Concept Granular Computing Based on Lattice Theoretic Setting

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Abstract. Based on the theory of concept lattice and fuzzy concept lattice, a mathematical model of a concept granular computing system is established, and relationships of the system and concept lattices, various variable threshold concept lattices and fuzzy concept lattices are then investigated. For this system, concept granules, sufficiency information granules and necessity information granules which are used to express different relations between a set of objects and a set of attributes are proposed. Approaches to construct sufficiency and necessity information granules are also shown. Some iterative algorithms to form concept granules are proposed. It is proved that the concept granules obtained by the iterative algorithms are the sub-concept granules or sup-concept granules under some conditions for this system. Finally, we give rough approximations based on fuzzy concept lattice in formal concept analysis.

1 Introduction

A concept is the achievement of human recognizing the world. It announces the essence to distinguish one object from the others. Meanwhile, a concept is also a unit of human thinking and reasoning. New concepts are often produced by the original known ones. Thus, a concept is regarded as an information granule, and it plays an important role in our perception and recognition. In 1979, Zadeh first introduced the notion of fuzzy information granules [45]. From then on, many researchers paid much attention to the thought of information granules, and applied it to many fields such as rough set, fuzzy set and evidence reasoning [14,19]. The notion of granularity was proposed by Hobbs in 1985 [20], and granular computing was first provided by Zadeh from 1996 to 1997 [46,47]. Since then, granular computing as a fundamental thought has stepped to soft computing, knowledge discovery and data mining, and has obtained some good results [26,32,41].

Formal concept analysis (FCA), proposed by Wille in 1982 [37], is a mathematical framework for discovery and design of concept hierarchies from a formal context. It is an embranchment of applied mathematics, which made it need mathematical thinking

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for applying FCA to data analysis and knowledge processing [13]. All formal concepts of a formal context with their specification and generalization form a concept lattice [18]. And the concept lattice can be depicted by a Hassen diagram, where each node expresses a formal concept. The concept lattice is the core structure of data in FCA. In essence, a formal concept represents a relationship between the extension of a set of objects and the intension of a set of attributes, and the extension and the intension are uniquely determined each other. The more the formal concepts can be obtained, the stronger the ability is to recognize the world. Thus FCA is regarded as a power tool for learning problems [11,21,22,24,25].

Recently, there has been much advance in the study for FCA, especially in the study of the combination of FCA with the theory of rough set [12,23,28,30,35,38]. Zhang etc. proposed the theory and approach of attribute reduction of concept lattice with the formal context being regarded as a 0-1 information table, and introduced the judgment theorems of attribute reduction [48-50]. In their paper, they also introduced a decision formal context, and then acquired decision rules from it. Yao studied relations between FCA and the theory of rough set [42-44]. Burusco and Belohlavek investigated fuzzy concept lattices of *L*-fuzzy formal context [1-10]. Fan etc. discussed reasoning algorithm of the fuzzy concept lattice based on a complete residuated lattice, studied the relationships among various variable threshold fuzzy concept lattices, and proposed fuzzy inference methods [17]. Ma etc. constructed relations between fuzzy concept lattices and granular computing [26]. Qiu gave the iterative algorithms of concept lattices [29]. Shao etc. established the set approximation in FCA [31].

In this paper, a mathematical model of a concept granular computing system is introduced based on the study of concept lattice and fuzzy concept lattice. Relationships among this system and concept lattice, fuzzy concept lattice and variable threshold concept lattice are investigated. Properties of the system are then studied. To describe the relations between a set of objects and a set of attributes, sufficiency information granules and necessity information granules are defined. Iterative algorithms of a concept granular computing system are proposed to obtain the information granules. And rough approximations of a set based on the concept lattice are studied. It may supply another way to study FCA.

This paper is organized as follows. In section 2, we review basic notions and properties of concept lattice and fuzzy concept lattice. Then we propose a mathematical model called a concept granular computing system in Section 3. Relationships among this system and concept lattice, variable threshold concept lattice and fuzzy concept lattice are investigated. In Section 4, we study properties of this system. And sufficiency information granules and necessity information granules are defined in Section 5. We propose some iterative algorithms to produce concept granules in Section 6. Finally, set approximation in FCA is studied in Section 7. The paper is then concluded with a summary in Section 8.

2 Preliminaries

To facilitate our discussion, this section reviews some notions and results related to concept lattice and fuzzy concept lattice. The following definitions and theorems are the relevant facts about concept lattice and fuzzy concept lattice [2,18,19].

In FCA, the data for analysis is described as a formal context, on which we can construct formal concepts. All formal concepts form a concept lattice which explains hierarchical relations of concepts.

Definition 1. A triplet (U, A, I) is called a formal context, where $U = \{x_1, \cdots, x_n\}$ is a nonempty and finite set called the universe of discourse, every element $x_i (i \le n)$ is an object; $A = \{a_1, \cdots, a_m\}$ is a nonempty and finite set of attributes, every element $a_j (j \le m)$ is an attribute; and $I \subseteq U \times A$ is a binary relation between U and A.

For a formal context (U, A, I), $x \in U$ and $a \in A$, we use $(x, a) \in I$, or xIa, denotes that the object x has the attribute a. If we use 1 and 0 to express $(x, a) \in I$ and $(x, a) \notin I$, respectively, then the formal context can be described as a 0-1 information table.

Let (U, A, I) be a formal context, $X \subseteq U$ and $B \subseteq A$, we define a pair of operators:

$$X^* = \{a : a \in A, \forall x \in X, xIa\} \tag{1}$$

$$B^* = \{x : x \in U, \forall a \in B, xIa\} \tag{2}$$

where X^* denotes the set of attributes common to the objects in X, and B^* is the set of objects possessing all attributes in B. For simplicity, for any $x \in U$ and $a \in A$, we use x^* and a^* instead of $\{x\}^*$ and $\{a\}^*$, respectively. For any $x \in U$ and $a \in A$, if $x^* \ne \varnothing$, $x^* \ne A$, and $a^* \ne \varnothing$, $a^* \ne U$, we call the formal context (U, A, I) is regular. In this paper, we suppose the formal contexts we discussed are regular.

Definition 2. Let (U, A, I) be a formal context, $X \subseteq U$ and $B \subseteq A$. A pair (X, B) is referred to as a formal concept, or a concept if $X^* = B$ and $X = B^*$. We call X the extension and B the intension of the concept (X, B).

Proposition 1. Let (U, A, I) be a formal context. Then for any $X_1, X_2, X \subseteq U$ and $B_1, B_2, B \subseteq A$, we can obtain that:

(P1) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$, $B_1 \subseteq B_2 \Rightarrow B_2^* \subseteq B_1^*$;
(P2) $X \subseteq X^{**}$, $B \subseteq B^{**}$;
(P3) $X^* = X^{***}$, $B^* = B^{***}$;
(P4) $X \subseteq B^* \Leftrightarrow B \subseteq X^*$;
(P5) $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$, $(B_1 \cup B_2)^* = B_1^* \cap B_2^*$;

(P6) $(X_1 \cap X_2)^* \supseteq X_1^* \cup X_2^*$, $(B_1 \cap B_2)^* \supseteq B_1^* \cup B_2^*$;
(P7) (X^{**}, X^*) and (B^*, B^{**}) are always concepts.

In Definition 2, concepts are constructed based on a classical formal context with the binary relation between objects and attributes being either 0 or 1. In the real world, however, the binary relation between objects and attributes are fuzzy and uncertain. Burusco etc. extended the classical model to a fuzzy formal context [8,40], on which fuzzy concepts are first established.

Let L be a complete lattice. We denote by L^U the set of all L-fuzzy sets defined on U. Then for any L-fuzzy sets $\tilde{X}_1, \tilde{X}_2 \in L^U$, for any $x \in U$, $\tilde{X}_1 \subseteq \tilde{X}_2 \Leftrightarrow \tilde{X}_1(x) \leq \tilde{X}_2(x)$. Then $(L^U, \subseteq)$ forms a poset. Obviously, $([0,1]^U, \subseteq)$ and $(\{0,1\}^U, \subseteq)$ are both posets.

We denote by $P(U)$ and $P(A)$ the power set on the universe of discourse U and the power set on the set of attributes A, respectively.

Definition 3. A triplet $(U, A, \tilde{I})$ is referred to as a L-fuzzy formal context, where U is a universe of discourse, A is a nonempty and finite set of attributes, and $\tilde{I}$ is a L-fuzzy relation between U and A, i.e. $\tilde{I} \in L^{U \times A}$.

$\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is referred to as a complete residuated lattice, if $(L, \vee, \wedge, 0, 1)$ is a complete lattice with the least element 0 and the great element 1; $(L, \otimes, 1)$ is a commutative semigroup with unit element 1; and $(\otimes, \rightarrow)$ is a residuated pair of L, i.e. $\otimes : L \times L \rightarrow L$ is monotone increasing , $\rightarrow : L \times L \rightarrow L$ is non-increasing for the first variable and non-decreasing for the second variable, and for any $a, b, c \in L, a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$.

