# Parameterized Complexity for Domination Problems on Degenerate Graphs*

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**Abstract.** Domination problems are one of the classical types of problems in computer science. These problems are considered in many different versions and on different classes of graphs. We explore the boundary between fixed-parameter tractable and W-hard problems on sparse graphs. More precisely, we expand the list of domination problems which are fixed-parameter tractable(FPT) for degenerate graphs by proving that Connected $k$-Dominating set and $k$-Dominating threshold set are FPT. From the other side we prove that there are domination problems which are difficult (W[1] or W[2]-hard) for this graph class. The Partial $k$-dominating set and $(k, r)$-Center for $r \geq 2$ are examples of such problems. It is also remarked that domination problems become difficult for graphs of bounded average degree.

**Keywords:** Parameterized complexity, algorithms, degenerate graphs, domination.

## 1 Introduction and Overview of Results

Domination problems are fundamental and widely studied problems in algorithms and complexity theory. This paper deals with parameterized complexity of different domination problems for sparse graphs (we refer the reader to monographs [13,14] for general information on parameterized complexity and algorithms). It is well known that these problems are W[1]-complete (like $k$-Perfect code [6,12]) or W[2]-complete (like $k$-Dominating set [11]) for general graphs. It is then natural to investigate complexity of these problems for restricted graph classes. By now there is an extensive literature about parameterized complexity of domination problems for different families of graphs.

Most versions of domination problems becomes fixed-parameter tractable (FPT) for planar graphs. The first such result was established by Alber et al. [1]. Later, these results were generalized for other families of sparse graphs. The newly developed theory of bidimensionality (see e.g. survey [10]) was used for construction of fixed-parameter algorithms for broad graph classes. By using this theory it is possible to construct such algorithms for different domination problems on apex-minor-free graphs. Moreover, these results can be extended

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to even larger classes of graphs. Particularly, it was proved by Demaine et al. that the $k$-Dominating set problem is FPT for $H$-minor-free graphs [9], $(k, r)$-Center if FPT for map graphs [8] and apex-minor-free graphs. By developing these ideas and applying new branching techniques Amini et al. proved that Partial $k$-Dominating set, Partial weighted $k$-Dominating set and Partial $(k, r)$-center are FPT for $H$-minor-free graphs [3]. Even more general results for probelms which can be expressed in the first-order logic were received by Dawar et al. [7].

A graph $G$ is called *$d$-degenerate* (with $d$ being a positive integer) if every induced subgraph of $G$ has a vertex of degree at most $d$. For example, trees are exactly connected 1-degenerate graphs, and every planar graph is 5-degenerate. Moreover, it is known that all $H$-minor-free graphs are degenerate (see [18,19,20]).

An ordering of vertices of a graph $G$ $v_1, v_2, \ldots, v_n$ is called a *$d$-degenerate ordering* if every vertex $v_i$ has at most $d$ neighbors among the vertices $v_1, v_2, \ldots, v_{i-1}$. It is well known that a graph is $d$-degenerate if and only if it allows a $d$-degenerate ordering of its vertices. By considering vertices in the backward $d$-degenerate ordering and using the method of bounded search trees it can be proved that $k$-Perfect code [5] is FPT for degenerate graphs. Similar fact can be easily established for Independent $k$-dominating set.

In [2] Alon and Gutner proved that $k$-Dominating set is FPT for $d$-degenerate graphs. They used the method of bounded search trees and the fact that degenerate graphs have bounded average degree. They also have shown that Weighted $k$-dominating set is FPT for degenerate graphs. Using same techniques it can be easily proved that some other domination problems (for example, Roman $k$-dominating set) are also FPT for this family of graphs.

All these results lead us to the following question: which domination problems are FPT for degenerate graphs, and which problems are W[1] or W[2]-hard?

