underline{\rho}$, there exist $T \in \mathbb{N}, A \in \Sigma^T$ such that $\rho(A)^{1/T} \le \underline{\rho} + \varepsilon/4$. Since $\rho(A) = \lim ||A^k||^{1/k}$, one gets $\lim ||A^k||^{1/kT} \le \underline{\rho} + \varepsilon/4$ and there exists an integer $k_0$ such that for all $k > k_0$, $||A^k||^{1/kT} \le \underline{\rho} + \varepsilon/2$.

Let us first define a real number $M$ such that for each length $t' \le T$, there is a product $C$ of length $t'$ such that $||C|| \le M$. Next, there is an integer $T_0$ large enough so that $M^{1/T_0} \le (\underline{\rho} + \varepsilon)/(\underline{\rho} + \varepsilon/2)$.

Now, for any length $t > \max\{k_0 T, T_0\}$, we define $t' < T$ such that $t = kT + t'$, and we construct a product of length $t : B = A^k C$, such that $C \in \Sigma^{t'}$, and $||C|| \le M$. Finally

$$||B||^{1/t} \le (\underline{\rho} + \varepsilon/2) \frac{\underline{\rho} + \varepsilon}{\underline{\rho} + \varepsilon/2} \le \underline{\rho} + \varepsilon.$$

We also have the equality between $\check{\rho}$ and $\underline{\rho}$; moreover in this case the set need not be bounded;

**Theorem 1.1.** *[111] For any set of matrices $\Sigma$,*

$$\lim_{t \to \infty} \inf\{\rho(A)^{1/t} : A \in \Sigma^t\} = \lim_{t \to \infty} \inf\{||A||^{1/t} : A \in \Sigma^t\} \triangleq \check{\rho}(\Sigma).$$

*Proof.* Clearly,

$$\lim_{t \to \infty} \inf\{\rho(A)^{1/t} : A \in \Sigma^t\} \le \lim_{t \to \infty} \inf\{||A||^{1/t} : A \in \Sigma^t\}$$

because for any matrix $A$, $\rho(A) \le ||A||$.

Now, for any matrix $A \in \Sigma^t$ with averaged spectral radius $r$ close to $\underline{\rho}(\Sigma)$, the product $A^k \in \Sigma^{kt}$ is such that $||A^k||^{1/kt} \to r$ so that

$$\lim_{k \to \infty} \inf\{||A||^{1/kt} : A \in \Sigma^{kt}\} \le r.$$

## *1.2.2 Basic Properties*

### 1.2.2.1 Scaling Property

**Proposition 1.2.** *For any set $\Sigma \in \mathbb{R}^{n \times n}$ and for any real number $\alpha$,*

$$\hat{\rho}(\alpha\Sigma) = |\alpha|\hat{\rho}(\Sigma),$$

$$\check{\rho}(\alpha\Sigma) = |\alpha|\check{\rho}(\Sigma).$$

*Proof.* This is a simple consequence of the relation $||\alpha A|| = |\alpha|\,||A||$.

#### 1.2.2.2 Complex Matrices vs. Real Matrices

From now on, all matrices are supposed to be real-valued. This is not a restriction as we can consider complex matrices acting on $\mathbb{C}^{n\times n}$ as real operators acting on $\mathbb{R}^{2n\times 2n}$.

#### 1.2.2.3 Invariance under Similarity

**Proposition 1.3.** *For any bounded set of matrices* $\Sigma$, *and any invertible matrix* $T$,

$$\rho(\Sigma) = \rho(T\Sigma T^{-1}).$$

$$\check{\rho}(\Sigma) = \check{\rho}(T\Sigma T^{-1}).$$

*Proof.* This is due to the fact that for any product $A_1 \dots A_t \in \Sigma^t$, the corresponding product in $T\Sigma T^{-1}$ is $TA_1 \dots A_t T^{-1}$, and has equal spectral radius.

#### 1.2.2.4 The Joint Spectral Radius as an Infimum over All Possible Norms

The following result has been known for long, since it was already present in the seminal paper of Rota and Strang [104]. Nevertheless, it is very interesting, as it characterizes the joint spectral radius in terms of the matrices in $\Sigma$, without considering any product of these matrices. We give here a simple self-contained proof due to Berger and Wang [5].