Let $(U, A, \tilde{I})$ be a L-fuzzy formal context, $\tilde{X} \in L^U$ and $\tilde{B} \in L^A$. We define two operators as follows:

$$\tilde{X}^+(a) = \wedge_{x \in U} (\tilde{X}(x) \rightarrow \tilde{I}(x, a)) \tag{3}$$

$$\tilde{B}^+(x) = \wedge_{a \in A} (\tilde{B}(a) \rightarrow \tilde{I}(x, a)) \tag{4}$$

Then $\tilde{X}^+ \in L^A$ and $\tilde{B}^+ \in L^U$.

Definition 4. Let $(U, A, \tilde{I})$ be a L-fuzzy formal context. $(\tilde{X}, \tilde{B})$ is referred to as a fuzzy formal concept, or a fuzzy concept if $\tilde{X}^+ = \tilde{B}$ and $\tilde{B}^+ = \tilde{X}$ for any $\tilde{X} \in L^U$ and $\tilde{B} \in L^A$.

Proposition 2. Let $(U, A, \tilde{I})$ be a L-fuzzy formal context, $\boldsymbol{L} = (L, \vee, \wedge, \otimes, 0, 1)$ be a complete residuated lattice. Then for any $\tilde{X}_1, \tilde{X}_2, \tilde{X} \in L^U$ and $\tilde{B}_1, \tilde{B}_2, \tilde{B} \in L^A$, we have the following properties:

(F1) $\tilde{X}_1 \subseteq \tilde{X}_2 \Rightarrow \tilde{X}_2^+ \subseteq \tilde{X}_1^+$, $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_2^+ \subseteq \tilde{B}_1^+$;

(F2) $\tilde{X} \subseteq \tilde{X}^{++}$, $\tilde{B} \subseteq \tilde{B}^{++}$;

(F3) $\tilde{X}^+ = \tilde{X}^{+++}$, $\tilde{B}^+ = \tilde{B}^{+++}$;

(F4) $\tilde{X} \subseteq \tilde{B}^+ \Leftrightarrow \tilde{B} \subseteq \tilde{X}^+$;

(F5) $(\tilde{X}_1 \bigcup \tilde{X}_2)^+ = \tilde{X}_1^+ \bigcap \tilde{X}_2^+$, $(\tilde{B}_1 \bigcup \tilde{B}_2)^+ = \tilde{B}_1^+ \bigcap \tilde{B}_2^+$;

(F6) $(\tilde{X}^{++}, \tilde{X}^+)$ and $(\tilde{B}^+, \tilde{B}^{++})$ are always fuzzy concepts.

Proposition 3. Let $(U, A, \tilde{I})$ be a L-fuzzy formal context. Note that

$$L_f(U, A, \tilde{I}) = \{(\tilde{X}, \tilde{B}) : \tilde{X}^+ = \tilde{B}, \tilde{B}^+ = \tilde{X}\}$$

For any $(\tilde{X}_1, \tilde{B}_1)$, $(\tilde{X}_2, \tilde{B}_2) \in L_f(U, A, \tilde{I})$, we define a binary relation "$\leq$" as follows:

$$(\tilde{X}_1, \tilde{B}_1) \leq (\tilde{X}_2, \tilde{B}_2) \Leftrightarrow \tilde{X}_1 \subseteq \tilde{X}_2 (\Leftrightarrow \tilde{B}_1 \supseteq \tilde{B}_2).$$

Then "$\leq$" is a partial order on $L_f(U, A, \tilde{I})$, and $(L_f(U, A, \tilde{I}), \leq)$ is a complete lattice, called fuzzy concept lattice, in which the meet and join operators are given by:

$$\wedge_{i \in T} (\tilde{X}_i, \tilde{B}_i) = (\cap_{i \in T} \tilde{X}_i, (\cup_{i \in T} \tilde{B}_i)^{++}),$$
$$\vee_{i \in T} (\tilde{X}_i, \tilde{B}_i) = ((\cup_{i \in T} \tilde{X}_i)^{++}, \cap_{i \in T} \tilde{B}_i).$$

where T is a finite index set.

Obviously, a classical formal context is a special L-fuzzy formal context, i.e. formula (1) and (2) are special situations of formula (3) and (4), respectively.

3 Mathematical Model of Concept Granular Computing System

For a formal context and a fuzzy formal context, by constructing operators between the set of objects and the set of attributes, we obtain concept lattice and fuzzy concept lattice. In this section, we extend the formal context to a generalized setting, and then obtain a mathematical model for concept granular computing system.

Let L be a complete lattice. We denote by 0_L and 1_L the zero element and the unit element of L, respectively.

Definition 5. Let L_1, L_2 be two complete lattices. We call any element in L_1 an extent element and any elements in L_2 an intent element. The mapping $G : L_1 \to L_2$ is referred to as an extent-intent operator if it satisfies:

(G1) $G(0_{L_1}) = 1_{L_2}, G(1_{L_1}) = 0_{L_2}$;

(G2) $G(a_1 \vee a_2) = G(a_1) \wedge G(a_2), \forall a_1, a_2 \in L_1$.

For any $a \in L_1$, $G(a)$ is called the intent element of a. The mapping $H : L_2 \to L_1$ is referred to as an intent-extent operator if it satisfies:

(H1) $H(0_{L_2}) = 1_{L_1}, H(1_{L_2}) = 0_{L_1}$;

(H2) $H(b_1 \vee b_2) = H(b_1) \wedge H(b_2), \forall b_1, b_2 \in L_2$.

For any $b \in L_2$, $H(b)$ is called the extent element of b.

Definition 6. Let G and H be the extent-intent and intent-extent operators on L_1 and L_2, respectively. Furthermore, if for any $a \in L_1$ and $b \in L_2$,

$$a \le H \circ G(a), b \le G \circ H(b),$$

the quadruplex (L_1, L_2, G, H) is referred to as a concept granular computing system, where $H \circ G(a), G \circ H(b)$ is described as $H(G(a))$ and $G(H(b))$ respectively.

Theorem 1. Let (U, A, I) be a formal context. Then the operators $(*, *)$ defined by formula (1) and (2) are extent-intent and intent-extent operators, respectively. And $(P(U), P(A), *, *)$ is a concept granular computing system.

Proof. It immediately follows from Proposition 1.

Theorem 2. Let $(U, A, \tilde{I})$ be a *L*-fuzzy formal context. Then the operators defined by formula (3) and (4) are extent-intent and intent-extent operators, respectively. And $(L^U, L^A, +, +)$ is a concept granular computing system.

Proof. It immediately follows from Proposition 2.

Theorem 3. Let $(P(U), P(A), G, H)$ be a concept granular computing system. Then there exists a binary relation $I \subseteq U \times A$ such that (U, A, I) is a formal context, and $(*, *) = (G, H)$.

Proof. Note that

$$I = \{(x, a) : x \in H(\{a\})\}$$

Then (U, A, I) is a formal context. For any $B \subseteq A$, we have $H(B) = \bigcap_{a \in B} H(\{a\})$ by (H2). Thus

$$\begin{aligned} B^* &= \{x \in U : \forall a \in B, x \in H(\{a\})\} \\ &= \{x \in U : x \in \bigcap_{a \in B} H(\{a\}) = H(B)\} = H(B) \end{aligned}$$

By (G2), for any $X_1 \subseteq X_2$, we have

$$G(X_2) = G(X_1 \cup X_2) = G(X_1) \cap G(X_2).$$

Then $X_1 \subseteq X_2$ implies $G(X_2) \subseteq G(X_1)$. Thus, for $x \in H(\{a\})$, we have $G(\{x\}) \supseteq G(H(\{a\})) \supseteq \{a\}$. That is, $a \in G(\{x\})$. Analogously, we can prove that $B_1 \subseteq B_2 \Rightarrow H(B_2) \subseteq H(B_1)$. By which and $a \in G(\{x\})$ we can get that $x \in H(\{a\})$. Thus $x \in H(\{a\})$ iff $a \in G(\{x\})$. Therefore,

$$I = \{(x,a) : a \in G(\{x\})\}.$$

For any $X \subseteq U$, by property (G2) we have $G(X) = \bigcap_{x \in X} G(\{x\})$. Thus,

$$\begin{aligned} X^* &= \{a \in A : \forall x \in X, a \in G(\{x\})\} \\ &= \{a \in A : a \in \bigcap_{x \in X} G(\{x\}) = G(X)\} = G(X). \end{aligned}$$

We denote by $L = [0,1]$ a unit interval. Then $\boldsymbol{L} = ([0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice. We call the L-fuzzy formal context $(U, A, \tilde{I})$ with $L = [0,1]$ a fuzzy formal context. Then for any $X \in P(U)$, $B \in P(A)$ and $0 < \delta \leq 1$, we define two operators as follows:

$$X^{\#} = \{a \in A : \wedge_{x \in X} (X(x) \rightarrow \tilde{I}(x,a)) \geq \delta\} \tag{5}$$

$$B^{\#} = \{x \in U : \wedge_{a \in B} (B(a) \rightarrow \tilde{I}(x,a)) \geq \delta\} \tag{6}$$

Theorem 4. A quadruplex $(P(U), P(A), \#, \#)$ is a concept granular computing system.

