A Game Theoretic Approach for Efficient Graph Coloring*

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Abstract. We give an efficient local search algorithm that computes a good vertex coloring of a graph G. In order to better illustrate this local search method, we view local moves as selfish moves in a suitably defined game. In particular, given a graph $G = (V, E)$ of n vertices and m edges, we define the *graph coloring game* $\Gamma(G)$ as a strategic game where the set of players is the set of vertices and the players share the same action set, which is a set of n colors. The payoff that a vertex v receives, given the actions chosen by all vertices, equals the total number of vertices that have chosen the same color as v, unless a neighbor of v has also chosen the same color, in which case the payoff of v is 0. We show:

- The game $\Gamma(G)$ has always pure Nash equilibria. Each pure equilibrium is a proper coloring of G. Furthermore, there exists a pure equilibrium that corresponds to an optimum coloring.
- We give a polynomial time algorithm $\mathcal{A}$ which computes a pure Nash equilibrium of $\Gamma(G)$.
- The total number, k, of colors used in *any* pure Nash equilibrium (and thus achieved by $\mathcal{A}$) is $k \leq \min\{\Delta_2 + 1, \frac{n+\omega}{2}, \frac{1+\sqrt{1+8m}}{2}, n - \alpha + 1\}$, where ω, α are the clique number and the independence number of G and Δ_2 is the maximum degree that a vertex can have subject to the condition that it is adjacent to at least one vertex of equal or greater degree. (Δ_2 is no more than the maximum degree Δ of G.)
- Thus, in fact, we propose here a new, efficient coloring method that achieves a number of colors satisfying (together) the known general upper bounds on the chromatic number χ. Our method is also an alternative general way of proving, constructively, all these bounds.
- Finally, we show how to strengthen our method (staying in polynomial time) so that it avoids "bad" pure Nash equilibria (i.e. those admitting a number of colors k far away from χ). In particular, we show that our enhanced method colors *optimally* dense random q-partite graphs (of fixed q) with high probability.

* Partially supported by the EU within the 6th Framework Programme under contract 015964 "Algorithmic Principles for Building Efficient Overlay Computers" (AEOLUS) and the ICT Programme under contract IST-2008-215270 (FRONTS), and by the General Secretariat for Research and Technology of the Greek Ministry of Development within the Programme PENED 2003.

S.-H. Hong, H. Nagamochi, and T. Fukunaga (Eds.): ISAAC 2008, LNCS 5369, pp. 183–195, 2008.

1 Introduction

Overview. One of the central optimization problems in Computer Science is the problem of *vertex coloring* of graphs: given a graph $G = (V, E)$ of n vertices, assign a color to each vertex of G so that no pair of adjacent vertices gets the same color and so that the total number of distinct colors used is minimized. The global optimum of vertex coloring (the *chromatic number*) is, in general, inapproximable in polynomial time unless a collapse of some complexity classes happens [7]. In this paper, we propose an efficient vertex coloring algorithm that is based on *local search*: Starting with an arbitrary proper vertex coloring (e.g. the trivial proper coloring where each vertex is assigned a unique color), we do local changes, by allowing each vertex (one at a time) to move to another color class of higher cardinality, until no further local moves are possible.

We choose to illustrate this local search method via a game-theoretic analysis; we do so because of the natural correspondence of the local optima of our proposed method to the pure Nash equilibria of a suitably defined *strategic game*. In particular, we view vertices of a graph $G = (V, E)$ as players in a strategic game. Each player has the same set of actions, which is a set of $|V|$ colors. In a certain profile $\mathbf{c}$ (where each vertex v has chosen a color), v gets a payoff of zero if its color is the same with the color of a neighbor of v. Else, v gets as a payoff the number of vertices having selected the same color as the color that v has chosen. In a pure Nash equilibrium of such a game (if such an equilibrium exists), no vertex can improve its payoff by unilaterally deviating. Note that, given a profile, one can compute payoffs in small polynomial time. Furthermore, a "better response" (i.e., a selfish improvement) of a vertex v, given a choice of colors by all the other vertices, can also be computed quickly by v and the only global information needed is the number of vertices per color in the graph.