**Proposition 1.4.** *For any bounded set* $\Sigma$ *such that* $\hat{\rho}(\Sigma) \neq 0$, *the joint spectral radius can be defined as*

$$\hat{\rho}(\Sigma) = \inf_{||\cdot||} \sup_{A\in\Sigma} \{||A||\}.$$

From now on, we denote by $\Sigma^*$ the *monoid generated by* $\Sigma$:

$$\Sigma^* \triangleq \cup_{t=0}^{\infty} \Sigma^t,$$

With $\Sigma^0 \triangleq I$. If we exclude $\Sigma^0$ from the above definition, we obtain $\Sigma^+$, the *semigroup generated by* $\Sigma$:

$$\Sigma^+ \triangleq \cup_{t=1}^{\infty} \Sigma^t.$$

*Proof.* Let us fix $\varepsilon > 0$, and consider the set $\tilde{\Sigma} = (1/(\hat{\rho}+\varepsilon))\Sigma$. Then, all products of matrices in $\tilde{\Sigma}^*$ are uniformly bounded, and one can define a norm $|\cdot|$ on $\mathbb{R}^n$ in the following way: $|x| = \max\{|Ax|_2 : A \in \tilde{\Sigma}^*\}$, where $|\cdot|_2$ is the Euclidean vector norm. Remark that in the above definition, the maximum can be used instead of the supremum, because $\rho(\tilde{\Sigma}) < 1$. The matrix norm induced by this latter vector norm, that is, the norm defined by

$$||A|| = \max_{|x|=1} \{|Ax|\},$$

clearly satisfies $\sup_{A\in\tilde{\Sigma}} \{||A||\} \leq 1$, and so $\sup_{A\in\Sigma} \{||A||\} \leq \hat{\rho} + \varepsilon$.

### 1.2.2.5 Common Reducibility

We will say that a set of matrices is *commonly reducible,* or simply *reducible* if there is a nontrivial linear subspace (i.e. different from $\{0\}$ and $\mathbb{R}^n$) that is invariant under all matrices in $\Sigma$. This property is equivalent to the existence of an invertible matrix $T$ that "block-triangularizes simultaneously" all matrices in $\Sigma$ :

$$\Sigma \text{ reducible} \quad \Leftrightarrow \quad \exists T, n' : \forall A_i \in \Sigma, TA_iT^{-1} = \begin{pmatrix} B_i & C_i \\ 0 & D_i \end{pmatrix} : D_i \in \mathbb{R}^{n' \times n'}.$$

We will say that a set of matrices is *commonly irreducible,* or simply *irreducible* if it is not commonly reducible.

**Proposition 1.5.** *With the notations defined above, if $\Sigma$ is bounded and reducible,*

$$\rho(\Sigma) = \max\{\rho(\{B_i\}), \rho(\{D_i\})\},$$

$$\check{\rho}(\Sigma) \geq \max\{\check{\rho}(\{B_i\}), \check{\rho}(\{D_i\})\}, \tag{1.5}$$
$$\hat{\rho}(\Sigma) = \max\{\hat{\rho}(\{B_i\}), \hat{\rho}(\{D_i\})\}.$$

*Proof.* The first two relations follow from the invariance under similarity (Proposition 1.3), together with the following elementary facts:

$$\begin{pmatrix} B_1 & C_1 \\ 0 & D_1 \end{pmatrix} \cdot \begin{pmatrix} B_2 & C_2 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} B_1B_2 & B_1C_2 + C_1D_2 \\ 0 & D_1D_2 \end{pmatrix},$$

$$\rho\left(\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\right) = \max\{\rho(B), \rho(D)\}.$$

The third relation is more technical, and is proved by showing that extradiagonal blocks cannot increase the exponent of growth. We can suppose $A_i \in \Sigma$ block-triangular, still by invariance under similarity. Let us denote $M$ the maximal joint spectral radius among the diagonal blocks:

$$M = \max\{\hat{\rho}(\{B_i\}), \hat{\rho}(\{D_i\})\}.$$

We define the norm $||\cdot||$ as the sum of the absolute values of the entries. Clearly $\hat{\rho}(\Sigma) \geq M$, and we now prove the reverse inequality.