Proof. Obviously, $P(U), P(A)$ are complete lattices. According to formula (5), the operator $\# : P(U) \rightarrow P(A)$ satisfies $\varnothing^{\#} = A$ and $U^{\#} = \varnothing$. Since $\vee x_i \rightarrow a = \wedge(x_i \rightarrow a)$, then for any $X_1, X_2 \in P(U)$, it follows that

$$\begin{aligned} (X_1 \cup X_2)^{\#} &= \{a \in A : \wedge_{x \in X_1 \cup X_2} ((X_1(x) \vee X_2(x)) \rightarrow \tilde{I}(x,a)) \geq \delta\} \\ &= \{a \in A : \wedge_{x \in X_1 \cup X_2} ((X_1(x) \rightarrow \tilde{I}(x,a)) \wedge (X_2(x) \rightarrow \tilde{I}(x,a))) \geq \delta\} \\ &= \{a \in A : \wedge_{x \in X_1} (X_1(x) \rightarrow \tilde{I}(x,a)) \geq \delta\} \cap \{a \in A : \wedge_{x \in X_2} (X_2(x) \rightarrow \tilde{I}(x,a)) \geq \delta\} \\ &= X_1^{\#} \cap X_2^{\#}. \end{aligned}$$

Thus, the operator $\# : P(U) \rightarrow P(A)$ is an extent-intent operator. Similarly, we can prove the operator $\# : P(A) \rightarrow P(U)$ is an intent-extent operator.

Meanwhile, for any $X \in P(U)$, since

$$\begin{aligned} X^{\#} &= \{a \in A : \wedge_{x \in X} (X(x) \rightarrow \tilde{I}(x,a)) \geq \delta\}, \\ X^{\#\#} &= \{x \in U : \wedge_{a \in X^{\#}} (X^{\#}(a) \rightarrow \tilde{I}(x,a)) \geq \delta\}, \end{aligned}$$

by formula (5) we have, for any $x \in X$, if $a \in X^{\#}$, we have $1 \to \tilde{I}(x,a) \geq \delta$. Because $X^{\#}$ is a crisp set, we have $X^{\#\#} = \{x \in U : \wedge_{a \in X^{\#}} (1 \to \tilde{I}(x,a)) \geq \delta\}$. Therefore, $x \in X$ implies $\wedge_{a \in X^{\#}} (X^{\#}(a) \to \tilde{I}(x,a)) \geq \delta$. That is, $x \in X^{\#\#}$. Thus, $X \subseteq X^{\#\#}$. Similarly, it can be proved that for any $B \in P(A)$, $B \subseteq B^{\#\#}$. Thus, $(P(U), P(A), \#, \#)$ is a concept granular computing system.

Let $(U, A, \tilde{I})$ be a fuzzy formal context. For any $X \in P(U)$, $\tilde{B} \in L^{A}$, and $0 < \delta \leq 1$, a pair of operators are defined as follows:

$$X^{\Delta}(a) = \delta \to \wedge_{x \in X} \tilde{I}(x,a) \qquad (a \in A) \tag{7}$$

$$\tilde{B}^{\nabla} = \{x \in U : \wedge_{a \in A} (\tilde{B}(a) \to \tilde{I}(x,a)) \geq \delta\} \tag{8}$$

Theorem 5. A quadruplex $(P(U), L^{A}, \Delta, \nabla)$ is a concept granular computing system.

Proof. It is similarly proved as Theorem 4.

Let $(U, A, \tilde{I})$ be a fuzzy formal context. For any $\tilde{X} \in L^{U}$, $B \in P(A)$, and $0 < \delta \leq 1$, a pair of operators are defined as follows:

$$\tilde{X}^{\nabla} = \{a \in A : \wedge_{x \in U} (\tilde{X}(x) \to \tilde{I}(x,a)) \geq \delta\} \tag{9}$$

$$B^{\Delta}(x) = \delta \to \wedge_{a \in B} \tilde{I}(x,a) \qquad (x \in U) \tag{10}$$

Theorem 6. A quadruplex $(L^{U}, P(A), \nabla, \Delta)$ is a concept granular computing system.

Proof. It is similarly proved as Theorem 4.

4 Properties of Concept Granular Computing System

Definition 7. Let (L_1, L_2, G, H) be a concept granular computing system. If for any $a \in L_1$ and $b \in L_2$, $G(a) = b$ and $H(b) = a$, then the pair (a,b) is called a concept. We call a the extension and b the intension of the concept (a,b).

For any concepts $(a_1, b_1), (a_2, b_2)$, we define a binary relation "$\leq$" as follows:

$$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 \leq a_2 .$$

Then "$\leq$" is a partial order.

Let (L_1, L_2, G, H) be a concept granular computing system. By the operators G and H, a bridge between the extent set and the intent set is constructed, which describe the transformation process of objects and attributes for the recognition.

Theorem 7. Let (L_1, L_2, G, H) be a concept granular computing system. Then the following conclusions hold:

(1) $a_1 \le a_2 \Rightarrow G(a_2) \le G(a_1)$, for any $a_1, a_2 \in L_1$;

(2) $b_1 \le b_2 \Rightarrow H(b_2) \le H(b_1)$, for any $b_1, b_2 \in L_2$;

(3) $G(a_1) \vee G(a_2) \le G(a_1 \wedge a_2)$ for any $a_1, a_2 \in L_1$;

(4) $H(b_1) \vee H(b_2) \le H(b_1 \wedge b_2)$, for any $b_1, b_2 \in L_2$;

(5) $G \circ H \circ G(a) = G(a)$ for any $a \in L_1$;

(6) $H \circ G \circ H(b) = H(b)$ for any $b \in L_2$.

Proof. (1) Suppose $a_1, a_2 \in L_1$, and $a_1 \le a_2$. Since G is an extent-intent operator, we have

$$G(a_2) = G(a_1 \vee a_2) = G(a_1) \wedge G(a_2).$$

Thus, $G(a_2) \le G(a_1)$.

(2) It is similarly proved as (1).

(3) Because $a_1 \wedge a_2 \le a_1$ and $a_1 \wedge a_2 \le a_2$, by (1) we can get that

$$G(a_1) \le G(a_1 \wedge a_2) \text{ and } G(a_2) \le G(a_1 \wedge a_2).$$

Then $G(a_1) \vee G(a_2) \le G(a_1 \wedge a_2)$.

(4) It is similarly proved as (3).

(5) Since for any $a \in L_1$, $a \le H \circ G(a)$, then by (1) we can get that $G \circ H \circ G(a) \le G(a)$. Meanwhile, let $b = G(a)$, we have $b \le G \circ H(b)$. Thus, $G(a) \le G \circ H \circ G(a)$, which leads to $G(a) = G \circ H \circ G(a)$.

(6) It is similarly proved as (5).

Theorem 8. Let (L_1, L_2, G, H) be a concept granular computing system. Note that

$$\mathfrak{B}(L_1, L_2, G, H) = \{(a,b) : G(a) = b, H(b) = a\}$$

Then $\mathfrak{B}(L_1, L_2, G, H)$ is a lattice with a great element and a least element, where the infimum and the supremum are defined as follows

$$\underset{i\in T}{\wedge}(a_i, b_i) = (\underset{i\in T}{\wedge} a_i, G \circ H(\underset{i\in T}{\vee} b_i)),$$

$$\underset{i\in T}{\vee}(a_i, b_i) = (H \circ G(\underset{i\in T}{\vee} a_i), \underset{i\in T}{\wedge} b_i),$$

where $(a_i, b_i) \in \mathfrak{B}(L_1, L_2, G, H)$ ($i \in T, T$ is a finite index set).

Proof. Since $(a_i, b_i) \in \mathfrak{B}(L_1, L_2, G, H)$, we have $G(a_i) = b_i, H(b_i) = a_i$. Thus,

$$G(\underset{i\in T}{\wedge} a_i) = G(\underset{i\in T}{\wedge} H(b_i)) = G(H(\underset{i\in T}{\vee} b_i)) = G \circ H(\underset{i\in T}{\vee} b_i),$$

$$H \circ G \circ H(\vee_{i\in T} b_i) = H(\vee_{i\in T} b_i) = \wedge_{i\in T} H(b_i) = \wedge_{i\in T} a_i .$$

Then $\wedge_{i\in T}(a_i, b_i) \in \mathfrak{B}(L_1, L_2, G, H)$. Similarly, we can prove that

$$\vee_{i\in T}(a_i, b_i) \in \mathfrak{B}(L_1, L_2, G, H) .$$

Since $G(0_{L_1}) = 1_{L_2}, H(1_{L_2}) = 0_{L_1}$, by the partial order $\le$ we have $(0_{L_1}, 1_{L_2})$ is the least element of $\mathfrak{B}(L_1, L_2, G, H)$. Similarly, $(1_{L_1}, 0_{L_2})$ is the great element of $\mathfrak{B}(L_1, L_2, G, H)$.

In order to prove $\mathfrak{B}(L_1, L_2, G, H)$ is a lattice, we need to prove that $\wedge_{i\in T}(a_i, b_i)$ is the great lower bound of $(a_i, b_i)(i \in T)$ and $\vee_{i\in T}(a_i, b_i)$ is the least upper bound of $(a_i, b_i)(i \in T)$. Since $\wedge_{i\in T} a_i \le a_i$, we have $\wedge_{i\in T}(a_i, b_i) = (\wedge_{i\in T} a_i, G \circ H(\vee_{i\in T} b_i)) \le (a_i, b_i)$. That is, $\wedge_{i\in T}(a_i, b_i)$ is the lower bound of $(a_i, b_i)(i \in T)$. Suppose $(a, b) \in \mathfrak{B}(L_1, L_2, G, H)$ and $(a, b) \le (a_i, b_i)$ for any $i \in T$. Then $(a, b) \le \wedge_{i\in T}(a_i, b_i)$, and we can get that $\wedge_{i\in T}(a_i, b_i)$ is the great lower bound of $(a_i, b_i)(i \in T)$. Similarly, we can prove that $\vee_{i\in T}(a_i, b_i)$ is the least upper bound of $(a_i, b_i)(i \in T)$. Therefore, $\mathfrak{B}(L_1, L_2, G, H)$ is a lattice with a great element and a least element.

