Affinities between Perceptual Granules: Foundations and Perspectives

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Abstract. This chapter gives a concise overview of the foundations of a perceptual near set approach to the discovery of affinities between perceptual objects and perceptual granules that provide a basis for perceptual systems useful in science and engineering. A *perceptual object* is something perceptible to the senses or knowable by the mind. Perceptual objects that have the same appearance are considered to be perceptually near each other, i.e., perceived objects that have perceived affinities or, at least, similar descriptions. A perceptual granule is a set of perceptual objects originating from observations of the objects in the physical world. Near set theory provides a basis for observation, comparison and classification of perceptual granules. By considering nearness relations in the context of a perceptual system, it is possible to gauge affinities (nearness) perceptual objects. Two kinds of indiscernibility relations and a tolerance relation make it possible to define various nearness relations. Examples of near images as perceptual systems are presented. The main contribution of this chapter is the introduction of a formal basis for discovering affinities between perceptual information granules.

Keywords: Affinities, near sets, perceptual granule, tolerance relations.

1 Introduction

The basis for perceptual systems hearkens back to the original notion of a deterministic information system introduced by Zdzisław Pawlak [20]. A *perceptual system* is a real-valued, total, deterministic information system. A *perceptual object* is something perceptible to the senses or knowable by the mind. Examples of perceptual objects include observable organism behaviour, growth rates, soil erosion, events containing the outcomes of experiments such as energizing a network, testing digital camera functions, microscope images, MRI scans, and the results of searches for relevant web pages. Granulation can be viewed as a human way of achieving data compression and it plays a key role in implementing the divide-and-conquer strategy in human problem-solving. A comprehensive study of granular computing can be found in [1]. A perceptual granule is a set of perceptual objects originating from observations of the objects in the physical world. Formally, a *perceptual granule* is a finite, non-empty set containing sample

A. Bargiela, W. Pedrycz (Eds.): Human-Centric Information Processing, SCI 182, pp. 49–66.
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perceptual objects with common descriptions and a set probe functions representing perceptual object features.

Another means of discovering perceptual granules was suggested by Charles Darwin, who called attention to affinities that one can observe between different members of the same species. The proposed approach to discovering affinities between perceptual granules is analogous to what Charles Darwin did during the voyage of the H.M.S. Beagle during the 1830s, starting in 1831 and ending in 1836. That is, Darwin kept adding to his collection of specimens and eventually, in some cases, found affinities between a set of specimens of interest and his expanding set of specimens found during the voyage of the Beagle [3].

Near set theory provides a basis for observation, comparison and measuring affinities of perceptual granules. Near sets have a human-centric character. Sensed physical characteristics of perceptual objects are identified with object features. It is our mind that identifies relationships between object feature values to form perceptions of sensed objects [7]. Human perceptions can be quantified through the use of near sets by providing a framework for comparing objects based on object descriptions. Objects that have the same appearance (*i.e.*, objects with matching descriptions) are considered *perceptually near each other*. Sets are considered near each other when they have “things” (perceived objects) in common. Specifically, near sets facilitate measurement of similarities between perceptual objects based on feature values (obtained by probe functions) that describe the objects. This approach is similar to the way humans perceive objects (see, *e.g.*, [4]) and as such facilitates pattern classification systems.

Near sets originally grew out of a study of images [5, 28, 30, 33] either by considering single images containing near sub images or segmented images containing perceptually near pixel windows. Two kinds of indiscernibility relations and a tolerance relation make it possible to define various nearness relations. A weak tolerance relation is also defined in this chapter. This tolerance relation is very important in discovering near sets, since it defines tolerance classes relative to a threshold ε, rather than require strict equality of probe function values in the case of the indiscernibility relations. The underlying assumption made here is that human perception relies on a limited view of perceived objects to discover affinities between samples. For this reason, the discovery of near objects begins with the perception of one or more matching characteristics, not a complete set of matching characteristics. Finding a multitude of matches between perceptual objects is not considered in arriving at the discovery threshold in detecting affinities between objects, *i.e.*, in discovering near sets. This approach is in keeping with the original view of tolerance spaces as models for human vision [37].

The Pal entropy measure defined in [12] provides a useful basis for probe functions used in the search for perceptual granules that are, in some sense, near each other. Other forms of entropy introduced by Sankar Pal *et al.* can be found in [9, 13, 14, 15, 16, 17, 18]. It has been shown that perceptual near sets are a generalization of rough sets introduced by Zdzisław Pawlak during the early 1980s. That is, every rough set is a near set but not every near set is a rough set. In addition, it can be shown that fuzzy sets with non-empty cores are near sets. The connections between these three forms of sets are briefly discussed in this chapter. By way of an illustration, affinities between microscope images (as elements in perceptual systems) of various leaves of trees are briefly explored.

This chapter is organized as follows. Section 2 presents the basis for perceptual systems. Indiscernibility relations and a tolerance relation are introduced in Section 3. Three basic nearness relations are presented and illustrated in Section 3 accompanied by an illustration of near images in Section 4.2. Examples of rough near sets and fuzzy near sets are presented in Sections 5 and 6, respectively.

2 Perceptual Systems: An Overview

This section briefly presents the basis for perceptual systems that hearkens back to the original notion of a deterministic information system introduced by Zdzisław Pawlak [20] and elaborated in [10, 11].

2.1 Perceptual Object Descriptions

Perceptual objects are known by their descriptions. An *object description* is defined by means of a tuple of function values $\phi(x)$ associated with an object $x \in X$ (see Table 1). The important thing to notice is the choice of functions $\phi_i \in \mathscr{B}$ used to describe an object of interest. Assume that $\mathscr{B} \subseteq \mathbb{F}$ (see Table 1) is a given set of functions representing features of sample objects $X \subseteq O$ and $\mathbb{F}$ is finite. Let $\phi_i \in \mathscr{B}$, where $\phi_i : O \longrightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an *object description* $\phi : O \longrightarrow \mathbb{R}^l$, a vector containing measurements (returned values) associated with each functional value $\phi_i(x)$ for $x \in X$, where $|\phi| = l$, i.e. the description length is l.

$$\textbf{Object Description:} \quad \phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_i(x), \ldots, \phi_l(x)).$$

The intuition underlying a description $\phi(x)$ is a recording of measurements from sensors, where each sensor is modeled by a function ϕ_i. Notice that all sensor values belong to the set of reals. That is, the perception of an object (*i.e.*, in effect, our knowledge about an object) depends on information gathered by our senses. The proposed approach to perception is feature-based and is similar to the one discussed in the introduction in [2].