**Our results.** Building on the ideas of Alon and Gutner we prove that it is possible to construct FPT-algorithms for some other domination problems with additional restrictions. Particularly, we prove that Connected $k$-dominating set is FPT for degenerate graphs. Because of the additional restrictions (which is connectivity of the dominating set in this case) it is impossible to apply the results of [2] directly. By using a similar approach we also show that $k$-dominating threshold set is FPT. Next we prove that there are domination problems which are more difficult for degenerate graphs. For instance we show that Partial $k$-dominating set is W[1]-hard, and $(k, r)$-Center is W[2]-hard for this class. We conclude our paper by the observation that domination problems become difficult for graphs of bounded average degree.

## 2 Preliminaries

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$ and its edge set by $E(G)$ (or simply $V$ and $E$ if it does not create a confusion). For $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of $G$ induced by $S$. A set $S \subseteq V(G)$ is called *connected* if $G[S]$ is a connected graph.

The *open neighborhood* of a vertex is denoted by $N_G(v) = \{x\colon xv \in E(G)\}$. For a positive integer $r$ we define *closed r-neighborhood* of a vertex $v$, denoted $N_G^r[v]$, to be the set of vertices of $G$ at distance at most $r$ from $v$. If $U \subset V(G)$, then $N_G^r[U]$ denotes the set $\bigcup_{v\in U} N_G^r[v]$. We use notations $N_G[v]$ and $N_G[U]$ for $r = 1$. Degree of a vertex $v$ is denoted by $\deg_G v$, $\Delta(G) = \max\{deg_G v\colon v \in V(G)\}$. We omit a subscript $G$ if it does not create a confusion. The *average degree* of $G$ is defined as $\frac{1}{|V(G)|}\sum_{v\in V(G)} \deg v = \frac{2|E(G)|}{|V(G)|}$. It is known (end easy to see) that every $d$-degenerate graph has the average degree no more than $2d$.

It is said that a vertex $v$ is *dominated* by the vertex $u$ if $v \in N[u]$, and it is said that the vertex $v$ is *dominated* by set $S \subseteq V(G)$ if $v \in N[S]$. Recall that a set $S \subseteq V(G)$ is called a *dominating set* if every vertex of $G$ is dominated by $S$.

The $k$-Dominating set is the following problem:

INSTANCE: A graph $G$.
PARAMETER: A positive integer $k$.
QUESTION: Is there a dominating set $S \subseteq V(G)$ such that $|S| \leq k$?

Recently Alon and Gutner [2] proved $k$-Dominating set to be FPT on $d$-degenerate graphs by proving the proposition below. This proposition will be required in both the FPT algorithms presented in the next sections. In order to apply the proposition the problem is rephrased in terms of black and white vertices. It is supposed that the vertex set $V$ is partitioned into two sets $W$ (white vertices) and $B$ (black vertices). The goal is to find a set $S \subset W \cup B$ which dominates $B$.

**Proposition 1.** *([2]) Let $G = (B \cup W, E)$ be a d-degenerate black and white graph. If $|B| > (4d + 2)r$, then there are at most $(4d + 2)r$ vertices in $G$ that dominate at least $|B|/r$ vertices of $B$.*

This proposition enable us to apply the method of bounded search trees for the $k$-Dominating set problem.

## 3 FPT Algorithms on *d*-Degenerate Graphs

### 3.1 Connected Domination

This subsection deals with the problem of finding a connected dominating set of size at most $k$ in a $d$-degenerate graph which we call Connected $k$-Dominating set. Formally we define the problem as follows:

INSTANCE: A $d$-degenerate graph $G$.
PARAMETER: Positive integers $d$ and $k$.
QUESTION: Does there exist a dominating set $S \subseteq V$ such that $G[S]$ is connected and $|S| \leq k$?

It is well known that finding a connected dominating set of size $k$ is $W[2]$ hard on general graphs (see e.g. [13]). The rest of this section presents an FPT algorithm for this problem.