According to the relationships between the concept granular computing system and concept lattice, variable threshold concept lattice and fuzzy concept lattice, we can get the following results from Theorem 8.

(1) Let (U, A, I) be a formal context. Then

$$B(P(U), P(A), *, *) = \{(X, B), X^* = B, B^* = X\}$$

is a complete lattice.

(2) Let $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a complete residuated lattice, and $(U, A, \tilde{I})$ be a L-fuzzy formal context. Then

$$B(L^U, L^A, +, +) = \{(\tilde{X}, \tilde{B}) : \tilde{X}^+ = \tilde{B}, \tilde{B}^+ = \tilde{X}\} = L_f(U, A, \tilde{I})$$

is a lattice with the great and the least elements.

(3) Let $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a complete residuated lattice, and $(U, A, \tilde{I})$ be a fuzzy formal context. Then

$$B(P(U), P(A), \#, \#) = \{(X, B) : X^\# = B, B^\# = X\}$$

is a lattice with the great and the least elements, and any element in it is called a crisp-crisp variable threshold concept, for simply, variable threshold concept.

(4) Let $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice, and $(U, A, \tilde{I})$ be a fuzzy formal context. Then

$$B(P(U), L^A, \Delta, \nabla) = \{(X, \tilde{B}) : X^{\Delta} = \tilde{B}, \tilde{B}^{\nabla} = X\}$$

is a lattice with the great and the least elements, and any element in it is called a crisp-fuzzy variable threshold concept, for simply, variable threshold concept.

(5) Let $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ be a complete residuated lattice, and $(U, A, \tilde{I})$ be a fuzzy formal context. Then

$$B(L^U, P(A), \nabla, \Delta) = \{(\tilde{X}, B) : \tilde{X}^{\nabla} = B, B^{\Delta} = \tilde{X}\}$$

is a lattice with the great and the least elements, and any element in it is called a fuzzy-crisp variable threshold concept, for simply, variable threshold concept.

Example 1. Table 1 shows a fuzzy formal context with $U = \{1, 2, 3, 4\}$ being a set of objects and $A = \{a, b, c, d\}$ being a set of attributes.

Table 1. The fuzzy formal context $(U, A, \tilde{I})$

U	a	B	C	d
1	0.5	1.0	0.7	0.5
2	0.6	0.7	1.0	0.5
3	1.0	0.9	1.0	0.1
4	1.0	0.9	0.9	0.1

We take Luksiewicz implication operators [15,33,34]

$$a \rightarrow_L b = \begin{cases} 1, & a \leq b, \\ 1 - a + b, & a > b. \end{cases}$$

Then the corresponding adjoin operator is:

$$a \otimes_L b = (a + b - 1) \vee 0 .$$

It is easy to prove that $\boldsymbol{L} = ([0,1], \vee, \wedge, \otimes_L, \rightarrow_L, 0, 1)$ is a residuated complete lattice.

For the fuzzy formal context $(U, A, \tilde{I})$ given in Table 1, take $\delta = 1$. Then for any $X \in P(U)$ and $B \in P(A)$, by formula (1) and (2) we can get $X^{\#}$ and $B^{\#}$. Thus, any crisp-crisp variable threshold concept (X, B) satisfying $X^{\#} = B$ and $B^{\#} = X$ can be obtained. Table 2 shows all crisp-crisp variable threshold concepts.

Analogously, for $\delta = 1$, $X \in P(U)$ and $\tilde{B} \in F(A)$, we can get all crisp-fuzzy variable threshold concepts by formula (7) and (8). Table 3 shows all crisp-fuzzy variable threshold concepts. And for $\delta = 1$, $\tilde{X} \in F(U)$ and $B \in P(A)$, by formula (9) and (10), we can obtain all fuzzy-crisp variable threshold concepts, see Table 4.

Fig. 1 depicts the three kinds of corresponding variable threshold concept lattices. For simplicity, a set is denoted by listing its elements in sequence. For example, the set $\{1,2,3,4\}$ is denoted by 1234.

Table 2. The crisp-crisp variable threshold concepts for $\delta = 1$

X	B
∅	{abcd}
{3}	{ac}
{34}	{a}
{23}	{c}

Table 3. The crisp-fuzzy variable threshold concepts for $\delta = 1$

X	a	b	c	d
∅	1.0	1.0	1.0	1.0
{3}	1.0	0.9	1.0	0.1
{34}	1.0	0.9	0.9	0.1
{2}	0.6	0.7	1.0	0.5
{23}	0.6	0.7	1.0	0.1
{234}	0.6	0.7	0.9	0.1
{1}	0.5	1.0	0.7	0.5
{134}	0.5	0.9	0.7	0.1
{12}	0.5	0.7	0.7	0.5
{1234}	0.5	0.7	0.7	0.1

Table 4. The fuzzy-crisp variable threshold concepts for $\delta = 1$

B	1	2	3	4
∅	1.0	1.0	1.0	1.0
{c}	0.7	1.0	1.0	0.9
{b}	1.0	0.7	0.9	0.9
{bc}	0.7	0.7	0.9	0.9
{a}	0.5	0.6	1.0	1.0
{ac}	0.5	0.6	1.0	0.9
{abc}	0.5	0.6	0.9	0.9
{abcd}	0.5	0.6	0.1	0.1

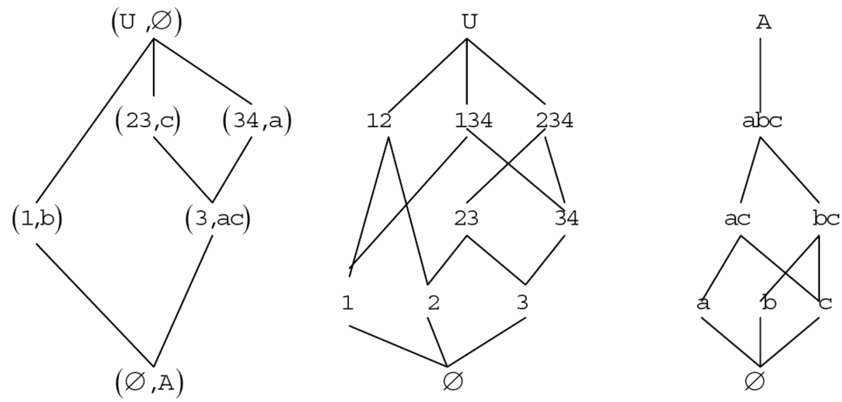

Fig. 1. The corresponding variable threshold concept lattices shown in Table 2-Table 4

5 Sufficiency and Necessity Information Granules

In order to reflect the granular idea of the concept granular computing system, we introduce information granules.

Definition 8. Let (L_1, L_2, G, H) be a concept granular computing system. Note that

$$\mathbb{G}_1 = \{(a,b) : b \le G(a), a \le H(b)\},$$

$$\mathbb{G}_2 = \{(a,b) : G(a) \le b, H(b) \le a\}.$$

If $(a,b) \in \mathbb{G}_1$, we call (a,b) a necessity information granule of the concept granular computing system, and call b the necessity attribute of a. Then $\mathbb{G}_1$ is the set of all necessity information granules of the concept granular computing system. (See Fig. 2.).

If $(a,b) \in \mathbb{G}_2$, we call (a,b) a sufficiency information granule of the concept granular computing system, and call b the sufficiency attribute of a. Then $\mathbb{G}_2$ is the set of all sufficiency information granules of the concept granular computing system. (See Fig. 3).

If $(a,b) \in \mathbb{G}_1 \cup \mathbb{G}_2$, we call (a,b) an information granule of the concept granular computing system. Then $\mathbb{G}_1 \cup \mathbb{G}_2$ is the set of all information granules of the concept granular computing system.

If $(a,b) \in \mathbb{G}_1 \cap \mathbb{G}_2$, then the pair (a,b) satisfies $b = G(a), a = H(b)$, we call (a,b) a sufficiency and necessity information granule of the concept granular computing system, and call b the sufficiency and necessity attribute of a .Then a sufficiency and necessity information granule is actually a concept of a concept granular computing system defined in Definition 7.

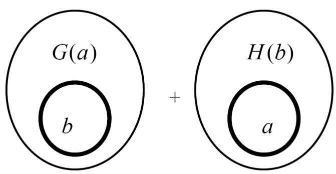

Fig. 2. Necessity information granule (a,b)

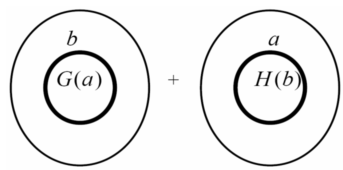

Fig. 3. Sufficiency information granule (a,b)

If $(a,b) \notin \mathbb{G}_1 \cap \mathbb{G}_2$, we call (a,b) a contradiction information granule.