Table 1. Description Symbols

Symbol	Interpretation
$\mathbb{R}$	Set of real numbers,
O	Set of perceptual objects,
X	$X \subseteq O$, set of sample objects,
x	$x \in O$, sample object,
$\mathbb{F}$	A set of functions representing object features,
$\mathscr{B}$	$\mathscr{B} \subseteq \mathbb{F}$,
ϕ	$\phi : O \rightarrow \mathbb{R}^l$, object description,
l	l is a description length,
i	$i \leq l$,
ϕ_i	$\phi_i \in \mathscr{B}$, where $\phi_i : O \longrightarrow \mathbb{R}$, probe function,
$\phi(x)$	$\phi(x) = (\phi_1(x), \ldots, \phi_i(x), \ldots, \phi_L(x))$, description,
$\langle X, \mathbb{F} \rangle$	$\phi(x_1), \ldots, \phi(x_{\lvert X \rvert})$, *i.e.*, perceptual information system.

In this view, our senses are likened to probe functions, *i.e.*, mappings of sensations to values assimilated by the mind.

Let $X,Y \subseteq O$ denote sets of perceptual objects. Sets $X,Y \subseteq O$ are considered near each other if the sets contain perceptual objects with at least partial matching descriptions. A *perceptual object* $x \in O$ is something presented to the senses or knowable by the mind [8]. In keeping with the approach to pattern recognition suggested by Pavel [19], the features of an object such as contour, colour, shape, texture, bilateral symmetry are represented by probe functions. A *probe function* can be thought of as a model for a sensor. A probe makes it possible to determine if two objects are associated with the same pattern without necessarily specifying which pattern (classification). A detailed explanation about probe functions vs. attributes in the classification of objects is given in [26].

2.2 Perceptual Systems: Specialized Deterministic Systems

For representing results of a perception, the notion of a perceptual system is briefly introduced in this section. In general, an *information system* is a triple $S = \langle Ob, At, \{Val_f\}_{f \in At} \rangle$ where *Ob* is a set of objects, *At* is a set of functions representing either object features or object attributes, and each Val_f is a value domain of a function $f \in At$, where $f : Ob \longrightarrow \mathscr{P}(Val_f)$, ($\mathscr{P}(Val_f)$ is a power set of Val_f) (see, *e.g.*, citePawlak1983). If $f(x) \neq \emptyset$ for all $x \in Ob$ and $f \in At$, then S is *total*. If $card(f(x)) = 1$ for every $x \in Ob$ and $f \in At$, then S is *deterministic*. Otherwise S is *non-deterministic*. In the case, when $f(x) = \{v\}$, $\{v\}$ is identified with v. An information system S is *real valued* iff $Val_f = \mathbb{R}$ for every $f \in At$. Very often a more concise notation is used: $\langle Ob, At \rangle$, especially when value domains are understood, as in the case of real valued information systems. Since we focus on sensed objects we consider each $f \in At$ to be a *probe functions*. Two examples of perceptual systems are given in Table 2 (see 3.1 for a discussion of the examples).

Table 2. Sample perceptual information systems

Sys. 1					Sys. 2				
X	ϕ_1	ϕ_2	ϕ_3	ϕ_4	**Y**	ϕ_1	ϕ_2	ϕ_3	ϕ_4
x_1	0	1	0.1	0.75	y_1	0	2	0.2	0.01
x_2	0	1	0.1	0.75	y_2	1	1	0.25	0.01
x_3	1	2	0.05	0.1	y_3	1	1	0.25	0.01
x_4	1	3	0.054	0.1	y_4	1	3	0.5	0.55
x_5	0	1	0.03	0.75	y_5	1	4	0.6	0.75
x_6	0	2	0.02	0.75	y_6	1	4	0.6	0.75
x_7	1	2	0.01	0.9	y_7	0	2	0.4	0.2
x_8	1	3	0.01	0.1	y_8	0	3	0.5	0.6
x_9	0	1	0.5	0.1	y_9	0	3	0.5	0.6
x_{10}	1	1	0.5	0.25	y_{10}	1	2	0.7	0.4
					y_{11}	1	4	0.6	0.8
					y_{12}	1	4	0.7	0.9
					y_{13}	1	1	0.25	0.01
					y_{14}	1	4	0.6	0.75

Definition 1. Perceptual System
A perceptual system $\langle O, \mathbb{F} \rangle$ *is a real valued total deterministic information system where* *O is a non-empty set of* perceptual objects*, while* $\mathbb{F}$ *a countable set of* probe functions.

The notion of a perceptual system admits a wide variety of different interpretations that result from the selection of sample perceptual objects contained in a particular sample space O. Perceptual objects are known by their descriptions. For simplicity, we consider only small sets of probe functions in this chapter. The question of countable (denumerable) sets of probe functions is not within scope of this paper.

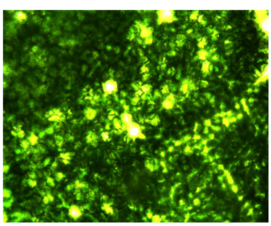
2.3 Sample Perceptual System

By way of an illustration, let $\langle P, \phi \rangle$ denote a perceptual system where P is a set of microscope images and ϕ is a probe function representing luminance contrast[1], respectively. A sample Shubert choke cherry leaf and Native Pin choke cherry leaf are shown in Figures 1.2 and 1.3. The National Optical DC3-163 microscope in Fig. 1.1 was used to produce the magnified leaf-section images shown in Figures 1.4 and 1.5 with a lens that magnifies the size of an object by a factor of 40. Intuitively, if we compare colour, luminance contrast or sub image shapes, the microscope leaf images are similar. By

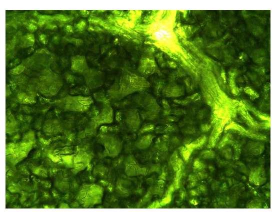
1.1: DC3-163 Scope

1.2: Shubert CC leaf

1.3: Pin CC leaf



1.4: Shubert CC slide



1.5: Pin CC slide

Fig. 1. Sample Percepts

[1] In digital images, luminance contrast can be controlled by converting irradiance (amount of light per unit area) into a grey value g using a function $g(E) = E^{\gamma}$, where E denotes irradiance level and luminance varies non-linearly with γ typically having a value of 0.4 [6].

considering nearness relations in the context of a perceptual system, it is possible to classify sets of perceptual objects. A formal basis for the discovery of near sets is the focus of the remaining sections of this chapter.

3 Relations, Partitions and Classes

The basic idea in the near set approach to object recognition is to compare object descriptions. Sample perceptual objects $x,y \in O, x \neq y$ are near each other if, and only if x and y have similar descriptions. Similarly, sets X,Y are perceptually near each other in the case where there is at least one pair of objects $x \in X, y \in Y$ that have similar descriptions. In this section, two kinds of indiscernibility relations and a tolerance relation are briefly introduced. These relations make it possible to define various nearness relations and make it possible to provide a formal foundation for near sets.