Writing

$$A_i = \begin{pmatrix} 0 & C_i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_i & 0 \\ 0 & D_i \end{pmatrix},$$

we have

$$||A_t \dots A_1|| = ||B_t \dots B_1|| + ||D_t \dots D_1|| + ||\sum_{r=1}^{t} B_t \dots B_r C_r D_{r-1} \dots D_1||.$$

Now for any $\varepsilon$ there is a natural $T$ such that for all $t \geq T$,

$$\hat{\rho}_t(\{B_i\}), \hat{\rho}_t(\{D_i\}) < (M+\varepsilon)^t.$$

Thus, for $t$ large enough we can bound each term in the summation above by $O((M+\varepsilon)^t)$ :

if $T < r < t-T$, then

$$||B_t \dots B_r C_r D_{r-1} D_1|| \leq ||C_r||(M+\varepsilon)^{t-1},$$

and in the other case (say, $r \leq T$, the other case is similar),

$$||B_t \dots B_r C_r D_{r-1} D_1|| < ||C_r||(\hat{\rho}_1)^r (M+\varepsilon)^{t-r-1} = O((M+\varepsilon)^t).$$

Recall that $\hat{\rho}_1$ is the supremum of the norms of the matrices in $\Sigma$. Finally, $||A_t \dots A_1|| \leq 2(M+\varepsilon)^t + tO((M+\varepsilon)^t)$, and $\hat{\rho}(\Sigma) \leq M+\varepsilon$.

It is straightforward that the above proposition generalizes inductively to the case where there are more than two blocks on the diagonal.

In the above proposition, Equation (1.5) enlightens a fundamental difference between the joint spectral radius and the joint spectral subradius. For this latter quantity, the inequality cannot be replaced by an equality. This is due to the fact that the joint spectral subradius is the *minimum* growth of a quantity (the spectral radius) which is by essence a *maximum* (over all eigenvalues of a matrix). Consider the next example:

$$\Sigma = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

The joint spectral subradius of the first diagonal entries is 2, and this is also the case for the set of the second diagonal entries. However, the joint spectral subradius of $\Sigma$ is equal to $\sqrt{8} > 2$.

#### 1.2.2.6 Three Members Inequalities

**Proposition 1.6.** *For any bounded set $\Sigma \in \mathbb{R}^{n\times n}$ and for any natural $t$,*

$$\rho_t(\Sigma) \leq \rho(\Sigma) \leq \hat{\rho}_t(\Sigma). \tag{1.6}$$

*Proof.* The left hand side inequality is due to the fact that $\rho(A^k) = \rho(A)^k$. The right hand side is from Fekete's lemma (Lemma 1.1).

Let us add that this has been generalized to unbounded sets to what is called the four members inequality[33, 35]:

$$\rho_t(\Sigma) \leq \rho(\Sigma) \leq \hat{\rho}(\Sigma) \leq \hat{\rho}_t(\Sigma).$$

For the joint spectral subradius, it appears that both quantities $\underline{\rho}_t$ and $\check{\rho}_t$ are in fact upper bounds:

**Proposition 1.7.** *For any bounded set $\Sigma \in \mathbb{R}^{n\times n}$ and for any natural $t$,*

$$\check{\rho}(\Sigma) \leq \underline{\rho}_t(\Sigma) \leq \check{\rho}_t(\Sigma).$$

*Proof.* The left hand side inequality is due to the fact that $\rho(A^k) = \rho(A)^k$, implying that $\underline{\rho}_{kt} \leq \underline{\rho}_t$. The right hand side is a straightforward consequence of the property $\rho(A) \leq ||A||$.

#### 1.2.2.7 Closure and Convex Hull

Taking the closure or the convex hull of a set does not change its joint spectral radius. For the closure, we also prove the result for the generalized spectral radius, since it will be needed in further developments.

**Proposition 1.8.** *[111] For any bounded set* $\Sigma \in \mathbb{R}^{n\times n}$

$$\hat{\rho}(\Sigma) = \hat{\rho}(conv\,\Sigma) = \hat{\rho}(cl\,\Sigma),$$

$$\rho(\Sigma) = \rho(cl\,\Sigma).$$

*Proof.* For the convex hull, observe that for all $t > 0 : \hat{\rho}_t(\mathrm{conv}\Sigma) = \hat{\rho}_t(\Sigma)$. Indeed, all products in $(\mathrm{conv}\Sigma)^t$ are convex combinations of products in $\Sigma^t$, and are thus less or equally normed. The equalities for the closure hold because for all $t$, $\rho_t(\mathrm{cl}\,\Sigma) = \rho_t(\Sigma)$, and $\hat{\rho}_t(\mathrm{cl}\,\Sigma) = \hat{\rho}_t(\Sigma)$, by continuity of the norm and the eigenvalues.