Theorem 9. Let $\mathbb{G}_1$ be a necessity information granule set. For any $(a_1,b_1),(a_2,b_2) \in \mathbb{G}_1$, we define the infimum and the supremum operators on $\mathbb{G}_1$ as follows:

$$(a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge a_2, G \circ H(b_1 \vee b_2)),$$

$$(a_1,b_1) \vee (a_2,b_2) = (H \circ G(a_1 \vee a_2), b_1 \wedge b_2).$$

Then $\mathbb{G}_1$ is closed under the infimum and supremum operators.

Proof. Suppose $(a_1,b_1),(a_2,b_2) \in \mathbb{G}_1$. Then

$$b_1 \le G(a_1), b_2 \le G(a_2), \text{ and } a_1 \le H(b_1), a_2 \le H(b_2).$$

Thus,

$$a_1 \wedge a_2 \le H(b_1) \wedge H(b_2) = H(b_1 \vee b_2) = H \circ G \circ H(b_1 \vee b_2),$$

$$G \circ H(b_1 \vee b_2) = G(H(b_1) \wedge H(b_2)) \le G(a_1 \wedge a_2).$$

Therefore, $(a_1,b_1) \wedge (a_2,b_2)$ is a necessity information granule. Similarly, we can prove that $(a_1,b_1) \vee (a_2,b_2)$ is a necessity information granule.

Theorem 10. Let $\mathbb{G}_2$ be a sufficiency information granule set. For any $(a_1,b_1),(a_2,b_2) \in \mathbb{G}_2$, we define the infimum and the supremum operators on $\mathbb{G}_2$ as follows:

$$(a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge a_2, G \circ H(b_1 \vee b_2)),$$

$$(a_1,b_1) \vee (a_2,b_2) = (H \circ G(a_1 \vee a_2), b_1 \wedge b_2).$$

Then $\mathbb{G}_2$ is closed under the infimum and supremum operators.

Proof. Suppose $(a_1, b_1), (a_2, b_2) \in \mathbb{G}_2$. Then

$$G(a_1) \le b_1, G(a_2) \le b_2, \text{ and } H(b_1) \le a_1, H(b_2) \le a_2 .$$

Thus,

$$H \circ G \circ H(b_1 \vee b_2) = H(b_1 \vee b_2) = H(b_1) \wedge H(b_2) \le a_1 \wedge a_2 ,$$

$$G(a_1 \wedge a_2) \le G(H(b_1) \wedge H(b_2)) = G \circ H(b_1 \vee b_2) .$$

Therefore, $(a_1, b_1) \wedge (a_2, b_2)$ is a sufficiency information granule. Similarly, we can prove that $(a_1, b_1) \vee (a_2, b_2)$ is a sufficiency information granule.

Example 2. Given a formal context (U, A, I) as Table 5, where $U = \{1, 2, 3, 4\}$ is the set of objects, and $A = \{a, b, c, d\}$ is a set of attributes.

Table 5. The formal context (U, A, I)

U	a	b	c	d
1	1	0	1	1
2	1	1	0	0
3	0	0	1	0
4	1	1	0	0

From Table 5, we can get the partial necessity information granules (See Fig. 4) and the partial sufficiency information granules (See Fig. 5).

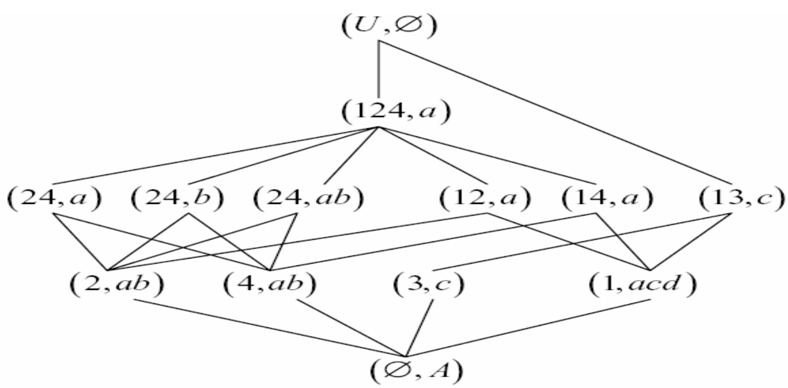

Fig. 4. Partial necessity information granule

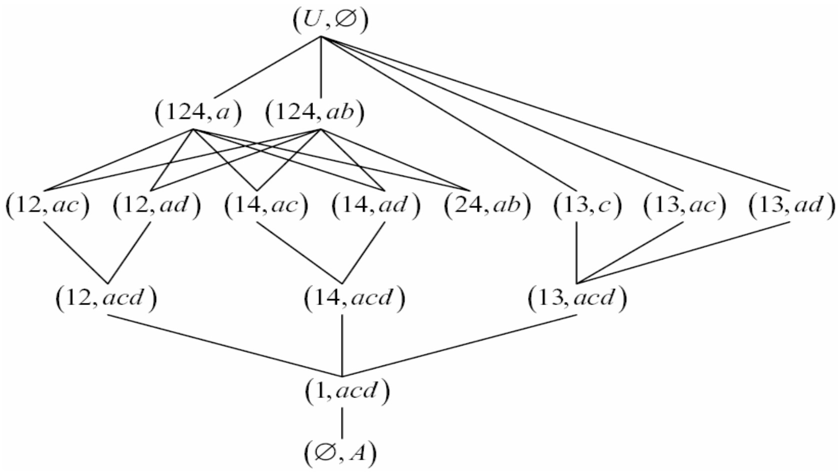

Fig. 5. Partial sufficiency information granules

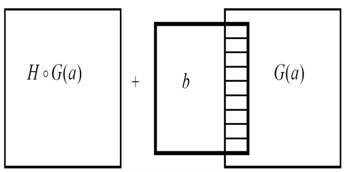

Fig. 6. $(H \circ G(a), b \wedge G(a))$

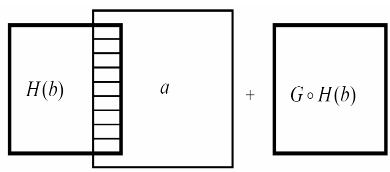

Fig. 7. $(a \wedge H(b), G \circ H(b))$

Now we introduce approaches to construct the sufficiency or necessity information granules.

Theorem 11. Let (L_1, L_2, G, H) be a concept granular computing system, $\mathbb{G}_1$ is the set of necessity information granules. Then for any $a \in L_1$ and $b \in L_2$, we have

$$(H \circ G(a), b \wedge G(a)) \in \mathbb{G}_1 \text{ and } (a \wedge H(b), G \circ H(b)) \in \mathbb{G}_1$$

(See Fig. 6 and Fig. 7).

Proof. Since (L_1, L_2, G, H) is a concept granular computing system, by Theorem 7 and Definition 8 we have $G \circ H \circ G(a) = G(a) \geq G(a) \wedge b$ and $H(b \wedge G(a)) \geq H \circ G(a) \vee H(b) \geq H \circ G(a)$. Thus, $(H \circ G(a), b \wedge G(a)) \in G_1$.

Similarly, we can prove $(a \wedge H(b), G \circ H(b)) \in G_1$.

Theorem 12. Let (L_1, L_2, G, H) be a concept granular computing system, $\mathbb{G}_2$ is the set of sufficiency information granules. Then for any $a \in L_1$ and $b \in L_2$, we have

$$(H \circ G(a), b \vee G(a)) \in \mathbb{G}_2 \text{ and } (a \vee H(b), G \circ H(b)) \in \mathbb{G}_2$$

(See Fig. 8 and Fig. 9).

Proof. Because (L_1, L_2, G, H) is a concept granular computing system, by Theorem 7 and Definition 8 we have $G \circ H \circ G(a) = G(a) \leq G(a) \vee b$ and $H(G(a) \vee b) = H \circ G(a) \wedge H(b) \leq H \circ G(a)$. Thus, $(H \circ G(a), b \vee G(a)) \in G_2$ .Similarly, we can prove that $(a \vee H(b), G \circ H(b)) \in G_2$.

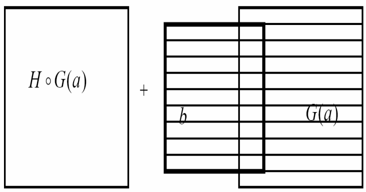

Fig. 8. $(H \circ G(a), b \vee G(a))$

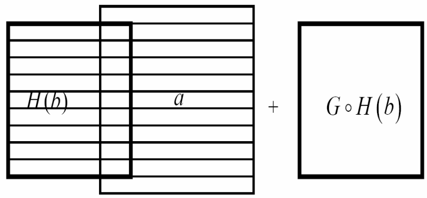

Fig. 9. $(a \vee H(b), G \circ H(b))$

Example 3. The formal context (U, A, I) is the one given as Example 2.

Then $(P(U), P(A), *, *)$ is a concept granular computing system, and we can obtain all formal concepts which form a concept lattice as Fig. 10.