3.1 Indiscernibility and Tolerance Relations

Recall that each ϕ defines the description of an object (see Table 1). To establish a nearness relation, we first consider the traditional indiscernibility relation. Let $\mathscr{B} \subseteq \mathbb{F}$ denote a set of functions representing perceptual object features. The indiscernibility relation $\sim_{\mathscr{B}}$ introduced by Zdzisław Pawlak [20] is distinguished from weak indiscernibility $\underline{\bowtie}$ introduced introduced by Ewa Orłowska [10]. In keeping with the original indiscernibility relation symbol $\sim_{\mathbb{F}}$ [20], the symbol $\underline{\bowtie}$ is used to denote weak indiscernibility instead of the notation *wind* [10].

Definition 2. Indiscernibility Relation
Let $\langle O, \mathbb{F} \rangle$ be a perceptual system. For every $\mathscr{B} \subseteq \mathbb{F}$ the indiscernibility relation $\sim_{\mathscr{B}}$ is defined as follows:

$$\sim_{\mathscr{B}} = \{(x,y) \in O \times O \mid \ \forall \phi_i \in \mathscr{B} \centerdot \ \phi_i(x) = \phi_i(y)\}.$$

If $\mathscr{B} = \{\phi\}$ for some $\phi \in \mathbb{F}$, instead of $\sim_{\{\phi\}}$ we write $\sim_{\phi}$.

Example 1. Sample Partitions
Let $\langle O_1, \mathbb{F}_1 \rangle$ denote perceptual system Sys. 1 with $O_1 = \{x_1, \ldots, x_9\}$, $\mathbb{F}_1 = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of probe functions from $\mathbb{F}_1$ are given in the lefthand side of table 2. Similarly, let $\langle O_2, \mathbb{F}_2 \rangle$ denote perceptual system Sys. 2 with $O_2 = \{y_1, \ldots, x_{14}\}$, $\mathbb{F}_2 = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of the probe functions from $\mathbb{F}_1$ are given in the righthand side of table 2. The perceptual systems $\langle O_1, \mathbb{F}_1 \rangle$, $\langle O_2, \mathbb{F}_2 \rangle$ have partitions (1) and (2-1.3) of the space of percepts defined by relations $\sim_{\mathbb{F}_1}$ and $\sim_{\mathbb{F}_2}$.

$$O_{1/\sim_{\mathbb{F}_1}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}, \{a_8\}, \{x_9\}, \{x_{10}\}\}, \tag{1}$$

$$O_{2/\sim_{\mathbb{F}_2}} = \{\{y_1\}, \{y_2, y_3, y_{13}\}, \{y_4\}, \{y_5, y_6\}, \{y_7\}, \{y_8, y_9\}, \{y_{10}\}, \tag{2}$$

$$\{y_{11}\}, \{y_{12}\}, \{y_{14}\}\}. \tag{3}$$

If we consider only probe function ϕ_3 relative to O_1, then we obtain, *e.g.*, several equivalence classes such as (4), each containing a pair of objects.

$$x_{1/\sim_{\phi_3}} = \{x_1, x_2\}, \quad (4)$$
$$x_{7/\sim_{\phi_3}} = \{x_7, x_8\}, \quad (5)$$
$$x_{9/\sim_{\phi_3}} = \{x_9, x_{10}\}. \quad (6)$$

Again, for example, if we probe O_2 with ϕ_3, we obtain, *e.g.*, a number of multi-object classes such as the one in (7).

$$y_{2/\sim_{\phi_3}} = \{y_2, y_3, y_{13}\}, \quad (7)$$
$$y_{4/\sim_{\phi_3}} = \{y_4, y_8, y_9\}, \quad (8)$$
$$y_{5/\sim_{\phi_3}} = \{y_5, y_6, y_{11}, y_{14}\}, \quad (9)$$
$$y_{10/\sim_{\phi_3}} = \{y_{10}, y_{12}\}. \quad (10)$$

Definition 3. Weak Indiscernibility Relation
Let $\langle O, \mathbb{F}\rangle$ be a perceptual system. For every $\mathscr{B} \subseteq \mathbb{F}$ the weak indiscernibility relation $\simeq_{\mathscr{B}}$ is defined as follows:

$$\simeq_{\mathscr{B}} = \{(x,y) \in O \times O \mid \exists \phi_i \in \mathscr{B} \centerdot \phi_i(x) = \phi_i(y)\}.$$

If $\mathscr{B} = \{\phi\}$ for some $\phi \in \mathbb{F}$, instead of $\simeq_{\{\phi\}}$ we write $\simeq_{\phi}$.

Example 2. Weak Indiscernibility Partitions
Let $\langle O_1, \mathbb{F}_1\rangle$ denote perceptual system Sys. 1 with $O_1 = \{x_1, \ldots, x_9\}$, $\mathbb{F}_1 = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of probe functions from $\mathbb{F}_1$ are given in the lefthand side of table 2. Similarly, let $\langle O_2, \mathbb{F}\rangle$ denote perceptual system Sys. 2 with $O_2 = \{y_1, \ldots, y_{14}\}$, $\mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of the probe functions from $\mathbb{F}$ are given in the right-hand side of table 2. Let $X \subset O_1, X = \{x_1, x_9, x_{10}\}$ and $Y \subset O_2, Y = \{y_1, y_8, y_{10}, y_{11}, y_{12}\}$. Consider partitions $X_{/\simeq_{\phi_3}}$ and $Y_{/\simeq_{\phi_3}}$ given in (11) and (12), respectively.

$$X_{/\simeq_{\phi_3}} = \{\{x_1\}, \{x_9, x_{10}\}\}, \quad (11)$$
$$Y_{/\simeq_{\phi_3}} = \{\{y_1\}, \{y_8\}, \{y_{10}\}, \{y_{11}\}, \{y_{12}\}\}, \quad (12)$$

Remark 1. Notice that the class $\{x_1\} \in X_{/\cong_{\phi_3}}$ contains only a single object, since there is no other object in $x \in X$ such that $\phi_3(x_1) = \phi_3(x)$. Similarly, each of the classes in $Y_{/\cong_{\phi_3}}$ contains only a single object.

Definition 4. Weak Tolerance Relation
Let $\langle O, \mathbb{F}\rangle$ be a perceptual system and let $\varepsilon \in \Re$ (reals). For every $\mathscr{B} \subseteq \mathbb{F}$ the weak tolerance relation $\cong_{\mathscr{B},\varepsilon}$ is defined as follows:

$$\cong_{\mathscr{B},\varepsilon} = \{(x,y) \in O \times O \mid \exists \phi_i \in \mathscr{B} \centerdot |\phi_i(x) - \phi_i(y)| \leq \varepsilon\}.$$

That is, in general, the relation $\cong_{\mathscr{B},\varepsilon}$ is reflexive and symmetric but not transitive. This relation is very important in discovering near sets, since it defines tolerance classes relative to a threshold ε, rather than require strict equality of probe function values in the case of the indiscernibility relations $\sim_{\mathscr{B}}$ and $\simeq_{\mathscr{B}}$ (see, *e.g.*, [30]).