We assume without loss of generality that $G$ is a connected graph with $n$ vertices and $n \geq k$. Like in the paper by Alon and Gutner [2] a new problem instance based on black and white vertices is created. Let $(S, W, B, q)$ be a problem instance for the graph $G$, where $S$ contains vertices that must be contained in the dominating set, the set $W \subseteq V \setminus S$ (white vertices) contains vertices that are dominated by $S$, the set $B$ (black vertices) contains the undominated vertices $V \setminus (S \cup W)$, and $q = k - |S|$. Our goal is to decide if $q$ vertices from $W \cup B$ can be added to $S$, such that $S$ becomes a connected dominating vertex set of size $k$. The initial problem instance for a graph $G$ will be $(S = \emptyset, W = \emptyset, B = V, q = k)$. Each time a vertex in $W \cup B$ is moved to $S$ the parameter $q$ will be reduced by one. From the initial problem instance, we will grow a tree of problem instances, where the final leafs either contains a solution, or claims that the choices made on the path from the root to the leaf is not consistent with any solution.

Connected or not, every vertex set of cardinality $q$ in $W \cup B$ that dominates $B$, contains a vertex that dominates at least $|B|/q$ of the vertices in $B$. By Proposition 1, $|B| \leq (4d+2)q$ or there exists at most $(4d+2)q$ vertices in $W \cup B$ that dominate at least $|B|/q$ vertices in $B$. As long as $|B| > (4d+2)q$, add a leaf $(S \cup \{u\}, W \cup (N(u) \cap B) \setminus \{u\}, B \setminus N[u], q-1)$ to the problem instance $(S, W, B, q)$ for every vertex $u \in W \cup B$ where $|N[u] \cap B| \geq |B|/q$. By Proposition 1 we can now conclude that problem instance $(S, W, B, q)$ contains a solution if and only if one of the added leafs contains a solution. Thus, the initial problem instance $(S, W, B, q)$ can be ignored from this point on. This process of reducing the size of the problem instance by creating several new problem instances where one of them contains a solution if and only if the initial problem instance contained a solution will be referred to as *branching*. The number of new problem instances is called the degree of the branching.

Like the algorithm for dominating set on $d$-degenerate graphs [2] we branch on the at most $(4d+2)q$ vertices that dominates $|B|/q$ vertices in $B$, until $q = 0$ or $|B| \leq (4d+2)q$. If $q = 0$ and $S$ is a connected dominating set, return the vertex set $S$ as the solution, otherwise mark the problem instance with no solution.

Let us now consider the case, where $q \geq 1$ and $|B| \leq (4d+2)q$. Denote by $v_1, v_2, ..., v_q$ the vertices of the connected dominating set, which are not contained in $S$. Clearly, the subgraph of $G$ induced by vertices from $S$ and $v_1, v_2, ..., v_q$ must have some spanning tree $T$. Label the vertices of $S$ and $v_1, v_2, ..., v_q$ in any order, this makes $T$ into a labeled spanning tree. We will now extend the our problem instance by adding a labeled tree containing $k$ vertices. By Cayley's theorem the number of labeled spanning trees containing $k$ vertices is $k^{k-2}$. Add $k^{k-2}$ leafs to the problem instance $(S, W, B, q)$, one for each possible labeled tree $T$ over $k$ vertices. Notice that one of these will be equal to the labeled tree obtained by the spanning tree of a connected dominating set and the labelling assigned to $S$ and $v_1, v_2, ..., v_q$. From now on a problem instance $(S, W, B, q, T)$ is considered.

In any connected dominating set of size $k$, the neighborhood of $v_i$ in $v_1, v_2, ..., v_q$ will define a subset $V_i$ of $B$ which is dominated by $v_i$. Branch on the no more than $q^{(4d+2)q}$ different ways of putting the vertices of $B$ into the $q$ vertex sets $V_1, V_2, ..., V_q$ (some sets can be empty).