In such a setting, if we start by the trivial proper coloring of G (where each v chooses its unique name as a color), then any selfish improvement sequence always produces proper colorings of G. This would give an efficient and general proper coloring heuristic provided that: (i) Pure equilibria exist; (ii) Such selfish improvement sequences reach an equilibrium in small time; and (iii) The number of colors at equilibrium is a good approximation of the chromatic number of G.

Our Results. Quite surprisingly, we show for our game that:

(1) Any selfish improvement sequence, when started with a *proper* (e.g., the trivial) coloring, always reaches an equilibrium in $O(n \cdot \alpha(G))$ selfish moves, where $\alpha(G)$ is the independence number of G. We prove this by a potential-based method [14].

(2) Any pure Nash equilibrium of the game is a *proper* coloring of G that uses a number of colors, k, bounded above by all the general known to us upper bounds on the chromatic number of G. Specifically, let $n, m, \chi(G)$ and $\omega(G)$ and $\Delta(G)$, denote the number of vertices, number of edges, chromatic number, clique number and maximum degree of G, respectively. Let $\Delta_2(G)$ be the maximum degree that a vertex v can have subject to the condition that v is adjacent to at least one vertex of degree no less than the degree of v (note that $\Delta_2(G) \leq \Delta(G)$). We show that k (in any pure Nash equilibrium) satisfies

$$k \leq \min\left\{\Delta_2(G)+1,\ \frac{n+\omega(G)}{2},\ n-\alpha(G)+1,\ \frac{1+\sqrt{1+8m}}{2}\right\}.$$

Note that $\Delta_2(G)+1$ is the bound of Stacho [16] and implies Brooks' bound [4] on $\chi(G)$. In fact, we get *constructively* all these bounds via a single polynomial time algorithm. For some of these bounds their proofs till now (in popular graph theory books, e.g. [9]) are not constructive and not based on a single unifying method.

(3) Since $\chi(G)$ is inapproximable in polynomial time (unless a collapse of complexity classes happens) it is natural to expect the existence of some pure equilibria in our game that use a number of colors k far away from $\chi(G)$. Indeed we were able to construct a class of (almost complete bipartite) graphs G which have equilibrium colorings of $k = \frac{n}{2}+1$, while $\chi(G) = 2$. However, our selfish improvement method does not have to go to such bad equilibria. For the same class of graphs we show that a *randomized* sequence of selfish improvements achieves $k = 2$ with high probability. In fact, our class of algorithms can be started by the proper colorings achieved by the *best till now* approximation methods. Then, it may improve on them, if their output is not an equilibrium of our game.

(4) Motivated by such thoughts, we investigated the following question: What kind of polynomial time "mechanisms" (e.g., some preprocessing, a particular order of selfish moves, e.t.c.) can help our coloring method to get closer to $\chi(G)$ in certain graph classes? We managed to provide such enhanced methods that e.g. are optimal with high probability for dense random q-partite graphs.

We believe that our game and its properties can serve also as an educational tool in introducing and proving general bounds on the chromatic number.

Previous work. The problem of coloring a graph using the minimum number of colors is NP-hard [13], and the best polynomial time approximation algorithm achieves an approximation ratio of $O(n(\log\log n)^2/(\log n)^3)$ [8]. It is known [7] that the chromatic number cannot be approximated to within $\Omega(n^{1-\epsilon})$ for any constant $\epsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{co\text{-}RP}$. Several vertex coloring heuristics have been proposed in the literature, such as Brelaz's heuristic [3]. To the best of our knowledge, none of these heuristics achieves all these bounds on the total number of colors that our algorithm guarantees. Graph coloring games have been studied before, but in a very different context than here. In these games there are 2 players, who are introduced with the graph to be colored and a color bound k. A legal move of either player consists of choosing an uncolored vertex v, and assign to it any of the k colors that has not been assigned to any neighbor of v. In one variant of such a game [1] the first player which is unable to move loses the game. In another variant [1] the first player wins if and only if the game ends with all vertices colored. Further variants have also been studied by e.g. [10,6].