Remark 2. Special Case
Notice that Def. 4 represents a special case. That is, in general, the sets X and Y represent sample sets of observations from distinct perceptual systems. In effect, it is possible to state a Proposition to this effect.

Definition 5. Weak Tolerance Relation Between Sets of Perceptual Objects
Let $P1 = \langle O_1, \mathbb{F}\rangle$ denote perceptual system P1. Similarly, let $P2 = \langle O_2, \mathbb{F}\rangle$ denote a second, distinct perceptual system. Also, let $\varepsilon \in \Re$. P_1 has a weak tolerance relation to P_2 if, and only if $O_1 \cong_{\mathbb{F},\varepsilon} O_2$.

Definition 6. Weak Tolerance Relation on Perceptual Systems
Let $Sys1 = \langle O_1, \mathbb{F}\rangle$ denote perceptual system Sys1. Similarly, let $Sys2 = \langle O_2, \mathbb{F}\rangle$ denote a second, distinct perceptual system with the same set of features $\mathbb{F}$. Let $\mathscr{B} \subseteq \mathbb{F}$ and choose ε. Then

$$Sys1 \cong_{\mathscr{B},\varepsilon} Sys1 \iff O_1 \cong_{\mathscr{B},\varepsilon} O_2.$$

Example 3. Weak Tolerance
Let $\langle O_1, \mathbb{F}\rangle$ denote perceptual system Sys. 1 with $O_1 = \{x_1, \ldots, x_9\}$, $\mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of probe functions from $\mathbb{F}$ are given in the lefthand side of table 2. Similarly, let $\langle O_2, \mathbb{F}\rangle$ denote perceptual system Sys. 2 with $O_2 = \{y_1, \ldots, y_{14}\}$, $\mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of the probe functions from $\mathbb{F}$ are given in the righthand side of table 2. Let $\varepsilon = 0.1$ for both perceptual systems. For example, let $\phi_3 \in \mathbb{F}_1$. The perceptual system $\langle O_1, \{\phi_3\}\rangle$ has tolerance classes (13), (14), (15) defined by relation $\cong_{\phi_{3,0.1}}$.

$$x_{1/\cong_{\phi_3,0.1}} = \{x_1, x_2, x_5, x_6, x_7, x_8\}, \tag{13}$$

$$x_{3/\cong_{\phi_3,0.1}} = \{x_3, x_4\}, \tag{14}$$

$$x_{9/\cong_{\phi_3,0.1}} = \{x_9, x_{10}\}. \tag{15}$$

For example, in $x_{3/\cong_{\phi_3,0.1}}$, we have

$$|\phi_3(x_3) - \phi_3(x_4)| = |0.05 - 0.054| \leq 0.1$$

Similarly, the perceptual system $\langle O_2, \{\phi_3\}\rangle$ has tolerance classes defined by relation $\cong_{\phi_{3,0.1}}$: (16), (17), (18), (19) .

$$y_{1/\cong_{\phi_3,0.1}} = \{y_1, y_2, y_3, y_{13}\}, \tag{16}$$

$$y_{4/\cong_{\phi_3,0.1}} = \{y_4, y_5, y_6, y_8, y_9, y_{11}, y_{14}\}, \tag{17}$$

$$y_{7/\cong_{\phi_3,0.1}} = \{y_7, y_4, y_8, y_9\}, \tag{18}$$

$$y_{10/\cong_{\phi_3,0.1}} = \{y_5, y_6, y_{10}, y_{11}, y_{12}, y_{14}\}, \tag{19}$$

For example, in $y_{7/\cong_{\phi_3,0.1}}$, we have

$$|\phi_3(y_7) - \phi_3(y_4)| = |0.4 - 0.5| \leq 0.1,$$
$$|\phi_3(y_7) - \phi_3(y_8)| = |0.4 - 0.5| \leq 0.1,$$
$$|\phi_3(y_7) - \phi_3(y_9)| = |0.4 - 0.5| \leq 0.1,$$
$$|\phi_3(y_8) - \phi_3(y_9)| = |0.5 - 0.5| \leq 0.1$$

4 Nearness Relations

Three basic nearness relations are briefly presented and illustrated in this section.

Definition 7. Nearness Relation [34]
Let $\langle O, \mathbb{F}\rangle$ be a perceptual system and let $X, Y \subseteq O$. The set X is perceptually near to the set Y ($X \bowtie_{\mathbb{F}} Y$), if and only if there are $x \in X$ and $y \in Y$ such that $x \sim_{\mathbb{F}} y$ (see Table 3).

Table 3. Relation Symbols

Symbol	Interpretation
$\mathscr{B}$	see Table 1,
ε	$\varepsilon \in [0, 1]$,
$\sim_{\mathscr{B}}$	$\{(x,y) \mid f(x) = f(y)\ \forall f \in \mathscr{B}\}$, indiscernibility relation [20],
$\simeq_{\mathscr{B}}$	weak indiscernibility relation [10],
$\cong_{\mathscr{B},\varepsilon}$	weak tolerance relation,
$x_{/\sim_{\mathscr{B}}}$	$x_{/\sim_{\mathscr{B}}} = \{y \in X \mid y \sim_{\mathscr{B}} x\}$, elementary set (class),
$O_{/\sim_{\mathscr{B}}}$	$O_{/\sim_{\mathscr{B}}} = \{x_{/\sim_{\mathscr{B}}} \mid x \in O\}$, quotient set,
$\bowtie$	nearness relation symbol,
$\underline{\bowtie}$	weak nearness relation symbol,
$\underline{\underline{\bowtie}}$	weak tolerance nearness relation symbol.

Example 4. Consider the perceptual systems $\langle O_1, \mathbb{F}\rangle$, $\langle O_2, \mathbb{F}\rangle$ given in Table 2. From Example 2, we obtain

$$\begin{aligned}
\mathscr{B} &= \{\phi_3\}, \text{ where } \phi_3 \in \mathbb{F},\\
X_{new} &= x_{9/\sim_{\phi_3}}, \text{ from Example 2},\\
&= \{x_9, x_{10}\},\\
Y_{new} &= y_{8/\sim_{\phi_3}}\\
&= \{y_4, y_8, y_9\},\\
X_{new} &\bowtie_{\phi_3} Y_{new}, \text{ since}\\
\phi_3(x_9) &= \phi_3(y_8) = 0.5
\end{aligned}$$

Definition 8. Weak Nearness Relation [34]
Let $\langle O, \mathbb{F}\rangle$ be a perceptual system and let $X, Y \subseteq O$. The set X is weakly near to the set Y *within* the perceptual system $\langle O, \mathbb{F}\rangle$ ($X \underline{\bowtie}_{\mathbb{F}} Y$) iff there are $x \in X$ and $y \in Y$ and there is $\mathscr{B} \subseteq \mathbb{F}$ such that $x \simeq_{\mathscr{B}} y$. If a perceptual system is understood, then we say shortly that a set X *is weakly near to* set Y (see Table 3).