Now a dynamic programming algorithm is used to find vertices $v_1, v_2, \ldots, v_q$ or prove that there are no such vertices. Choose a vertex $z$ of $T$ as a root of the tree. This induces a parent-child relation in the tree, which defines the set of leafs. For every vertex $x \in V(T)$ a vertex set $U(x)$ is defined. For every leaf $x$ of $T$, the vertex set $U(x) = \{x\}$ if $x \in S$ and $U(x) = \{v \in W \cup B \colon V_i \subseteq N_G[v]\}$ if $x = v_i$ for $i \in \{1, 2, ..., q\}$. Assume now that $x$ in $T$ has children $y_1, y_2, \ldots, y_t$ such that $U(y_i)$ is defined for $i \in \{1, 2, \ldots, t\}$. There are two cases, let us first assume that $x \in S$. If $N_G[x] \cap U(y_j) \neq \emptyset$ for all $j \in \{1, 2, \ldots, t\}$ then $U(x) = \{x\}$, otherwise $U(x) = \emptyset$. Let us now assume that $x = v_i$ for $i \in \{1, 2, ..., q\}$. Then, $U(x) = \{v \in W \cup B \colon V_i \subseteq N_G[v]$ and $N_G[v] \cap U(y_j) \neq \emptyset$ for all $j \in \{1, 2, \ldots, t\}\}$. If $U(z) = \emptyset$ then the considered problem instance has no solution, and if $U(z) \neq \emptyset$ then by the dynamic programming it is easy to chose vertices from $U(v_1), U(v_2), \ldots, U(v_q)$ so that they together with vertices of $S$ compose a connected dominating set of cardinality no more than $k$. Note that it is possible that same vertices are chosen from different sets $U(v_i)$, but it only means that we have a connected dominating set of lesser cardinality.

The properties of our algorithm are summarized in the following theorem.

**Theorem 1.** *The described algorithm decides in $O(k^{O(dk)} \cdot n^{O(1)})$ time if a $d$-degenerate graph contains a connected dominating set of size $k$.*

*Proof.* The correctness of the algorithm follows from the description above. From the initial problem instance the algorithm above create a branching tree, which has the following properties. For every internal vertex of the branching tree, there exists a child that has a solution if and only if the parent contains a solution. For every leaf instance, we can decide in $O(n^{O(1)})$ time if there exists a solution to the instance, and every problem instance is created in $O(n^{O(1)})$ time.

It remains to bound the number of instances in the branching tree, let us count them. At most $((4d+2)k)^k$ problem instances are created when branching on vertices that dominates at least $|B|/k$ of the vertices in $B$. There exists $k^{k-2}$ trees containing $k$ vertices, and these trees can be listed with complexity $O(k^{k-1})$ [15]. At most $k^{(4d+2)k}$ problem instances are created when distributing vertices of $B$ into the vertex sets $V_1, V_2, ..., V_k$. Total number of created problem instances is obtained by multiplying these numbers, which give the total $O(k^{O(dk)})$. Thus, the running time is $O(k^{O(dk)} \cdot n^{O(1)})$.

### 3.2 Dominating Threshold Set

For a given graph $G = (V, E)$, and integers $k$ and $r$, a vertex set $S \subset V$ is a *dominating threshold set* if the closed neighborhood $N[v]$ contains at least $r$

vertices in $S$ for every vertex $v \in V$. Formally the $k$-Dominating threshold set problem is defined as follows:

INSTANCE: A graph $G$ and a positive integer $r$.
PARAMETER: A positive integer $k$.
QUESTION: Is there a set $S \subseteq V(G)$ such that $|S| \leq k$ and for every vertex $u \in V(G)$ $|N[u] \cap S| \geq r$?

This problem can be solved on $d$-degenerate graphs in a similar way as the Connected $k$-dominating set problem was solved on $d$-degenerate graphs.

We assume that $n = |V|$ and $k \leq n$. If $r > d+1$ or $r > k$ then the graph has no dominating threshold set. So we suppose that $r \leq min(d+1, k)$.

Let us rephrase the problem in terms of black and white vertices. Consider the problem instance $(S, W, B, q)$, where $S$ are vertices in the dominating threshold set, $W$ are vertices that are dominated by at least $r$ vertices in $S$, $B = V \setminus W$, and $q = k - |S|$. Notice that even vertices in $S$ has to be dominated by $r$ vertices, so adding a vertex $v$ to $S$, do not enable us to remove it from $B$. The initial problem instance will be $(S = \emptyset, W = \emptyset, B = V, q = k)$.