2 The Model

Notation. For a finite set A we denote by $|A|$ the cardinality of A. For an event E in a sample space, denote $\Pr\{E\}$ the probability of E occurring. Denote $G = (V, E)$ a simple, undirected graph with vertex set V and set of edges E. For a vertex $v \in V$ denote $N(v) = \{u \in V : \{u, v\} \in E\}$ the set of its neighbors, and let $\deg(v) = |N(v)|$ denote its degree. Let $\Delta(G) = \max_{v \in V} \deg(v)$ be the maximum degree of G. Let $\Delta_2(G) = \max_{u \in V} \max_{v \in N(u): d(v) \leq d(u)} \deg(v)$ be the maximum degree that a vertex v can have, subject to the condition that v is adjacent to at least one vertex of degree no less than $\deg(v)$. Clearly, $\Delta_2(G) \leq \Delta(G)$. Let $\chi(G)$ denote the chromatic number of G, i.e. the minimum number of colors needed to color the vertices of G such that no adjacent vertices get the same color (i.e., the minimum number of colors used by a *proper coloring* of G). Let $\omega(G)$ and $\alpha(G)$ denote the clique number and independence number of G, i.e. the number of vertices in a maximum clique and a maximum independent set of G.

The Graph Coloring Game. Given a finite, simple, undirected graph $G = (V, E)$ with $|V| = n$ vertices, we define the *graph coloring game* $\Gamma(G)$ as the game in strategic form where the set of players is the set of vertices V, and the action set of each vertex is a set of n colors $X = \{x_1, \ldots, x_n\}$. A *configuration* or *pure strategy profile* $\mathbf{c} = (c_v)_{v \in V} \in X^n$ is a combination of actions, one for each vertex. That is, c_v is the color chosen by vertex v. For a configuration $\mathbf{c} \in X^n$ and a color $x \in X$, we denote by $n_x(\mathbf{c})$ the number of vertices that are colored x in $\mathbf{c}$, i.e. $n_x(\mathbf{c}) = |\{v \in V : c_v = x\}|$. The *payoff* that vertex $v \in V$ receives in the configuration $\mathbf{c} \in X^n$ is

$$\lambda_v(\mathbf{c}) = \begin{cases} 0 & \text{if } \exists u \in N(v) : c_u = c_v \\ n_{c_v}(\mathbf{c}) & \text{else} \end{cases} .$$

A *pure Nash equilibrium* [15] (PNE in short) is a configuration $\mathbf{c} \in X^n$ such that no vertex can increase its payoff by unilaterally deviating. Let $(x, \mathbf{c}_{-v})$ denote the configuration resulting from $\mathbf{c}$ if vertex v chooses color x while all the remaining vertices preserve their colors. Then

Definition 1. *A configuration $\mathbf{c} \in X^n$ of the graph coloring game $\Gamma(G)$ is a pure Nash equilibrium if, for all vertices $v \in V$, $\lambda_v(x, \mathbf{c}_{-v}) \leq \lambda_v(\mathbf{c}) \quad \forall x \in X$.*

A vertex $v \in V$ is *unsatisfied* in the configuration $\mathbf{c} \in X^n$ if there exists a color $x \neq c_v$ such that $\lambda_v(x, \mathbf{c}_{-v}) > \lambda_v(\mathbf{c})$; else we say that v is *satisfied.* For an unsatisfied vertex $v \in V$ in the configuration $\mathbf{c}$, we say that v performs a *selfish step* if v unilaterally deviates to some color $x \neq c_v$ such that $\lambda_v(x, \mathbf{c}_{-v}) > \lambda_v(\mathbf{c})$.