Example 5. Consider the perceptual systems $\langle O_1, \mathbb{F}\rangle$, $\langle O_2, \mathbb{F}\rangle$ given in Table 2.

$$\mathscr{B} = \{\phi_3\}, \text{ where } \phi_3 \in \mathbb{F},$$
$$X = \{x_1, x_2, x_7, x_8, x_9, x_{10}\},$$
$$Y = \{y_4, y_5, y_6, y_8, y_9, y_{11}\},$$
$$X \bowtie_{\phi_3} Y, \text{ since we can find } x \in X, y \in Y \text{ where } x \simeq_{\phi_3} y, \text{ e.g.,}$$
$$\phi_3(x_9) = \phi_3(y_8) = 0.5.$$

Definition 9. Weak Tolerance Nearness Relation [30]
Let $\langle O, \mathbb{F}\rangle$ be a perceptual system and let $X, Y \subseteq O, \varepsilon \in [0,1]$. The set X is perceptually near to the set Y *within* the perceptual system $\langle O, \mathbb{F}\rangle$ ($X \underline{\underline{\bowtie}}_{\mathbb{F}} Y$) iff there exists $x \in X$, $y \in Y$ and there is a $\phi \in \mathbb{F}, \varepsilon\ \Re$ such that $x \underline{\underline{\simeq}}_{\mathscr{B},\varepsilon} y$ (see Table 3). If a perceptual system is understood, then we say shortly that a set X *is perceptually near to* a set Y in a weak tolerance sense of *nearness*.

Example 6. Sample Weak Tolerance Nearness
Let $\langle O_1, \mathbb{F}\rangle$ denote perceptual system Sys. 1 with $O_1 = \{x_1, \ldots, x_9\}, \mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of probe functions from $\mathbb{F}$ are given in the lefthand side of table 2. Similarly, let $\langle O_2, \mathbb{F}\rangle$ denote perceptual system Sys. 2 with $O_2 = \{y_1, \ldots, y_{14}\}$, $\mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of the probe functions from $\mathbb{F}$ are given in the righthand side of table 2. Now choose ε and arbitrary samples X_1 and Y_1 so that they are also weak tolerance near sets.

$$\varepsilon = 0.1,$$
$$\mathscr{B} = \{\phi_3\}, \text{ where } \phi_3 \in \mathbb{F},$$
$$X_1 \in O_1, Y_1 \in O_2,$$
$$X_1 = \{x_1, x_2, x_7, x_8, x_9, x_{10}\},$$
$$Y_1 = \{y_4, y_5, y_6, y_8, y_9, y_{11}\},$$
$$X_1 \underline{\underline{\bowtie}}_{\phi_3} Y_1, \text{ since we can find } x \in X, y \in Y \text{ where } x \underline{\underline{\simeq}}_{\phi_3,\varepsilon} y, \text{ e.g.,}$$
$$|\phi_3(x_9) - \phi_3(y_8)| = |0.5 - 0.5| = 0 \leq 0.1; \text{ again, e.g.,}$$
$$|\phi_3(x_{10}) - \phi_3(y_{11})| = |0.1 - 0.2| = 0.1$$

Remark 3. *In Example 6, we know that* $X \underline{\underline{\bowtie}}_{\mathbb{F}} Y$*, since there exists an* $x \in X, y \in Y$ *(namely,* x_9, y_8*) such that*

$$|\phi_3(x) - \phi_3(y)| \leq \varepsilon$$

We can generalize the result from Example 6 in Prop 1 by extending the idea in Prop. 6.

Proposition 1. *Let* $Sys1 = \langle O_1, \mathbb{F}\rangle$ *denote perceptual system Sys1. Similarly, let* $Sys2 = \langle O_2, \mathbb{F}\rangle$ *denote a second, distinct perceptual system. Then*

$$Sys1 \underline{\underline{\bowtie}}_{\mathbb{F}} Sys1 \iff O_1 \underline{\underline{\bowtie}}_{\mathbb{F}} O_2.$$

4.1 Tolerance Perceptual Near Sets

Object recognition problems, especially in images [5], and the problem of the nearness of objects have motivated the introduction of near sets (see, *e.g.*, [28]). Since we are mainly interested in real-valued probe functions in comparing swarm behaviours, perceptual near sets are briefly considered in this section based on the weak tolerance nearness relation [30] $\underline{\underline{\bowtie}}_{\mathbb{F}}$ in Def. 9. Other forms of near sets are introduced in [27, 34].

Definition 10. Tolerance Perceptual Near Sets
Let $\langle O, \mathbb{F} \rangle$ be a perceptual system and let $X \subseteq O$. A set X is a tolerance perceptual near set iff there is $Y \subseteq O$ such that $X \underline{\underline{\bowtie}}_{\mathbb{F}} Y$. The family of near sets of a perceptual system $\langle O, \mathbb{F} \rangle$ is denoted by $\mathsf{Near}_{\mathbb{F}}(O)$.

In effect, tolerance perceptual near sets are those sets that are defined by the nearness relation $\underline{\underline{\bowtie}}_{\mathbb{F}}$.

Example 7. Sample Tolerance Perceptual Near Sets
Let $\langle O_1, \mathbb{F} \rangle$ denote perceptual system Sys. 1 with $O_1 = \{x_1, \ldots, x_9\}, \mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of probe functions from $\mathbb{F}$ are given in the lefthand side of table 2. Similarly, let $\langle O_2, \mathbb{F} \rangle$ denote perceptual system Sys. 2 with $O_2 = \{y_1, \ldots, y_{14}\}$, $\mathbb{F} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where the values of the probe functions from $\mathbb{F}$ are given in the righthand side of table 2. Now choose samples X and Y that are also weak tolerance near sets. Sets X, Y in Example 6 are near sets, since $X \underline{\underline{\bowtie}}_{\phi_3} Y$. Again, for example, consider the following near sets extracted from Table 2.