We now show the counterpart for the joint spectral subradius. The property still holds for the closure, but not for the convex hull:

**Proposition 1.9.** *For any bounded set* $\Sigma \in \mathbb{R}^{n\times n}$

$$\check{\rho}(\Sigma) = \check{\rho}(cl\,\Sigma),$$

*but the equality* $\check{\rho}(\Sigma) = \check{\rho}(conv\,\Sigma)$ *does not hold in general.*

*Proof.* The equality $\check{\rho}_t(\mathrm{cl}\Sigma) = \check{\rho}_t(\Sigma)$ still holds for all $t$ by continuity of the norm and the matrix multiplication.

On the other hand, consider the simple example $\Sigma = \{1, -1\} \subset \mathbb{R}$. All products have norm one, and so $\check{\rho} = \rho = 1$, but $0 \in \mathrm{conv}\,\Sigma$, and so $\check{\rho}(\mathrm{conv}\,\Sigma) = 0$.

#### 1.2.2.8 Continuity

We show here that the joint spectral radius of bounded sets of matrices is continuous in their entries. Recall that the Hausdorff distance measures the distance between sets of points in a metric space:

$$d(\Sigma, \Sigma') \triangleq \max\{\sup_{A\in\Sigma}\{\inf_{A'\in\Sigma'} ||A - A'||\}, \sup_{A'\in\Sigma'}\{\inf_{A\in\Sigma} ||A - A'||\}\}.$$

**Proposition 1.10.** *The joint spectral radius of bounded sets of matrices is continuous with respect to the Hausdorff distance in* $\mathbb{R}^{n\times n}$.

*That is, for any bounded set of matrices* $\Sigma \in \mathbb{R}^{n\times n}$, *and for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that*

$$d(\Sigma,\Sigma') < \delta \Rightarrow |\hat{\rho}(\Sigma) - \hat{\rho}(\Sigma')| < \varepsilon.$$

*Proof.* Let us fix $\varepsilon > 0$. By Proposition 1.4, there exists a norm $||\cdot||$ such that

$$\hat{\rho}_1(\Sigma) = \sup\{||A|| : A \in \Sigma\} \leq \hat{\rho}(\Sigma) + \varepsilon/2.$$

Let us now pick a set $\Sigma'$ close enough to $\Sigma$ : $d(\Sigma,\Sigma') < \varepsilon/2$. By definition of the Hausdorff distance, we have

$$\forall A' \in \Sigma', \exists A \in \Sigma : ||A' - A|| < \varepsilon/2,$$

and we can bound the norm of any matrix in $\Sigma'$ :

$$||A'|| = ||A + (A' - A)|| \leq \hat{\rho}(\Sigma) + \varepsilon/2 + \varepsilon/2 = \hat{\rho}(\Sigma) + \varepsilon.$$

By applying the same argument to $\Sigma$, we obtain $|\hat{\rho}(\Sigma) - \hat{\rho}(\Sigma')| \leq \varepsilon$.

Let us note that this proposition does not generalize to unbounded sets, as shown by the next example:

$$\Sigma = \left\{ \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} : n \in \mathbb{N} \right\}.$$

Indeed for $\varepsilon = 0$ we have $\hat{\rho}(\Sigma) = 0$, while for any $\varepsilon > 0$ we have $\hat{\rho}(\Sigma) = \infty$.

Let us add that Wirth has proved that the joint spectral radius is even locally Lipschitz continuous on the space of compact irreducible sets of matrices endowed with the Hausdorff topology [115, 117].

Surprisingly, a similar continuity result for the joint spectral subradius is not possible. It appears that this quantity is only lower semicontinuous:

**Proposition 1.11.** *The joint spectral subradius of bounded sets of matrices is lower semicontinuous with respect to the Hausdorff distance in* $\mathbb{R}^{n\times n}$.