Dominating threshold set are also dominating sets, so Proposition 1 applies here as well. Thus, $|B| \leq (4d+2)q$ or there exists at most $(4d+2)q$ vertices in $W \cup B$ that dominates at least $|B|/q$ of the vertices in $B$. While $|B| \geq (4d+2)q$ we branch and create one new problem instance $(S \cup \{u\}, W \cup U, B \setminus U, q-1)$ for each vertex $u \in (W \cup B) \setminus S$ that dominates at least $|B|/q$ vertices in $B$, where $U$ is the set of vertices in $N[u] \cap B$ that are dominated by $r-1$ vertices in $S$. Repeat this branching until $q = 0$ or $|B| \leq (4d+2)q$. If $q = 0$ and $B = \emptyset$ return $S$ as the solution, otherwise if $B \neq \emptyset$ then mark the problem instance with no solution.

Consider now the case where $|B| \leq (4d+2)q$. It is not enough to find a dominating set of $B$ in this case since every vertex requires $r$ neighbors in $S$. Like the connected domination set problem we define the vertex sets $V_1, V_2, ..., V_q \subseteq B$, but this time every vertex of $B$ can be added to several sets. The reason for this is that it might be missing more than one dominator.

Now, branch on the at most $q^{(4d+2)qr}$ different ways of adding the at most $(4d+2)q$ vertices of $B$ to the vertex sets $V_1, V_2, ..., V_q$ in such a way that for every vertex $v \in B$ $|\{i \colon v \in V_i\}| + |\{u \in S \colon v \in N[u]\}| \geq r$. If there are no such sets then the problem instance has no solution. Otherwise for every new instance we are trying to add to our set $S$ vertices $v_1, v_2, \ldots, v_q$ such that $v_i$ dominates exactly set $V_i$.

Define $U_i = \{w \colon w \in (W \cup B) \setminus S \text{ and } V_i = N[w] \cap B\}$ for $i \in \{1, 2, \ldots, q\}$. Clearly there is no solution if some vertex set $U_i$ is empty. Also if $V_i \neq V_j$ then $U_i \cap U_j = \emptyset$. Let $s_i = |\{j \colon V_j = V_i\}|$ for every $i \in \{1, 2, \ldots, q\}$. If $|U_i| < s_i$ for some $i \in \{1, 2, \ldots, q\}$ then the problem instance has no solution. Otherwise we consider all pairwise different sets $U_i$. From every such set $U_i$ we chose $s_i$ different vertices and add them to the set $S$.

**Theorem 2.** *The described algorithm decides in $O(k^{O(dkr)} \cdot n^{O(1)})$ time if a $d$-degenerate graph contains a dominating $r$-threshold set of size $k$.*

*Proof.* The correctness of the algorithm follows from the description above. From the initial problem instance the algorithm create a branching tree, which has the following properties. For every internal vertex of the branching tree, there exists a child that has a solution if and only if the parent contains a solution. For every leaf instance, we can decide in $O(n^{O(1)})$ time if there exists a solution to the instance, and every instance is created in $O(n^{O(1)})$ time.

It remains to bound the number of instances in the branching tree. At most $((4d+2)k)^k$ problem instances are created when branching on vertices that dominates at least $|B|/k$ of the vertices in $B$. There exists no more than $k^{(4d+2)kr} = k^{O(dkr)}$ different ways of adding vertices from $B$ to the vertex sets $V_1, V_2, ..., V_q$. The total number of created problem instances is then $O(k^{O(dkr)})$. Thus, the running time is $O(k^{O(dkr)} \cdot n^{O(1)})$.

## 4 Partial Domination

Here we consider a variant of domination problem, in which it is not necessary to dominate all vertices of a graph, but at least the given number of vertices. The PARTIAL $k$-DOMINATING SET problem is formulated as follows:

INSTANCE: A graph $G$ and a positive integer $N$.
PARAMETER: A positive integer $k$.
QUESTION: Is there a set $S \subseteq V(G)$ such that $|S| \leq k$ and which dominates at least $N$ vertices?