$$\begin{aligned}
&\varepsilon = 0.3, \\
&\mathscr{B} = \{\phi_3\}, \\
&X_1 \in O_1, Y_1 \in O_2, \\
&X_1 = \{x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \\
&Y_1 = \{y_4, y_5, y_6, y_8, y_9, y_{10}, y_{11}, y_{12}\}, \\
&X_1 \underline{\underline{\bowtie}}_{\phi_3} Y_1, \text{ since we can find } x \in X_1, y \in Y_1, \text{ where}
\end{aligned}$$

$$x \cong_{\phi_{3,0.3}} y, \; e.g., x_9 \cong_{\phi_{3,0.3}} y_{10}, \text{ since } |\phi_3(x_9) - \phi_3(y_{10})| = |0.5 - 0.7| = 0.2 \leq 0.3$$

The basic idea here is to look for sets of objects containing at least one pair of objects that satisfy the weak tolerance relation. Consider, for example, sets $X_2 \in O_2, Y_1 \in O_2$ extracted from Table 2 in (23) and (24).

$$\varepsilon = 0.3 \tag{20}$$

$$\mathscr{B} = \{\phi_4\}, \tag{21}$$

$$X_2 \in O_2, Y_1 \in O_2, \tag{22}$$

$$X_2 = \{x_1, x_2, x_5, x_6, x_7, x_8, x_9\}, \tag{23}$$

$$Y_2 = \{y_5, y_6, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{14}\}, \tag{24}$$

$$X_2 \underline{\underline{\bowtie}}_{\phi_3} Y_2, \text{ since we can find } x \in X_2, y \in Y_2, \text{ where} \tag{25}$$

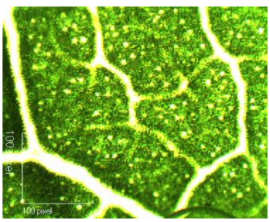
$x \cong_{\phi_{4,0.3}} y$, *e.g.*,

$x_1 \cong_{\phi_{4,0.3}} y_8$, since $|\phi_4(x_1) - \phi_4(y_8)| = |0.75 - 0.6| = 0.15 \leq 0.3$; again, *e.g.*,

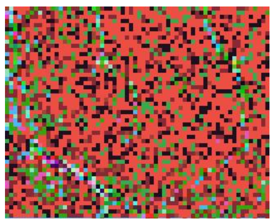
$x_7 \cong_{\phi_{4,0.3}} y_{11}$, since $|\phi_4(x_7) - \phi_4(y_{11})| = |0.9 - 0.8| = 0.1 \leq 0.3$

4.2 Sample Near Images

By way of an illustration of near images, let $\langle Im, H \rangle$ denote a perceptual system where Im is a set of segmented microscope images and H is a probe function representing image entropy[2], respectively. A sample Shubert choke cherry leaf and Native Pin choke cherry leaf are shown in Figures 1.2 and 1.3. For small segments of two sample choke cherry leaves, the National Optical DC3-163 microscope in Fig. 1.1 was used to produce the magnified images in Figures 2, 3 and 4. For this example, it was found that $\gamma = 0.4239$ worked best to show the contrast between areas of the leaf fragment at the 10× level of magnification in Fig. 2.1 and Fig. 2.3. Higher values of γ were used higher levels of magnification ($\gamma = 0.874$ for 20× magnification and $\gamma = 0.819$ for 40× magnification).

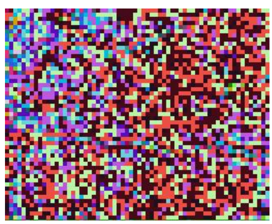


2.1: Shubert 10×



2.2: Shubert $10\times, \varepsilon = 0.01$

2.3: Pin 10×



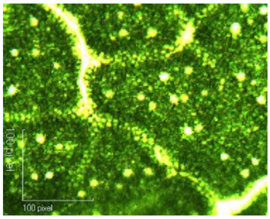
2.4: Pin $10\times, \varepsilon = 0.01$

Fig. 2. Sample Segmented 10× Images

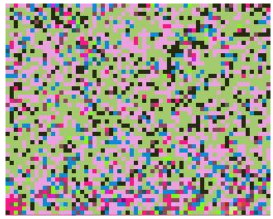
Let im1, im2 denote the Shubert choke cherry leaf image in Fig. 2.1 and Native pin choke cherry leaf in Fig. 2.3, respectively, each shown at magnification 10. The segmentation of these images obtained by separating image areas[3] representing tolerance classes are shown in Fig. 2.2 and Fig. 2.4. Let $\varepsilon = 0.01$ in the definition of the weak

[2] Entropy defined in the context of images is explained in [12].

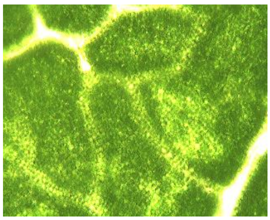
[3] Christopher Henry wrote the matlab program used to obtain the image segmentations shown in this section.



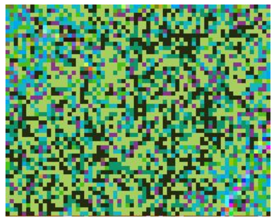
3.1: Shubert 20×



3.2: Shubert 20×, $\varepsilon = 0.01$



3.3: Pin 20×



3.4: Pin 20×, $\varepsilon = 0.01$

Fig. 3. Sample Segmented 20× Images

tolerance relation (see Def. 4). Let X denote a greyscale image, $x \in X$ a sequence of grey levels in X. In addition, let $p(x_i)$ denotes the probability of the i^{th} sequence of grey levels). For greyscale image X, Pal [12] entropy $H^{(1)}$ is defined by

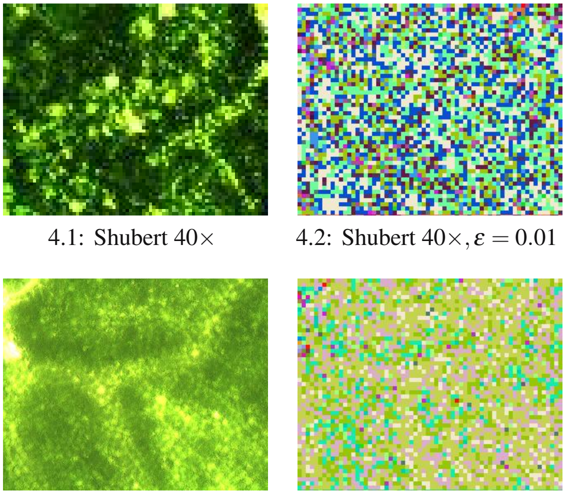
$$H(X) = \sum_{i=0}^{|X|} p(x_i) e^{1-p(x_i)}.$$

Intuitively, $H(X)$ represents the expected value of the gain in information resulting from the occurrence of different sequences of grey levels in an image. Let x, y denote a pair of $n \times n$ pixel windows in an image, *i.e.*, each pixel window contains $n \times n$ pixels (picture elements). Then all pairs of pixel windows having Pal entropy within $\varepsilon = 0.01$ belong to the same tolerance class. In other words,

$$|H(x) - H(y)| \leq \varepsilon.$$

The tolerance classes represented in a segmented image are each assigned a different color. For example, the Shubert choke cherry 10× microscopic image in Fig. 2.1 is dominated by one tolerance class (visualized with tiny rectangles with the colour orange in Fig. 2.2). It can be observed that a small number of pixels windows in have the same colour. Notice that the windows in a single tolerance class are scattered throughout the image in Fig. 2.2.