*That is, for any bounded set of matrices* $\Sigma \in \mathbb{R}^{n\times n}$, *and for any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that*

$$d(\Sigma,\Sigma') < \delta \Rightarrow \check{\rho}(\Sigma') < \check{\rho}(\Sigma) + \varepsilon.$$

*Proof.* Let us fix $\varepsilon > 0$. By Proposition 1.1, there exists a $t$ and a product $A \in \Sigma^t$ such that

$$\rho(A)^{1/t} \leq \check{\rho}(\Sigma) + \varepsilon/2.$$

Let us now pick a set $\Sigma'$ close enough to $\Sigma$. By continuity of the eigenvalues there exists a product $A' \in \Sigma'^t$ with averaged spectral radius $\rho(A')^{1/t} < \rho(A)^{1/t} + \varepsilon/2$, and $\check{\rho}(\Sigma') < \check{\rho}(\Sigma) + \varepsilon$.

To prove that the joint spectral subradius is not continuous, we introduce the following example.

**Example 1.1.** *Consider the set*

$$\Sigma = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\frac{1}{k} & 1 \end{pmatrix} \right\}.$$

*Where $k \in \mathbb{N}$. When $k \to \infty$, the joint spectral subradius of these sets is equal to zero (the product $(A_1 A_0^k)^2$ is the zero matrix). However these sets tend to*

$$\Sigma = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

*whose joint spectral subradius is equal to* 1. *Indeed any matrix in the semigroup is nonnegative, and has the lower right entry equal to one.*

#### 1.2.2.9 Zero Spectral Radius

The case where the joint spectral radius (resp. joint spectral subradius) is equal to zero is of practical importance for obvious reasons. In order to state them, following to [44], we introduce the following definition, which holds for the rest of this monograph, unless specified otherwise. A *polynomial time algorithm* is an algorithm that takes an instance and delivers an answer "yes" or "no", after having performed a number of elementary operations that is bounded by a fixed polynomial in the size of the instance, where the size of the instance is its "bit size", that is, the number of bits necessary to encode it.

The following two results are not trivial. Their proofs are to be found in Chapter 2:

**Proposition 1.12.** *There is a polynomial time algorithm allowing to decide whether the joint spectral radius of a set of matrices is zero.*

**Proposition 1.13.** *There is no algorithm allowing to decide whether the joint spectral subradius of a set of matrices is zero, that is, this problem is undecidable.*

## 1.3 Stability of Dynamical Systems

As explained in the introduction, one possible use of the joint spectral radius is to characterize the maximal asymptotic behavior of a dynamical system. But is this exactly what we are doing, when we compute a joint spectral radius? The notion of stability of a dynamical system (like the system defined in Equation (1.1)) is somewhat fuzzy in the literature, and many different (and not equivalent) definitions appear. According to the natural intuition, and to the more commonly used definition, we introduce the next definition:

**Definition 1.3.** *A switched dynamical system*

$$\begin{aligned} x_{t+1} &= A_t x_t : \quad A_t \in \Sigma, \\ x_0 &\in \mathbb{R}^n, \end{aligned} \tag{1.7}$$

*is* stable *if for any initial condition* $x_0 \in \mathbb{R}^n$, *and any sequence of matrices* $\{A_t\}$, $\lim_{t\to\infty} x_t = 0$.

Clearly, if $\rho(\Sigma) < 1$, then the dynamical system is stable, because $x_t = Ax_0$, with $A \in \Sigma^t$, and so $|x_t| \leq ||A||\,|x_0| \to 0$. But the converse statement is less obvious: could the condition $\rho < 1$ be too strong for stability? Could it be that for any length, one is able to provide a product of this length that is not too small, but yet that any *actual trajectory,* defined by an infinite sequence of matrices, is bound to tend to zero? The next example shows that such a case appears with unbounded sets:

**Example 1.2.** *Let*

$$\Sigma = \left\{ A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ B_k = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, k \in \mathbb{R} \right\}.$$

*For any length* $t$, $\hat{\rho}_t = \infty$, *but one can check easily that every infinite product tends to zero. To see this, observe that a left-infinite product must be of one of these three forms, each of which tends to zero*

$$\begin{aligned} ||\ldots AA|| &\approx (1/2)^t, \\ ||\ldots A \ldots AB_kA|| &\approx k(1/2)^{t-1}, \\ ||\ldots A \ldots AB_kA \ldots AB_{k'}A|| &= 0. \end{aligned}$$

The following theorem ensures that such a pathological situation does not appear with bounded sets:

**Theorem 1.2.** *[5] For any bounded set of matrices* $\Sigma$, *there exists a left-infinite product* $\ldots A_2A_1$ *that does not converge to zero if and only if* $\rho(\Sigma) \geq 1$.

