Conclusion

At the time of writing these lines, *Linear Algebra and its Applications* was editing a special issue devoted to the joint spectral radius. This is another indication of the growing interest and the increasing number of applications of the joint spectral radius. The goal of this work was twofold: to present a survey on the joint spectral radius, and to report the research that had been done during our Ph.D. on this topic. In this conclusion, we quickly recall some points developed in this thesis. We then try to put this work in perspective. We end with a personal conclusion.

Brief summary

Chapters 1 and 2 constitute a survey on the joint spectral radius.

In Chapter 1 we present elementary or fundamental results. Since it was possible to derive their counterpart concerning the joint spectral subradius, we have decided to present them.

In Chapter 2 we present more advanced results and try to understand the very nature of the joint spectral radius. In a first section, we have seen that the whole behavior is simple at first sight: The joint spectral radius is simply reached by commonly irreducible components, and for these components there exists an extremal norm, that is, a common norm that bounds individually the norm of each matrix with the exact value of the joint spectral radius. Moreover, these irreducible components can effectively be computed by quantifier elimination. In Section 2.2 we have seen that the reality is more complex: It is impossible to compute exactly the joint spectral radius. In Section 2.3 we show that despite these infeasibility results, it is possible to approximate the joint spectral radius up to an arbitrary accuracy, and that several algorithms exist, which often appear to be complementary. We end by saying a word on the finiteness property.

Concerning our own research work, two theoretical points were more deeply analyzed: First, the case of nonnegative integer matrices, for which we have delineated the polynomial time feasible questions, versus the infeasible ones. Second, the fascinating finiteness property: in the course of trying to prove that it holds for nonnegative rational (resp. rational) matrices, we have shown that it suffices to prove it for pairs of binary (resp. signed binary) matrices. In addition, we have shown that the property holds for 2×2 binary matrices.

We have also studied a number of applications of the joint spectral radius: We start with a classical one: the continuity of wavelet functions. We then turn to the capacity of codes, for which we have proved some convergence results that are more accurate than for general matrices. We have shown that the question of zero capacity is solvable in polynomial time, but that this is at the border of polynomial time feasibility, since adding don't care characters makes the problem NP-hard. We have then presented a new application of the joint spectral radius to the computation of the asymptotics of overlap-free words, a longstanding question that arises in combinatorics on words. It has been shown recently that our results can be generalized to wider applications in this area, but this still needs further investigations. We finally studied trackable sensor networks, and showed that they are recognizable in polynomial time.

What is next?

To our knowledge, the theoretical questions analyzed in Chapter 2 have not been studied for the joint spectral subradius. Some of them are perhaps not as deep as for the joint spectral radius. Indeed for instance, it is not difficult to show that the finiteness property does not hold for the joint spectral subradius: simple counterexamples exist for which the joint spectral subradius is not reached by a finite product. Nevertheless, we have the feeling that the joint spectral subradius has not been studied as much as it deserves, for instance for what concerns approximation algorithms. Perhaps the negative results mentioned in Chapter 1 are responsible for this situation, but they should not put an end to the analysis of this quantity. In this way of thinking, we present in Chapter 7 new algorithms for estimating the joint spectral subradius, that exhibit good performance in practice, at least on the particular matrices that we studied. We think that future research should analyze these algorithms and their convergence properties.

Research on the joint spectral radius is certainly not an ended story, and we have tried all along this book to emphasize questions that remain unsolved today. Some of them have been studied by several researchers from different communities, like for instance the finiteness conjecture for binary matrices (see Chapter 4). Some others have (to our knowledge) been less studied, like for instance the maximal growth of the products when the joint spectral radius is equal to one. In both cases, we felt it was worth to enlighten them, because they would have important implications in practice. These questions are summarized at the end of each chapter.

An important work that remains to be done, according to us, is a deeper understanding of the algorithms existing to approximate the joint spectral radius. One should most probably try to classify these algorithms, looking closely at their differences and similarities. The presentation of several approximation algorithms in Chapter 2 is intended to be a first step in this direction, but is definitely not a completed work. As mentioned in that chapter, it seems that a fair amount of both theoretical and numerical work is still needed in order to properly understand the different ways of approximating the joint spectral radius.

Finally, from the point of view of applications, we are wondering whether or not the joint spectral radius could be useful for more applied fields of mathematics. Indeed, as soon as a linear dynamical system is involved, and if the generalization to a switched dynamical system makes sense, the use of a joint spectral radius (and related quantities) is very natural. We have the feeling that some applications could benefit from the theoretical advances that researchers have done these last decades on such complex systems.

Personal conclusion

Before ending this book, and to summarize this work, we would like to stress one point: At first sight, and in view of the profusion of negative results on the joint spectral characteristics (undecidability, NP-hardness, non algebraicity,... see Chapter 2), one could have the impression that studying the joint spectral radius is useless. He or she could think that hoping to get an information on a system via a joint spectral radius computation is an utopia.

This is not the case at all.

On the one hand, despite all the infeasibility results, recent contributions have provided several approximation algorithms that appear to be very efficient in practice. Clearly, they require a certain amount of time in order to reach a high precision, but their flexibility often allows one to reach the precision needed. Indeed, a number of facts are of great help in practice and allow computations up to a reasonable accuracy. For instance, some algorithms allow to compute a priori the time needed to reach a given accuracy; also, algorithms of very different nature exist; finally, some algorithms can be tuned depending on algebraic properties of the particular set of matrices under study (non-negative matrices, cone-preserving matrices, commonly irreducible matrices,...). Let us mention for example the case of overlap-free words: even though the size of the matrices was relatively large (twenty by twenty), we have been able to reach a very satisfactory accuracy for the bounds on the joint spectral radius and the other related quantities. What is more, the bounds we have derived significantly outperform preexisting bounds in the literature, that had been derived with other tools.

On the other hand, the theoretical study of joint spectral characteristics is indispensable to understand the intrinsic behavior of complex systems such as switching linear systems. In this more theoretical point of view, the joint spectral radius can be seen as a first step in the understanding of these complex dynamical systems, leading to a number of questions that remain a source of beautiful results nowadays.