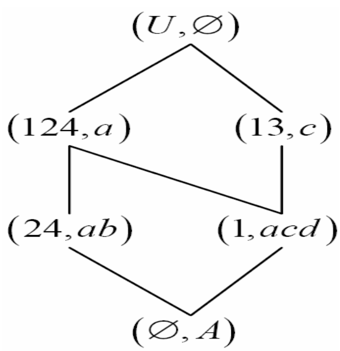

Fig. 10. Concept lattice of Example 2

Take $a_0 = \{1,4\}$ and $b_0 = \{a,b\}$. Then (a_0, b_0) is a contradiction granule. By Theorem 11, we can calculate that $(H \circ G(a_0), b_0 \wedge G(a_0)) = (\{1,4\}^{**}, \{a,b\} \bigcap \{1,4\}^*) = (\{1,2,4\}, \{a\})$ and $(a_0 \wedge H(b_0), G \circ H(b_0)) = (\{1,4\} \cap \{a,b\}^*, \{a,b\}^{**}) = (\{4\}, \{a,b\})$ are two necessity information granules. Similarly, we can construct the contradiction granule to a sufficiency information granule $(124, ab)$ by using Theorem 12.

From Example 3 we know, for any set of objects and any set of attributes, we can construct necessity or sufficiency information granules by using Theorem 11-Theorem 12, which support a way to construct a sufficiency and necessity information granules, i.e. concepts.

6 Iterative Algorithms and Their Optimizations in Concept Granular Computing System

In this section, we establish iterative algorithms to produce concepts from any extent element and intent element.

Theorem 13. Let (L_1, L_2, G, H) be a concept granular computing system, and $|L_1| < \infty$ ($|L_1|$ denotes the number of elements in L_1). For any $a_1 \in L_1$ and $b_1 \in L_2$, an iterative algorithm is given as follows:

$$\begin{cases} a_n = a_{n-1} \vee H(b_{n-1}), (n \geq 2) \\ b_n = G(a_n) \end{cases} \tag{11}$$

Then for the series of pairs $\{(a_n, b_n)\}_{n\geq 1}$, there exists $n_0 \geq 1$ such that

(1) $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$;

(2) For any $(a', b') \in B(L_1, L_2, G, H)$, if $a' \leq a_1 \vee H(b_1)$, then $(a', b') \leq (a_{n_0}, b_{n_0})$.

Proof. (1) By the iterative algorithm given by formula (11) we know the sequence $\{a_n\}_{n\geq 1}$ is monotone non-decreasing. Due to $|L_1| < \infty$, there exists a natural number $n_0 \geq 1$ such that for any $n \geq n_0$, we have $a_n = a_{n_0}$. Again using formula (11) we have $a_{n_0} = a_{n_0+1} = a_{n_0} \vee H(b_{n_0})$ and $b_{n_0} = G(a_{n_0})$. Then $a_{n_0} \geq H(b_{n_0})$. By $b_{n_0} = G(a_{n_0})$ we can get that $H(b_{n_0}) = H \circ G(a_{n_0}) \geq a_{n_0}$. Thus $a_{n_0} = H(b_{n_0})$ and $b_{n_0} = G(a_{n_0})$. Therefore, $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$.

(2) Suppose $(a', b') \in B(L_1, L_2, G, H)$, and $a' \leq a_1 \vee H(b_1)$. If $(a_1, b_1) \notin B(L_1, L_2, G, H)$, we have $a' \leq a_1 \vee H(b_1) = a_2$. Suppose $a' \leq a_n (n \geq 2)$.

Then $b_n = G(a_n) \leq G(a') = b'$. So $a' = H(b') \leq H(b_n)$. Thus, $a' \leq a_n \vee H(b_n) = a_{n+1}$. By the Inductive law we can obtain that for any $n \geq 2$, $a' \leq a_n$. If $(a_1, b_1) \in B(L_1, L_2, G, H)$, then $a' \leq a_1 \vee H(b_1) = a_1$. Therefore, if $a' \leq a_1 \vee H(b_1)$, we have $a' \leq a_n$ for any $n \geq 1$. For the series of pairs

$(a_n, b_n) \in B(L_1, L_2, G, H)$, by (1) there exists a natural number $n_0 \geq 1$ such that $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$. Then by $a' \leq a_n$ for any $n \geq 1$ we have $(a', b') \leq (a_{n_0}, b_{n_0})$.

Theorem 14. Let (L_1, L_2, G, H) be a concept granular computing system, and $|L_2| < \infty$. For any $a_1 \in L_1$ and $b_1 \in L_2$, an iterative algorithm is given as follows:

$$\begin{cases} b_n = b_{n-1} \vee G(a_{n-1}), (n \geq 2) \\ a_n = H(b_n) \end{cases} \tag{12}$$

Then for the series of pairs $\{(a_n, b_n)\}_{n\geq 1}$, there exists $n_0 \geq 1$ such that

(1) $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$;

(2) For any $(a', b') \in B(L_1, L_2, G, H)$, if $b' \leq b_1 \vee G(a_1)$, then $(a_{n_0}, b_{n_0}) \leq (a', b')$.

Proof. (1) By the iterative algorithm given by formula (2.12) we know the sequence $\{b_n\}_{n\geq 1}$ is monotone non-decreasing. Since $|L_2| < \infty$, there exists a natural number $n_0 \geq 1$ such that for any $n \geq n_0$, we have $b_n = b_{n_0}$. Then $b_{n_0} = b_{n_0+1} = b_{n_0} \vee G(a_{n_0})$ and $a_{n_0} = H(b_{n_0})$. Thus, $b_{n_0} \geq G(a_{n_0})$. By $a_{n_0} = H(b_{n_0})$ we can get that $G(a_{n_0}) = G \circ H(b_{n_0}) \geq b_{n_0}$.

So, $b_{n_0} = G(a_{n_0})$ and $a_{n_0} = H(b_{n_0})$. Therefore, $(a_{n_0}, b_{n_0}) \in \mathfrak{B}(L_1, L_2, G, H)$.

(2) Suppose $(a', b') \in B(L_1, L_2, G, H)$, and $b' \leq b_1 \vee G(a_1)$. If $(a_1, b_1) \notin B(L_1, L_2, G, H)$, we have $b' \leq b_1 \vee G(a_1) = b_2$. Suppose $b' \leq b_n (n \geq 2)$. Then $a_n = H(b_n) \leq H(b') = a'$. So $b' = G(a') \leq G(a_n)$. Thus, $b' \leq b_n \vee G(a_n) = b_{n+1}$. By the Inductive law we can obtain that for any $n \geq 2$, $b' \leq b_n$. If $(a_1, b_1) \in B(L_1, L_2, G, H)$, then $b' \leq b_1 \vee G(a_1) = b_1$. Therefore, if $b' \leq b_1 \vee G(a_1)$, we have $b' \leq b_n$ for any $n \geq 1$. For the series of pairs $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$, by (1) there exists a natural number $n_0 \geq 1$ such that $(a_{n_0}, b_{n_0}) \in B(L_1, L_2, G, H)$. Then by $b' \leq b_n$ for any $n \geq 1$ we have $(a_{n_0}, b_{n_0}) \leq (a', b')$.

In what follows, we show the iterative algorithms for a formal context and a fuzzy formal context.

Theorem 15. Let (U, A, I) be a formal context. For any $X \subseteq U$ and $B \subseteq A$, an iterative algorithm is given as follows:

$$\begin{cases} X_n = X_{n-1} \cup B_{n-1}{}^*, (n \geq 2) \\ B_n = X_n{}^* \end{cases}$$

Then for the series of pairs $\{(X_n, B_n)\}_{n \geq 1}$, there exists $n_0 \geq 1$ such that

(1) $(X_{n_0}, B_{n_0}) \in B(P(U), P(A), *, *)$;

(2) For any $(X', B') \in B(P(U), P(A), *, *)$, if $X' \leq X_1 \cup B_1^*$, then $(X', B') \leq (X_{n_0}, B_{n_0})$.

Proof. It is proved by Theorem 1 and Theorem 13.

Theorem 16. Let (U, A, I) be a formal context. For any $X \subseteq U$ and $B \subseteq A$, an iterative algorithm is given as follows:

$$\begin{cases} B_n = B_{n-1} \cup X_{n-1}{}^*, (n \geq 2) \\ X_n = B_n{}^* \end{cases}$$

Then for the series of pairs $\{(X_n, B_n)\}_{n \geq 1}$, there exists $n_0 \geq 1$ such that

(1) $(X_{n_0}, B_{n_0}) \in B(P(U), P(A), *, *)$;

(2) For any $(X', B') \in B(P(U), P(A), *, *)$, if $B' \leq B_1 \cup X_1^*$, then $(X_{n_0}, B_{n_0}) \leq (X', B')$.

Proof. It is proved by Theorem 1 and Theorem 14.