The proof of this theorem is not trivial, and makes use of results developed in the next chapter. The reader will find a proof of this important result in Section 2.1.

This proves that the joint spectral radius rules the stability of dynamical systems:

**Corollary 1.1.** *For any bounded set of matrices* $\Sigma$, *the corresponding switched dynamical system is stable if and only if* $\rho(\Sigma) < 1$.

In the above theorem, the boundedness assumption cannot be removed, as shown by Example 1.2.

The equivalent problem for the joint spectral subradius is obvious: For any bounded set of matrices $\Sigma$, the corresponding switched dynamical system is stabilizable (*i.e.* there exists an infinite product of matrices whose norm tends to zero) if and only if $\check{\rho}(\Sigma) < 1$. Indeed, if $\check{\rho} < 1$, there exists a real $\gamma$, and a finite product $A \in \Sigma^t$ such that $||A|| \leq \gamma < 1$, and $\lim_{k\to\infty} A^k = 0$. On the other hand, if $\check{\rho} \geq 1$, then for all $A \in \Sigma^t : ||A|| \geq 1$, and so no long product of matrices tends to zero. There is however a nontrivial counterpart to Corollary 1.1. To see this, let us rephrase Theorem 1.2 in the following corollary:

**Corollary 1.2.** *For any bounded set of matrices $\Sigma$, there is an infinite product of these matrices reaching the joint spectral radius. More precisely, there is a sequence of matrices $A_0, A_1, \ldots$ of $\Sigma$ such that*

$$\lim_{t\to\infty} ||A_t \ldots A_1||^{1/t} = \rho(\Sigma).$$

*Proof.* The proof is a direct consequence of the proof of Theorem 1.2, see Section 2.1.

The idea of this corollary can be transposed to the following result on the joint spectral subradius:

**Theorem 1.3.** *For any (even unbounded) set of matrices $\Sigma$, there is an infinite product of these matrices reaching the joint spectral subradius:*

$$\exists A_{t_j} \in \Sigma : \lim_{i\to\infty} ||A_{t_i} \ldots A_{t_1}||^{1/t} = \check{\rho}(\Sigma).$$

*Proof.* Let $\Sigma$ be a set of matrices. Thanks to the definition and Theorem 1.1, for every natural $k$ there exists a product $B_k \in \Sigma^{n_k}$ of a certain length $n_k$ such that

$$||B_k||^{1/n_k} < \check{\rho} + \frac{1}{2^k}.$$

Now the sequence $||B_t \ldots B_1||^{1/\sum_{1\le k\le t} n_k}$ tends to $\check{\rho}$. However, this only provides a product $\ldots A_2 A_1$ such that $\liminf ||A_t \ldots A_1||^{1/t} = \check{\rho}$. In order to replace the liminf by a limit, for all $k$ we define $c_k$ to be the maximal norm of all the suffixes of $B_k$, and one can raise the matrix $B_k$ to a sufficiently large power $p_k$ such that for any suffix $C$ of $B_{k+1}$,

$$||CB_k^{p_k}||^{1/t} < c_{k+1}^{1/t} ||B_k^{p_k}||^{1/t} < \check{\rho} + \frac{1}{2^{k-1}},$$

and finally the sequence $||\Pi_t||$ converges, where $\Pi_t$ is the suffix of length $t$ of the left infinite product $\ldots B_2^{p_2} B_1^{p_1}$.

## 1.4 Conclusion

The goal of this chapter was to understand properly the notions of joint spectral radius and joint spectral subradius in a glance. As the reader has seen, even some basic facts, such as the equivalence between the joint and generalized spectral radii, require some advanced results. We have thus decided to postpone this proof to Chapter 2. There, the result will naturally follow from a careful study of a particular problem related to the joint spectral radius, namely the *defectiveness* of a set of matrices.

Further elementary properties of the joint spectral radius of sets of matrices can be found in [20, 94, 115, 116].