The *Social Cost* $\mathsf{SC}(G, \mathbf{c})$ of a configuration $\mathbf{c} \in X^n$ of $\Gamma(G)$ is the number of distinct colors in $\mathbf{c}$, i.e., $\mathsf{SC}(G, \mathbf{c}) = |\{x \in X \mid n_x(\mathbf{c}) > 0\}|$. Given a graph G, the *Approximation Ratio* $\mathsf{R}(G)$ is the ratio of the worst, over all pure Nash equilibria of $\Gamma(G)$, Social Cost to the chromatic number: $\mathsf{R}(G) = \max_{\mathbf{c}\,:\,\mathbf{c} \text{ is a PNE}} \frac{\mathsf{SC}(G,\mathbf{c})}{\chi(G)}$.

3 Existence and Tractability of Pure Nash Equilibria

Theorem 1. *Every graph coloring game $\Gamma(G)$ possesses at least one pure Nash equilibrium, and there exists a pure Nash equilibrium* $\mathbf{c}$ *with* $\mathsf{SC}(G, \mathbf{c}) = \chi(G)$.

Proof. Consider any optimum coloring $\mathbf{o} = (o_v)_{v \in V} \in X^n$ of G. Then $\mathbf{o}$ uses $k = \chi(G)$ colors. For each optimum coloring $\mathbf{o}$ consider the vector $\mathbf{L_o} = (\ell_{\mathbf{o}}(1), \ldots, \ell_{\mathbf{o}}(k))$, where $\ell_{\mathbf{o}}(j)$ is the number of vertices that are assigned the color that is jth in the decreasing ordering of colors according to the number of vertices that use them. Let $\hat{\mathbf{o}}$ correspond to the lexicographically greatest vector $\mathbf{L}_{\hat{\mathbf{o}}}$. We will show that $\hat{\mathbf{o}}$ is a pure Nash equilibrium. First, since $\hat{\mathbf{o}}$ is a proper coloring, all vertices receive payoff no less than 1, so no vertex has any incentive to choose a new color other than those already used. Now consider a vertex v which is assigned color $\hat{o}_v$ and let i be the coordinate that corresponds to $\hat{o}_v$ in $\mathbf{L}_{\hat{\mathbf{o}}}$. If v had an incentive to choose a color that corresponds to the jth coordinate of $\mathbf{L}_{\hat{\mathbf{o}}}$ for some $j < i$, then this would yield an optimum coloring that would be lexicographically greater than $\hat{\mathbf{o}}$, a contradiction. If v had an incentive to choose a color that corresponds to the jth coordinate of $\mathbf{L}_{\hat{\mathbf{o}}}$ for some $j > i$, then it must essentially hold that $\ell_{\hat{\mathbf{o}}}(i) = \ell_{\hat{\mathbf{o}}}(j)$. So, if v deviates, this would again yield an optimum coloring that would be lexicographically greater that $\hat{\mathbf{o}}$, a contradiction. Therefore $\hat{\mathbf{o}}$ is a pure Nash equilibrium and $\mathsf{SC}(G, \hat{\mathbf{o}}) = \chi(G)$. □

Lemma 1. *Every pure Nash equilibrium* $\mathbf{c}$ *of $\Gamma(G)$ is a proper coloring of G.*

Proof. Assume, by contradiction, that $\mathbf{c}$ is not a proper coloring. Then there exists some vertex $v \in V$ such that $\lambda_v(\mathbf{c}) = 0$. Clearly, there exists some color $x \in X$ such that $c_u \neq x$ for all $u \in V$. Therefore $\lambda_v(x, \mathbf{c}_{-v}) = 1 > 0 = \lambda_v(\mathbf{c})$, which contradicts the fact that $\mathbf{c}$ is an equilibrium. □

Corollary 1. *It is* NP*-complete to decide whether there exists a pure Nash equilibrium of $\Gamma(G)$ that uses at most k colors.*

Proof (Sketch). Follows by reduction to the NP-complete problem of deciding whether there exists a proper coloring of a graph that uses at most k colors. □

Theorem 2. *For any graph coloring game $\Gamma(G)$, a pure Nash equilibrium can be computed in $O(n \cdot \alpha(G))$ selfish steps, where n is the number of vertices of G and $\alpha(G)$ is the independence number of G.*