A Native Pin choke cherry 10× microscopic image is shown in Fig. 2.3. The entropic pixel window values represented by the tiny rectangular regions in Fig. 2.4 are compared with the information gain (entropic image value) for each of the pixel windows shown in Fig. 2.4. For this pair of sample segmented images, roughly 20% of the pixel windows in the 10× Pin cherry segmentation have a colour (*i.e.*, information gain) that is similar to the colouring of the pixel windows in Fig. 2.2. That is, the



4.1: Shubert $40\times$ 4.2: Shubert $40\times, \varepsilon = 0.01$

4.3: Pin $40\times$ 4.4: Pin $40\times, \varepsilon = 0.01$

Fig. 4. Sample Segmented $40\times$ Images

degree-of-nearness of this pair of images is approximately 20 percent. From Def. 4, we can conclude that $im1$ and $im2$ are near images relative to the entropic image function H and for $\varepsilon = 0.01$, *i.e.*,

$$im1 \ \underline{\underline{\bowtie}}_{\{H\}} \ im2.$$

Similar results were obtained for $20\times$ and $40\times$ magnification levels for the segmentations shown in Fig. 3 and Fig. 4.

5 Rough Near Sets

The germ of the idea for near sets first appeared within a poem by Zdzisław Pawlak and this author in a poem entitled *Near To* written in 2002 and later published in English and Polish [21, 31]. In later years, the foundations for near sets grew out of a rough set approach to classifying images [5, 28, 29, 32]. It is fairly easy to show that every rough set is also a near set. This section briefly presents some fundamental notions in rough set theory resulting from the seminal work by Zdisław Pawlak during the early 1980s [20] and elaborated in [22, 23, 24]. An overview of the mathematical foundations of rough sets is given by Lech Polkowski in [35].

Let $\langle O, \mathbb{F} \rangle$ denote a perceptual system containing a set of perceptual objects O and a set of functions $\mathbb{F}$ representing features of the objects in O. Further, let $O_{\sim_{\mathscr{B}}}$ denote the set of all classes in the partition of O defined by $\sim_{\mathscr{B}}$ for $\mathscr{B} \subseteq \mathbb{F}$. Recall that $x_{/\sim_{\mathscr{B}}}$ denotes an equivalence class relative $x \in O$. For $X \subseteq O, \mathscr{B} \subseteq \mathbb{F}$, a sample perceptual granule X can be approximated with a $\mathscr{B}$-lower $\mathscr{B}_* X$ and $\mathscr{B}$-upper approximation $\mathscr{B}^* X$ defined by

$$\mathscr{B}_* X = \bigcup_{x : [x]_B \subseteq X} [x]_B,$$

$$\mathscr{B}^*X = \bigcup_{x:[x]_B \cap X \neq \emptyset} [x]_B.$$

Whenever $\mathscr{B}_*X$ is a proper subset of $\mathscr{B}^*X$, *i.e.*, $\mathscr{B}^*X - \mathscr{B}_*X \neq \emptyset$, the sample X has been classified imperfectly and X is considered a rough set. Notice, from Def. 7,

$$\mathscr{B}_*X \bowtie_{\mathscr{B}} X, \text{ and}$$
$$\mathscr{B}^*X \bowtie_{\mathscr{B}} X,$$

since the classes in an approximation of X contain objects with descriptions that match the description of at least one object in X. Hence, the pairs $\mathscr{B}_*X, X$ and $\mathscr{B}^*X, X$ are examples of near sets. In general,

Proposition 2. (Peters [27]) The pairs $(\mathscr{B}_*X, X)$ and $(\mathscr{B}^*X, X)$ are near sets.

Proposition 3. (Peters [27]) Any equivalence class $x_{/\sim_{\mathscr{B}}}$, $\left|x_{/\sim_{\mathscr{B}}}\right| > 2$ is a near set.

6 Fuzzy Near Sets

Fuzzy sets A1 and A2 shown in Fig. 5 are also near sets inasmuch as each fuzzy set has a non-empty core. Let X be a problem domain for a fuzzy set A. By definition [25], the core of a fuzzy set A_μ is a function defined relative to complete and full membership in the set A_μ prescribed by the membership function μ [36]. Specifically,

$$core(A_\mu) = \{x \in X \mid \mu(x) = 1\}.$$

The core of A_μ is an example of a probe function that defines the class

$$x_{/\simeq_{core(A_\mu)}} = \left\{y \in X \mid y \in core(A_\mu)\right\}.$$

It can also be argued that $\langle X, core(A_\mu)\rangle$ is a perceptual system. In the case where a pair of fuzzy sets has non-empty cores, then the fuzzy sets satisfy the condition for the

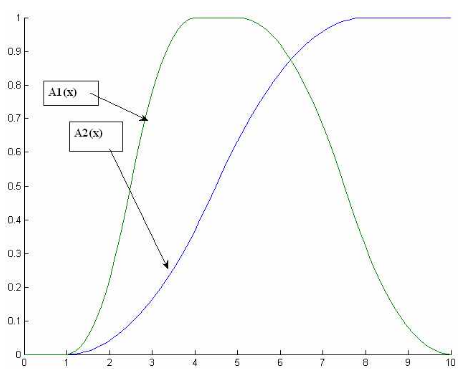


Fig. 5. Sample Fuzzy Near Sets

weak nearness relation, *i.e.*, we can find $x \in X, y \in Y$ for $(X, A1_{\mu_1}), (Y, A2_{\mu_2})$ relative to $A1_{\mu_1}, A2_{\mu_2}$, for membership functions μ_1, μ_2, where

$$x \in x_{/\simeq_{core(A1_{\mu_1})}},$$
$$y \in y_{/\simeq_{core(A2_{\mu_2})}},$$
$$\mu_1(x) = \mu_2(y) = 1.$$

Proposition 4. Fuzzy sets with non-empty cores are near sets.

7 Conclusion

The main contribution of this chapter is the introduction of a formal basis for discovering affinities between perceptual granules. This is made possible by the introduction of various forms of indiscernibility relations that define partitions and tolerance relations that define coverings of perceptual granules and lead to a number of useful nearness relations. A weak tolerance nearness relation is also defined in this chapter. This tolerance relation is has proven to be quite useful in discovering affinities between perceptual granules. The degree of affinity between microscope images as a perceptual system is measured with a form of entropic image function, has been briefly presented in an informal way in this chapter. Future work includes the introduction of various probe functions and nearness useful in image analysis.