It can be easily seen that this problem is W[2]-complete on general graphs(if $N = |V(G)|$ then PARTIAL $k$-DOMINATING SET is the $k$-DOMINATING SET problem). Note that here $N$ is a part of the instance, but is not a parameter of the problem. If $N$ is supposed to be a parameter of the problem then it is FPT [17]. Recall that Amini et al. proved that PARTIAL $k$-DOMINATING SET is FPT for H-minor-free graphs [3]. We prove that PARTIAL $k$-DOMINATING SET is difficult for degenerate graphs.

**Theorem 3.** PARTIAL $k$-DOMINATING SET *is W[1]-hard for 2-degenerate graphs.*

*Proof.* We reduce the $k$-PERFECT CODE problem. A *perfect code* in a graph $G$ is a set of vertices $S \subseteq V(G)$ with the property that for every vertex $v \in V(G)$, there is exactly one vertex from $S$ in $N[v]$. The $k$-PERFECT CODE is the following problem:

INSTANCE: A graph $G$.
PARAMETER: A positive integer $k$.
QUESTION: Is there a perfect code $S \subseteq V(G)$ of size $k$?

It is known [6,12] that this problem is W[1]-complete.

Let $G$ be a graph with $n$ vertices and $m$ edges. It can be assumed without loss of generality that this graph is connected and has at least 2 vertices. We construct graph $G'$ starting with the vertex set $V = V(G)$. Let $t = n^3$ and

$r = \Delta(G)^2 t$. If vertices $u, v \in V$ are adjacent in $G$ or are at distance 2 in this graph then $u$ and $v$ are joined by $t$ paths of length two. Then for every vertex $v \in V$ add $r - t(|N_G^2[v]| - 1)$ adjacent pendant vertices. Denote by $U$ the set of vertices of degree 1 or 2 which were included to the vertex set of $G'$ at this stage of our construction. Now for every vertex $v \in V$ we execute the following operation: for every two different vertices $x, y \in N_G[v]$ a $x, y$-path of length two is added to $G'$. Denote by $W(v)$ the set of vertices of degree 2 which were added during this operation for the vertex $v$, and let $W = \bigcup_{v \in V} W(v)$. Note that some vertices can be joined by several paths after these operations for all vertices of $V$, but since any two different vertices of $V$ can belong to closed neighborhoods of no more than $n$ vertices, the number of such paths is no more than $n$. Then $|W| \leq \frac{n^2(n-1)}{2}$. Clearly $G'$ is 2-degenerate, and our construction of $G'$ is polynomial. Now we define $N = (r+1)k + 2m$.

Suppose that $S \subseteq V(G)$ is a perfect code in $G$. It can be easily seen that $N_{G'}[S] \cap V = S$. Since $S$ is a perfect code, vertices of $S$ are at distance at least 3 in the graph $G$. It follows immediately that $|N_{G'}[S] \cap U| = kr$. For every vertex $v \in V$ exactly one vertex $x \in N_G[v]$ belongs to $S$. Then $N_{G'}[S] \cap W(v) = N_{G'}(x) \cap W(v)$, and $|N_{G'}(x) \cap W(v)| = \deg_G v$. So $|N_{G'}[S] \cap W| = \sum_{v \in V} \deg_G v = 2m$. Now $|N_{G'}[S]| = |N_{G'}[S] \cap V| + |N_{G'}[S] \cap U| + |N_{G'}[S] \cap W| = (1+r)k + 2m = N$.