Theorem 17. Let $(U, A, \tilde{I})$ be a *L*-fuzzy formal context. For any $\tilde{X} \subseteq L^U$ and $\tilde{B} \subseteq L^A$,

(1) if an iterative algorithm is given as follows:

$$\begin{cases} \tilde{X}_n = \tilde{X}_{n-1} \cup \tilde{B}_{n-1}{}^+, (n \geq 2) \\ \tilde{B}_n = \tilde{X}_n{}^+ \end{cases}$$

Then for the series of pairs $\{(\tilde{X}_n, \tilde{B}_n)\}_{n \geq 1}$, there exists $n_0 \geq 1$ such that $(\tilde{X}_{n_0}, \tilde{B}_{n_0})$ $\in B(L^U, L^A, +, +)$. And for any $(\tilde{X}', \tilde{B}') \in B(L^U, L^A, +, +)$, if $\tilde{X}' \leq \tilde{X}_1 \cup \tilde{B}_1^+$, then $(\tilde{X}', \tilde{B}') \leq (\tilde{X}_{n_0}, \tilde{B}_{n_0})$;

(2) if an iterative algorithm is given as follows:

$$\begin{cases} \tilde{B}_n = \tilde{B}_{n-1} \cup \tilde{X}_{n-1}{}^+, (n \geq 2) \\ \tilde{X}_n = \tilde{B}_n{}^+ \end{cases}$$

Then for the series of pairs $\{(\tilde{X}_n, \tilde{B}_n)\}_{n\geq 1}$, there exists $n_0 \geq 1$ such that $(\tilde{X}_{n_0}, \tilde{B}_{n_0}) \in B(L^U, L^A, +, +)$. And for any $(\tilde{X}', \tilde{B}') \in B(L^U, L^A, +, +)$, if $\tilde{B}' \leq \tilde{B}_1 \cup \tilde{X}_1^{\ +}$, then $(\tilde{X}_{n_0}, \tilde{B}_{n_0}) \leq (\tilde{X}', \tilde{B}')$.

Proof. It is proved by Theorem 2, Theorem 13 and Theorem 14.

Example 4. Let (U, A, I) be a formal context given in Example 2. Take $X_0 = \{1,4\}$ and $B_0 = \{a,b\}$. (X_0, B_0) is not a formal context. By Theorem 15 and Theorem 16 we can get that $(24, ab)$ and $(124, a)$ are concepts.

7 Rough Set Approximations in Formal Concept Analysis

A structure $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is referred to as a complete involutive residuated lattice if L is a complete residuated lattice and the operator $c: L \rightarrow L$ satisfies $a_1 \leq a_2 \Rightarrow a_2^c \leq a_1^c$ and $a^{cc} = a$ for any $a_1, a_2, a \in L$, where c represents the complement operator of any element of L.

A *L*-fuzzy formal context $(U, A, \tilde{I})$ is called an involutive *L*-fuzzy formal context if $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a complete involutive residuated lattice. Then for any $\tilde{X} \in L^U$ and $\tilde{B} \in L^A$, we define the following operators[27,39]:

$$\tilde{X}^{+}(a) = \wedge_{x\in U} (\tilde{X}(x) \rightarrow \tilde{I}(x,a))$$

$$\tilde{B}^{+}(x) = \wedge_{a\in A} (\tilde{B}(a) \rightarrow \tilde{I}(x,a))$$

$$\tilde{X}^{\uparrow}(a) = \vee_{x\in U} (\tilde{X}^{c}(x) \otimes \tilde{I}^{c}(x,a))$$

$$\tilde{B}^{\downarrow}(x) = \vee_{a\in A} (\tilde{B}^{c}(a) \otimes \tilde{I}^{c}(x,a))$$

$$\tilde{X}^{\Diamond}(a) = \vee_{x\in U} (\tilde{X}(x) \otimes \tilde{I}(x,a))$$

$$\tilde{X}^{\square}(a) = \wedge_{x\in U} (\tilde{I}(x,a) \rightarrow \tilde{X}(x))$$

$$\tilde{B}^{\Diamond}(x) = \vee_{a\in A} (\tilde{B}(a) \otimes \tilde{I}(x,a))$$

$$\tilde{B}^{\square}(x) = \wedge_{a\in A} (\tilde{I}(x,a) \rightarrow \tilde{B}(a))$$

Definition 10. Let $(U, A, \tilde{I})$ be an involutive *L*-fuzzy formal context. For any $\tilde{X} \in L^U$, we define

$$\underline{Apri}(\tilde{X}) = \tilde{X}^{\uparrow\downarrow} \text{ and } \overline{Apri}(\tilde{X}) = \tilde{X}^{++} \tag{13}$$

$\underline{Apri}(\tilde{X})$ and $\overline{Apri}(\tilde{X})$ are referred to as the lower and upper approximations of $\tilde{X}$, respectively. And the operators $\uparrow\downarrow: L^U \rightarrow L^U$ and $++: L^U \rightarrow L^U$ are referred to as the *i*-model lower and upper approximation operators, respectively. The pair $(\underline{Apri}(\tilde{X}), \overline{Apri}(\tilde{X}))$ is referred to as a generalized *i*-model rough fuzzy set.

Theorem 18. Let $(U, A, \tilde{I})$ be an involutive *L*-fuzzy formal context. Then for any $\tilde{X}, \tilde{Y} \in L^U$, the *i*-model lower and upper approximation operators satisfy the following properties:

(FL_1) $\underline{Apri}(\tilde{X}) = (\overline{Apri}(\tilde{X}^c))^c$;

(FU_1) $\overline{Apri}(\tilde{X}) = (\underline{Apri}(\tilde{X}^c))^c$;

(FL_2) $\underline{Apri}(\varnothing) = \overline{Apri}(\varnothing) = \varnothing$;

(FU_2) $\overline{Apri}(U) = \underline{Apri}(U) = U$;

(FL_3) $\underline{Apri}(\tilde{X} \cap \tilde{Y}) \subseteq \underline{Apri}(\tilde{X}) \cap \underline{Apri}(\tilde{Y})$;

(FU_3) $\overline{Apri}(\tilde{X} \cup \tilde{Y}) \supseteq \overline{Apri}(\tilde{X}) \cup \overline{Apri}(\tilde{Y})$;

(FL_4) $\tilde{X} \subseteq \tilde{Y} \Rightarrow \underline{Apri}(\tilde{X}) \subseteq \underline{Apri}(\tilde{Y})$;

(FU_4) $\tilde{X} \subseteq \tilde{Y} \Rightarrow \overline{Apri}(\tilde{X}) \subseteq \overline{Apri}(\tilde{Y})$;

(FL_5) $\underline{Apri}(\tilde{X} \cup \tilde{Y}) \supseteq \underline{Apri}(\tilde{X}) \cup \underline{Apri}(\tilde{Y})$;

(FU_5) $\overline{Apri}(\tilde{X} \cap \tilde{Y}) \subseteq \overline{Apri}(\tilde{X}) \cap \overline{Apri}(\tilde{Y})$;

(FL_6) $\underline{Apri}(\tilde{X}) \subseteq \tilde{X}$;

(FU_6) $\tilde{X} \subseteq \overline{Apri}(\tilde{X})$;

(FL_7) $\underline{Apri}(\underline{Apri}(\tilde{X})) = \underline{Apri}(\tilde{X})$;

(FU_7) $\overline{Apri}(\overline{Apri}(\tilde{X})) = \overline{Apri}(\tilde{X})$.

Proof. (FL_1) and (FU_1) show that the approximation operators $\underline{Apri}$ and $\overline{Apri}$ are dual to each other. Then we only need to prove (FL_i) or (FU_i), by the duality we can easily get (FU_i) or (FL_i) $(i = 1, \cdots, 7)$.

For any $\tilde{X} \in L^U$ and $x \in U$,

$$
\begin{aligned}
(\underline{Apri}(\tilde{X}^c))^c(x) &= ((\tilde{X}^c)^{\uparrow\downarrow}(x))^c \\
&= \vee_{a\in A} (\vee_{y\in U} (\tilde{X}^{cc}(y) \otimes \tilde{I}^c(y,a))^c \otimes \tilde{I}^c(x,a))^c \\
&= \vee_{a\in A} (\wedge_{y\in U} (\tilde{X}(y) \to \tilde{I}(y,a)) \otimes \tilde{I}^c(x,a))^c \\
&= \wedge_{a\in A} (\wedge_{y\in U} (\tilde{X}(y) \to \tilde{I}(y,a)) \to \tilde{I}(x,a)) \\
&= \tilde{X}^{++}(x)
\end{aligned}
$$

Thus, (FU_1) holds. (FL_2) immediately follows by Definition 10.

According to $(\tilde{X} \cup \tilde{Y})^+ = \tilde{X}^+ \cap \tilde{Y}^+$, $\tilde{X}^+ \cap \tilde{Y}^+ \subseteq \tilde{X}^+$ and $\tilde{X}^+ \cap \tilde{Y}^+ \subseteq \tilde{Y}^+$, we have

$$\tilde{X}^{++} \subseteq (\tilde{X}^+ \cap \tilde{Y}^+)^+ \text{ and } \tilde{Y}^{++} \subseteq (\tilde{X}^+ \cap \tilde{Y}^+)^+ .$$

Thus,

$$\overline{Apri}(\tilde{X}) \cup \overline{Apri}(\tilde{Y}) = \tilde{X}^{++} \cup \tilde{Y}^{++} \subseteq (\tilde{X}^+ \cap \tilde{Y}^+)^+ = \overline{Apri}(\tilde{X} \cup \tilde{Y}),$$

from which we can get (FU_3).

(FL_4) follows immediately from $\tilde{X}_1 \subseteq \tilde{X}_2 \Rightarrow \tilde{X}_2^{\uparrow} \subseteq \tilde{X}_1^{\uparrow}$ and $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_2^{\downarrow} \subseteq \tilde{B}^{\downarrow}$.