Acknowledgements. The comments, suggestions and questions by anonymous reviewer for this article were very helpful. Many thanks to Christopher Henry for providing the MatLab code for the image segmentations presented in this chapter. This research has been funded by grant SRI-BIO-05 from the Canadian Arthritis Network (CAN) and by discovery grants 185986 and 194376 from the Natural Sciences & Engineering Research Council of Canada (NSERC).

References

1. Bargiela, A., Pedrycz, W.: Granular Computing. Kluwer, Boston (2003)
2. Calitoiu, D., Oommen, B., Nussbaum, D.: Desynchronizing a chaotic pattern recognition neural network to model inaccurate perception. IEEE Trans. on Systems, Man, and Cybernetics, Part B: Cybernetics 37(3), 692–704 (2007)
3. Darwin, C.: On the Origin of the Species by Means of Natural Selection. J. Murray, Oxford (1859)
4. Fahle, M., Poggio, T.: Perceptual Learning. The MIT Press, Cambridge (2002)
5. Henry, C., Peters, J.F.: Image pattern recognition using approximation spaces and near sets. In: An, A., Stefanowski, J., Ramanna, S., Butz, C.J., Pedrycz, W., Wang, G. (eds.) RSFDGrC 2007. LNCS, vol. 4482, pp. 475–482. Springer, Heidelberg (2007)
6. Jähne, B.: Digital Image Processing, 6th edn. Springer, Heidelberg (2005)
7. Merleau-Ponty, M.: Phenomenology of Perception, trans. by Colin Smith. Smith, Callimard and Routledge & Kegan Paul, Paris and New York (1945, 1956)

8. Murray, J., Bradley, H., Craigie, W., Onions, C.: The Oxford English Dictionary. Oxford University Press, Oxford (1933)
9. Nasrabadi, N., Pal, S., King, R.: Entropy-coded hybrid differential pulse-code modulation. Electronics Letters 16(2), 63–65 (1983)
10. Orłowska, E.: Incomplete Information: Rough Set Analysis. Studies in Fuzziness and Soft Computing, vol. 13. Physica-Verlag, Heidelberg (1998)
11. Orłowska, E., Pawlak, Z.: Representation of nondeterministic information. Theoretical Computer Science 29, 27–39 (1984)
12. Pal, N.R., Pal, S.K.: Entropy: A new definition and its applications. IEEE Trans. on Sys., Man and Cybernetics 21(5), 1260–1270 (1981)
13. Pal, N.R., Pal, S.K.: Entropic thresholding. Signal Processing 16(2), 97–108 (1983)
14. Pal, N.R., Pal, S.K.: Object-background segmentation using new definitions of entropy. IEE Proceedings-E 136(4), 284–295 (1989)
15. Pal, N.R., Pal, S.K.: Entropy: A new definition and its applications. IEEE Trans. Syst, Man and Cybernetics SMC-21(5), 1260–1270 (1991)
16. Pal, N.R., Pal, S.K.: Higher order fuzzy entropy and hybrid entropy of a set. Information Sciences 61(3), 211–231 (1992)
17. Pal, N.R., Pal, S.K.: Some properties of the exponential entropy. Information Sciences 66(3), 119–137 (1992)
18. Pal, S.K., King, R.A., Hashim, A.A.: Automatic gray level thresholding through index of fuzziness and entropy. Pattern Recognition Letters 1, 141–146 (1983)
19. Pavel, M.: Fundamentals of Pattern Recognition, 2nd edn. Marcel Dekker, Inc., N.Y (1993)
20. Pawlak, Z.: Classification of objects by means of attributes. Polish Academy of Sciences 429 (1981)
21. Pawlak, Z., Peters, J.F.: Jak blisko. Systemy Wspomagania Decyzji I, 57 (2007)
22. Pawlak, Z., Skowron, A.: Rough sets and boolean reasoning. Information Sciences 177, 41–73 (2007)
23. Pawlak, Z., Skowron, A.: Rough sets: Some extensions. Information Sciences 177, 28–40 (2007)
24. Pawlak, Z., Skowron, A.: Rudiments of rough sets. Information Sciences 177, 3–27 (2007)
25. Pedrycz, W., Gomide, F.: An Introduction to Fuzzy Sets. The MIT Press, Cambridge (1998)
26. Peters, J.F.: Classification of objects by means of features. In: Proc. IEEE Symposium Series on Foundations of Computational Intelligence (IEEE SCCI 2007), Honolulu, Hawaii, pp. 1–8 (2007)
27. Peters, J.: Near sets. general theory about nearness of objects. Applied Mathematical Sciences 1(53), 2609–2629 (2007)
28. Peters, J.F.: Near sets. special theory about nearness of objects. Fundamenta Informaticae 75(1-4), 407–433 (2007)
29. Peters, J.F.: Near sets. Toward approximation space-based object recognition. In: Yao, J., Lingras, P., Wu, W.-Z., Szczuka, M.S., Cercone, N.J., Ślęzak, D. (eds.) RSKT 2007. LNCS (LNAI), vol. 4481, pp. 22–33. Springer, Heidelberg (2007)
30. Peters, J.: Discovery of perceputally near information granules. In: Novel Developments in Granular Computing: Applications of Advanced Human Reasoning and Soft Computation. Information Science Reference, Hersey, N.Y., U.S.A (to appear, 2008)
31. Peters, J.F., Skowron, A.: Zdzisław pawlak: Life and work. In: Peters, J.F., Skowron, A. (eds.) Transactions on Rough Sets V. LNCS, vol. 4100, pp. 1–25. Springer, Heidelberg (2006)
32. Peters, J.F., Skowron, A., Stepaniuk, J.: Nearness in approximation spaces. In: Proc. Concurrency, Specification and Programming (CS&P 2006), Humboldt Universität, pp. 435–445 (2006)

33. Peters, J.F., Skowron, A., Stepaniuk, J.: Nearness of objects: Extension of approximation space model. Fundamenta Informaticae 79(3-4), 497–512 (2007)
34. Peters, J.F., Wasilewski, P.: Foundations of near sets. Information Sciences (submitted, 2008)
35. Polkowski, L.: Rough Sets. Mathematical FoundationsGermany. Springer, Heidelberg (2002)
36. Ross, T.J.: Fuzzy Logic with Engineering Applications. John Wiley & Sons, New York (2004)
37. Zeeman, E.: The topology of the brain and visual perception. In: Fort Jr., M.K. (ed.) Topology of 3-Manifolds and Related Topics, University of Georgia Institute, pp. 240–256. Prentice-Hall, Inc., Englewood Cliffs (1961)