Assume now that $S \subseteq V(G')$, $|S| \leq k$ and $|N_{G'}[S]| \geq N$. Suppose that $|S| < k$. Then $|N_{G'}[S]| = |N_{G'}[S] \cap (V \cup U)| + |N_{G'}[S] \cap W| \leq |S|(r+1) + |W| \leq |S|(r+1) + \frac{n^2(n-1)}{2} < (|S|+1)(r+1) \leq N$. So $|S| = k$. If the set $S$ contains a vertex from $U \cup W$ then $|N_{G'}[S]| = |N_{G'}[S] \cap (V \cup U)| + |N_{G'}[S] \cap W| \leq (k-1)(r+1) + |W| + 3 \leq (k-1)(r+1) + \frac{n^2(n-1)}{2} + 3 < k(r+1) \leq N$. This contradiction means that $S \subseteq V$. Suppose that $S$ contains vertices that are adjacent or 2-distant in $G$. In this case $|N_{G'}[S]| = |N_{G'}[S] \cap (V \cup U)| + |N_{G'}[S] \cap W| \leq k(r+1) - t + |W| \leq k(r+1) + \frac{n^2(n-1)}{2} - t < k(r+1) \leq N$, and we conclude that for every $v \in V$ $N_G[v]$ contains no more than one vertex from $S$. If there is a vertex $v \in V$ such that there are no vertices from $S$ in $N_G[v]$ then $|N_{G'}[S]| = |N_{G'}[S] \cap (V \cup U)| + |N_{G'}[S] \cap W| \leq k(r+1) + 2m - \deg_G v < N$. It follows immediately that $S$ is a perfect code of the size $k$ in $G$.

## 5 $(k, r)$-Center Problem

The $(k, r)$-Center (see e.g. [4] for the background of this problem) is another example of domination problem which becomes difficult for degenerate graphs. Let $r$ be a positive integer. The set $S \subseteq V(G)$ is called a *$r$-center* if $N^r[S] = V(G)$. The $(k, r)$-Center is the following problem:

INSTANCE: A graph $G$.
PARAMETER: Positive integers $k$ and $r$.
QUESTION: Is there a $r$-center $S \subseteq V(G)$ such that $|S| \leq k$?

Clearly $k$-DOMINATING SET is a special case of this problem for $r = 1$, hence $(k, r)$-CENTER is W[2]-hard for general graphs. We prove that for $r \geq 2$ results by Amini et al. [3] can not be extended for degenerate graphs if FPT $\neq$ W[2].

**Theorem 4.** *For any $r \geq 2$ the $(k, r)$-CENTER is W[2]-hard for 2-degenerate graphs.*

*Proof.* We reduce $k$-DOMINATING SET problem. Let $G$ be a connected nonempty graph, and $V = V(G)$. Every edge of $G$ is replaced by the path of length $r$. We call a vertex $x$ of such a path the *central* vertex if it is at distance $\lfloor \frac{r}{2} \rfloor$ from one of endpoints (if $r$ is even every path has one central vertex, and there are two central vertices if $r$ is odd). Then a new vertex $u$ is introduced and joined by paths of length $\lceil \frac{r}{2} \rceil + 1$ with all central vertices. At the final stage of the construction a vertex $v$ is added and joined with $u$ by the path $P$ of length $r$. Denote the obtained graph by $G'$. Clearly $G'$ is 2-degenerate.

We prove that $G$ has a dominating set of a size at most $k$ if and only if $G$ has a $r$-center of a size at most $k+1$. Suppose that $S$ is the dominating set in $G$ and $|S| \leq k$. It can be easily seen that $S \cup \{u\}$ is a $r$-center of $G'$ and $|S| \leq k+1$. Assume now that $S'$ is a $r$-center of $G'$ and $|S'| \leq k+1$. At least one vertex of the path $P$ is included to $S'$. Without loss of generality we can assume that $u$ is a unique vertex of this path which belongs to $S'$. Note that $V(G') \setminus V = N^r_{G'}[u]$. Let $S = S' \setminus \{u\}$. Suppose that there is a vertex $x \in S$ such that $x \notin V$. Then either $x$ is a vertex of the path which replaced some edge $ab \in E(G)$ or it belongs to the path which connects $u$ with the central vertex of such a path. Only vertices $a$ and $b$ in $V$ are at distance at most $r$ from $x$. Then we can replace $x$ in $S$ by $a$ or $b$. It means that we can assume that $S \subseteq V$. It can be easily seen that $S$ is a dominating set of $G$, and $|S| \leq k$.