Since $\tilde{X}^{++} \cap \tilde{Y}^{++} = (\tilde{X}^+ \cup \tilde{Y}^+)^+$, $\tilde{X}^+ \subseteq (\tilde{X} \cap \tilde{Y})^+$ and $\tilde{Y}^+ \subseteq (\tilde{X} \cap \tilde{Y})^+$, we have

$$\tilde{X}^{++} \supseteq (\tilde{X} \cap \tilde{Y})^{++} \text{ and } \tilde{Y}^{++} \supseteq (\tilde{X} \cap \tilde{Y})^{++} .$$

Therefore,

$$\overline{Apri}(\tilde{X}) \cap \overline{Apri}(\tilde{Y}) = \tilde{X}^{++} \cap \tilde{Y}^{++} \supseteq (\tilde{X} \cap \tilde{Y})^{++} = \overline{Apri}(\tilde{X} \cap \tilde{Y}) .$$

Thus, (FU_5) holds.

(FL_6) follows directly by $\tilde{X} \subseteq \tilde{X}^{\uparrow\downarrow}$ and $\tilde{B} \subseteq \tilde{B}^{\downarrow\uparrow}$.

Since $\underline{Apri}(\underline{Apri}(\tilde{X})) = (\tilde{X}^{\uparrow\downarrow})^{\uparrow\downarrow}$ and $\tilde{X}^{\uparrow\downarrow\uparrow} = \tilde{X}^{\uparrow}$, we can get ($FL_7$).

Definition 11. Let $(U, A, \tilde{I})$ be an involutive L-fuzzy formal context. For any $\tilde{X} \in L^U$, we define the lower and upper approximations of $\tilde{X}$ as follows:

$$\underline{Aprii}(\tilde{X}) = \tilde{X}^{\Box\Diamond} \text{ and } \overline{Apri}(\tilde{X}) = \tilde{X}^{\Diamond\Box} \tag{14}$$

Then the operators $\Box\Diamond : L^U \to L^U$ and $\Diamond\Box : L^U \to L^U$ are referred to as the *ii*-model lower and upper approximation operators, respectively. The pair ($\underline{Aprii}(\tilde{X})$, $\underline{Aprii}(\tilde{X})$) is referred to as a generalized *ii*-model rough fuzzy set.

Theorem 19. Let $(U, A, \tilde{I})$ be an involutive L-fuzzy formal context. Then for any $\tilde{X}, \tilde{Y} \in L^U$, the *ii*-model lower and upper approximation operators satisfy the following properties:

(FL'_1) $\underline{Aprii}(\tilde{X}) = (\overline{Aprii}(\tilde{X}^c))^c$;

(FU'_1) $\overline{Aprii}(\tilde{X}) = (\underline{Aprii}(\tilde{X}^c))^c$;

(FL'_2) $\underline{Aprii}(\varnothing) = \overline{Aprii}(\varnothing) = \varnothing$;

(FU'_2) $\overline{Aprii}(U) = \underline{Aprii}(U) = U$;

(FL'_3) $\underline{Aprii}(\tilde{X} \cap \tilde{Y}) \subseteq \underline{Aprii}(\tilde{X}) \cap \underline{Aprii}(\tilde{Y})$;

(FU'_3) $\overline{Aprii}(\tilde{X} \cup \tilde{Y}) \supseteq \overline{Aprii}(\tilde{X}) \cup \overline{Aprii}(\tilde{Y})$;

(FL'_4) $\tilde{X} \subseteq \tilde{Y} \Rightarrow \underline{Aprii}(\tilde{X}) \subseteq \underline{Aprii}(\tilde{Y})$;

(FU'_4) $\tilde{X} \subseteq \tilde{Y} \Rightarrow \overline{Aprii}(\tilde{X}) \subseteq \overline{Aprii}(\tilde{Y})$;

(FL'_5) $\underline{Aprii}(\tilde{X} \cup \tilde{Y}) \supseteq \underline{Aprii}(\tilde{X}) \cup \underline{Aprii}(\tilde{Y})$;

(FU'_5) $\overline{Aprii}(\tilde{X} \cap \tilde{Y}) \subseteq \overline{Aprii}(\tilde{X}) \cap \overline{Aprii}(\tilde{Y})$;

(FL'_6) $\underline{Aprii}(\tilde{X}) \subseteq \tilde{X}$;

(FU'_6) $\tilde{X} \subseteq \overline{Aprii}(\tilde{X})$;

(FL'_7) $\underline{Aprii}(\underline{Aprii}(\tilde{X})) = \underline{Aprii}(\tilde{X})$;

(FU'_7) $\overline{Aprii}(\overline{Aprii}(\tilde{X})) = \overline{Aprii}(\tilde{X})$.

Proof. We still prove (FL'_i) or (FU'_i), by the duality we can easily get (FU'_i) or (FL'_i) $(i = 1, \cdots, 7)$.

For any $\tilde{X} \in L^U$,

$$(\overline{Aprii}(\tilde{X}^c))^c = ((\tilde{X}^c)^{\Diamond\Box})^c = (((\tilde{X}^c)^{c\Box c})^{\Box})^c$$
$$= (\tilde{X}^{\Box c \Box})^c = \tilde{X}^{\Box c \Box c} = (\tilde{X}^{\Box})^{c\Box c} = \tilde{X}^{\Box\Diamond} = \underline{Aprii}(\tilde{X})$$

Thus, (FU'_1) holds.

For any $x \in U$,

$$\overline{Aprii}(\varnothing)(x) = \varnothing^{\Diamond\Box}(x)$$
$$= \wedge_{a\in A}(\tilde{I}(x,a) \to \varnothing^{\Diamond}(a))$$
$$= \wedge_{a\in A}(\tilde{I}(x,a) \to (\vee_{y\in U}(\varnothing(y) \otimes \tilde{I}(y,a))))$$
$$= \wedge_{a\in A}(\tilde{I}(x,a) \to 0)$$
$$= 0.$$

By $\tilde{X}^{\Box\Diamond} \subseteq \tilde{X} \subseteq \tilde{X}^{\Diamond\Box}$ we can get that $\underline{Aprii}(\varnothing) = \varnothing$.

For any $\tilde{X}, \tilde{Y} \in L^U$, according to

$(\tilde{X} \cap \tilde{Y})^{\Box\Diamond} = (\tilde{X}^{\Box} \cap \tilde{Y}^{\Box})^{\Diamond}$, $\tilde{X}^{\Box} \cap \tilde{Y}^{\Box} \subseteq \tilde{X}^{\Box}$ and $\tilde{X}^{\Box} \cap \tilde{Y}^{\Box} \subseteq \tilde{Y}^{\Box}$,

we have

$$(\tilde{X}^{\Box} \cap \tilde{Y}^{\Box})^{\Diamond} \subseteq \tilde{X}^{\Box\Diamond} \text{ and } (\tilde{X}^{\Box} \cap \tilde{Y}^{\Box})^{\Diamond} \subseteq \tilde{Y}^{\Box\Diamond},$$

from which we can get (FL'_3).

(FL'_4) follows immediately by $\tilde{X}_1 \subseteq \tilde{X}_2 \Rightarrow \tilde{X}_1^{\Diamond} \subseteq \tilde{X}_2^{\Diamond}, \tilde{X}_1^{\Box} \subseteq \tilde{X}_2^{\Box}$.

By $\tilde{X}^{\Box} \subseteq (\tilde{X} \cup \tilde{Y})^{\Box}$, $\tilde{Y}^{\Box} \subseteq (\tilde{X} \cup \tilde{Y})^{\Box}$, $\tilde{X}^{\Box\Diamond} \subseteq (\tilde{X} \cup \tilde{Y})^{\Box\Diamond}$ and $\tilde{Y}^{\Box\Diamond} \subseteq (\tilde{X} \cup \tilde{Y})^{\Box\Diamond}$ we can get (FL'_5).

(FL'_6) follows directly by $\tilde{X}^{\Box\Diamond} \subseteq \tilde{X} \subseteq \tilde{X}^{\Diamond\Box}$.

Since $\underline{Aprii}(\underline{Aprii}(\tilde{X})) = (\tilde{X}^{\Box\Diamond})^{\Box\Diamond}$ and $\tilde{X}^{\Box\Diamond\Box} = \tilde{X}^{\Box}$, we can get (FL'_7).

The approach of rough set approximation in concept analysis gives a way for studying concept lattice via rough set.

8 Conclusions

Since FCA was introduced by Wille in 1982, many researches studied it from various points and extended it to more complex situations such as a *L*-fuzzy formal context which is appropriated to the real world. In this paper, a concept granular computing system is established based on the study of concept lattice and *L*-fuzzy concept lattice. Relationships between this system and concept lattice, variable threshold concept lattice and fuzzy concept lattice and properties of the system are discussed. In order to reflect different relations between a set of objects and a set of attributes, sufficiency information granules and necessity information granules are defined. Properties of them are then studied. Later, iterative algorithms for constructing concepts for any extent element or intent element are introduced, and the optimization of the iterative algorithms are investigated. Finally, set approximations in FCA are studied, which shows a way to study FCA by using the theory of rough set.

Learning and application for concepts is the key question in the field of artificial intelligence. In order to process information via computers, a kind of mathematical model needs to be built. This paper is a try to build concept granular computing system by introducing an algebra structure. It has more benefits for further studies, such as concept generalization and specialization, sufficiency and necessity concept, more generalized concept and more special concept. And using this framework, a researcher can conclude some axiomic characterizations from various kinds of concept systems. Therefore, this model may supply an important tool for the further study of the formation and learning of concepts.

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