Introduction

The *joint spectral radius* characterizes the maximal asymptotic growth rate of a point submitted to a switching linear system in discrete time. In the last decades it has been the subject of intense research due to its role in the study of wavelets, switching systems, approximation algorithms, curve design, and many other topics. In parallel with these practical engineering applications, beautiful theoretical challenges have arisen in the effort to understand the joint spectral radius. These two facts make the study of the joint spectral radius a dream of a subject for a Ph.D. thesis, but perhaps a not so easy task.

Indeed by its natural essence, this notion appears in a number of very different fields of mathematics. For instance, since its definition uses norms and eigenvalues, the joint spectral radius is undoubtedly a linear algebra concept, but not only. It has been defined for purposes of analysis of dynamical systems, and the boost of research on this topic came in the middle 90's from its use in numerical analysis: the joint spectral radius appeared to be the concept needed to determine the continuity of wavelets, a tool of high practical importance nowadays. But the range of applications in which the joint spectral radius has proved useful is much wider; it goes from number theory to network security management, from combinatorics on words to signal processing, etc.

Also, the spectrum of theoretical problems one has to cope with when analyzing the joint spectral radius is wide. In order to solve these problems, results from very different disciplines have been put together: Dynamical systems theory, numerical analysis, theoretical computer science and computability theory, abstract algebra and group theory, graph theory, convex optimization and semidefinite programming (SDP), combinatorics, are a few examples of fields of mathematics that have proved helpful for improving our understanding of problems related to the joint spectral radius. A beautiful example is the contribution of SDP-programming whose usefulness to approximate a joint spectral radius has been progressively understood in the last ten years. This particular contribution is still a subject of research on itself, and seems by now not only to be a state-of-the-art way of approximating the joint spectral radius, but also to bring interesting insight on the very nature of the joint spectral radius.

Undoubtedly, this profusion of different fields of mathematics that have been involved in “the joint spectral radius conquest” does not make its understanding easy. Many researchers with they own (very) personal background, conventions, motivations, notations and definitions have made progress that one who wants to properly understand the joint spectral radius cannot honestly ignore. However, the ideas behind the mathematical constructions are sometimes simpler than they look at first sight. In view of this, we provide in the first part of this monograph a survey on the subject.

In the theoretical survey, which constitutes the first two chapters, we tried to be exhaustive, self-contained, and easily readable at the same time. In order to do that, some proofs differ from the ones given in the literature. Also, the order of presentation of the results does not follow their chronological apparition in the literature, because it allowed sometimes to simplify the proofs. Finally, we decided to split the survey in two chapters: the first one is intended to help the reader to understand the notion of joint spectral radius, by describing its behavior without confusing him with long proofs and theoretical developments, while the second chapter brings the mathematical study of advanced results, and the rigorous demonstrations.

Outline. This monograph is separated in two parts, the first one is dedicated to theoretical and general problems on the joint spectral radius, while the second part is applications-oriented.

The first two chapters form the above mentioned survey: Chapter 1 presents elementary and fundamental results, while Chapter 2 is more involved, and brings the theory necessary to prove the fundamental theorems. In Chapter 1, we compare the results available for the joint spectral radius to its minimum-growth counterpart: the *joint spectral subradius*. Though very interesting and useful in practice, this latter quantity has received far less attention in the literature, perhaps because it has been introduced later. We had the feeling that a rigorous analysis of the basic behavior of this notion was missing.

The remainder of the monograph presents our personal research. We start with two particular theoretical questions: In chapter 3 we analyze the case of nonnegative integer matrices. We show that for these particular sets, it is possible to decide in polynomial time whether the joint spectral radius is exactly equal to zero, exactly equal to one, or larger than one. Moreover it is possible to precisely characterize the growth of the products in the case where the joint spectral radius is exactly equal to one.

In Chapter 4, we analyze the finiteness property. We show that this property holds for nonnegative rational matrices if and only if it holds for pairs of binary matrices. We give a similar result for matrices with negative entries, and we show that the property holds for pairs of 2×2 binary matrices.

The second part of this monograph presents applications of the joint spectral radius. We first present in Chapter 5 the continuity of wavelet functions. Then, in Chapter 6 we go to the capacity of codes submitted to forbidden differences constraints, that can be expressed in terms of a joint spectral radius. We propose two approximation algorithms for the capacity, we show how to efficiently decide whether the capacity is zero, and exhibit a closely related problem that we prove to be NP-hard.

We then turn to a problem in combinatorics on words: estimating the asymptotic growth of the overlap-free language (Chapter 7). We show how this problem is related with the joint spectral radius and related quantities. Thanks to this, we provide accurate estimates for the rate of growth of the number of overlap-free words, a classical problem in combinatorics on words. We also provide algorithms to estimate the joint spectral subradius and the Lyapunov exponent that appear to perform extremely well in practice.

We finally analyze a problem related to graph theory and network security: we present the trackability of sensor networks (Chapter 8) and show how this problem is efficiently tractable.