## 6 Domination Problems for Graphs of Bounded Average Degree

It is known that some graph covering problems (like $k$-INDEPENDENT SET) are FPT for graphs of bounded average degree, but it can be simply proved that domination problems are W[1] or W[2]-hard for this class.

**Proposition 2.** *The $k$-DOMINATING SET problem is W[2]-hard for graph of bounded average degree.*

*Proof.* We reduce $k$-DOMINATING SET for general graphs. Let $G$ be a graph with $n$ vertices and $m$ edges. Define $G'$ as a union of $G$ and a star $K_{1,r}$ for $r = n^2$. The average degree of $G'$ is equal to $\frac{2m+2r}{n+r+1} \leq \frac{3n^2}{n^2} = 3$, i.e. this graph has bounded average degree. It can be easily seen that $G$ has a dominating set of a size $k$ if and only $G'$ has a $k+1$-element dominating set.

By same reduction it can be easily proved that $k$-PERFECT CODE is W[1]-hard for graphs of bounded average degree and INDEPENDENT $k$-DOMINATING SET is W[2]-hard for this class. For connected dominating set reduction is slightly different.

**Proposition 3.** *The* CONNECTED $k$-DOMINATING SET *problem is W[2]-hard for graph of bounded average degree.*

*Proof.* We reduce $k$-DOMINATING SET for general graphs. Let $G$ be a graph with $n$ vertices and $m$ edges. Suppose that $V(G) = \{v_1, v_2, \dots, v_n\}$. We introduce two copies of the set $V(G)$: $U = \{u_1, u_2, \dots, u_n\}$ and $W = \{w_1, w_2, \dots, w_n\}$. Vertices $u_i$ and $w_j$ are joined by an edge if $i = j$ or $v_iv_j \in E(G)$. Then a new vertex $z$ is added and joined by edges with all vertices of $W$. At the final stage of the construction $n^2$ pendant vertices adjacent to $z$ are added. Denote the obtained graph $G'$. The average degree of $G'$ is equal to $\frac{4m+2n+2n^2}{2n+1+n^2} \leq 4$, i.e. this graph has bounded average degree. It can be easily seen that $G$ has a dominating set of a size $k$ if and only $G'$ has a $k+1$-element connected dominating set. If $S$ is a dominating set in $G$ then $S' = \{w_i : v_i \in S\} \cup \{z\}$ is a connected dominating set in $G'$. Suppose that $S'$ is a connected dominating set in $G'$ of size at most $k+1$. Clearly, $z \in S'$, and we can assume that all other vertices of this set belong to $U \cup W$. If some vertex $u_i \in S'$ then it can be replaced by $w_i$ in our dominating set. So we can also assume that $S' \setminus \{z\} \subseteq W$. We have only note that $S = \{v_i \in V(G) : w_i \in S' \setminus \{z\}\}$ is a dominating set in $G$.

## 7 Conclusion

We proved that the $k$-domination problem remains FPT for degenerate graphs, even if additional restrictions like connectivity or a threshold boundary is added. On the other side the $k$-domination problem becomes W[1] or W[2]-hard on degenerate graphs, when a partial or $r$-center domination is required. It could be interesting to obtain a sharper boundary between the FPT and W-hardness for different classes of sparse graphs. For example, it easily follows from the results of [3] that the PARTIAL $k$-VERTEX COVER problem is FPT for degenerate graphs, but this problem is W[1]-complete for general graphs [16]. By using the same reduction as in Theorem 2, the PARTIAL $k$-VERTEX COVER is W[1]-hard for graphs of bounded average degree. At the same time it is well known that the $k$-INDEPENDENT SET which is W[1]-hard for general graphs is FPT for this class. Another interesting problem is a construction of more efficient FPT-algorithms for domination problems on $d$-degenerate graphs.

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