Proof. We define the function $\Phi : P \to \mathbb{R}$, where $P \subseteq X^n$ is the set of all configurations that correspond to proper colorings of the vertices of G, as $\Phi(\mathbf{c}) = \frac{1}{2} \sum_{x \in X} n_x^2(\mathbf{c})$, for all proper colorings $\mathbf{c}$. Fix a *proper* coloring $\mathbf{c}$. Assume that vertex $v \in V$ can improve its payoff by deviating and selecting color $x \neq c_v$. This implies that the number of vertices colored c_v in $\mathbf{c}$ is at most the number of vertices colored x in $\mathbf{c}$, i.e. $n_{c_v}(\mathbf{c}) \leq n_x(\mathbf{c})$. If v indeed deviates to x, then the resulting configuration $\mathbf{c}' = (x, \mathbf{c}_{-v})$ is again a proper coloring (vertex v can only decrease its payoff by choosing a color that is already used by one of its neighbors, and v is the only vertex that changes its color). The improvement on v's payoff will be $\lambda_v(\mathbf{c}')-\lambda_v(\mathbf{c}) = n_x(\mathbf{c}')-n_{c_v}(\mathbf{c}) = n_x(\mathbf{c})+1-n_{c_v}(\mathbf{c})$. Moreover,

$$\begin{aligned}\Phi(\mathbf{c}') - \Phi(\mathbf{c}) &= \frac{1}{2}\left(n_x^2(\mathbf{c}') + n_{c_v}^2(\mathbf{c}') - n_x^2(\mathbf{c}) - n_{c_v}^2(\mathbf{c})\right) \\ &= \frac{1}{2}\left((n_x(\mathbf{c})+1)^2 + (n_{c_v}(\mathbf{c})-1)^2 - n_x^2(\mathbf{c}) - n_{c_v}^2(\mathbf{c})\right) \\ &= n_x(\mathbf{c}) + 1 - n_{c_v}(\mathbf{c}) = \lambda_v(\mathbf{c}') - \lambda_v(\mathbf{c}).\end{aligned}$$

Therefore, if any vertex v performs a selfish step (i.e. changes its color so that its payoff is increased) then the value of Φ is increased as much as the payoff of v is increased. Now, the payoff of v is increased by at least 1. So after any selfish step the value of Φ increases by at least 1. Now observe that, for all proper colorings $\mathbf{c} \in P$ and for all colors $x \in X$, $n_x(\mathbf{c}) \leq \alpha(G)$. Therefore $\Phi(\mathbf{c}) = \frac{1}{2}\sum_{x\in X} n_x^2(\mathbf{c}) \leq \frac{1}{2}\sum_{x\in X}(n_x(\mathbf{c})\cdot\alpha(G)) = \frac{1}{2}\alpha(G)\sum_{x\in X} n_x(\mathbf{c}) = \frac{n\cdot\alpha(G)}{2}$. Moreover, the minimum value of Φ is $\frac{1}{2}n$. Therefore, if we allow any unsatisfied vertex (but only one each time) to perform a selfish step, then after at most $\frac{n\cdot\alpha(G)-n}{2}$ steps there will be no vertex that can improve its payoff (because Φ will have reached a local maximum, which is no more than the global maximum, which is no more than $(n\cdot\alpha(G))/2$), so a pure Nash equilibrium will have been reached. Of course, we have to start from an initial configuration that is a proper coloring so as to ensure that $\mathcal{A}$ will terminate in $O(n\cdot\alpha(G))$ selfish steps; this can be found easily since there is always the trivial proper coloring that assigns a different color to each vertex of G. □

The above proof implies the following simple algorithm $\mathcal{A}$ that computes a pure Nash equilibrium of $\Gamma(G)$ (and thus a proper coloring of G):

Input: Graph G with vertex set $V = \{v_1,\ldots,v_n\}$; a set of colors $X = \{x_1,\ldots,x_n\}$
Output: A pure Nash equilibrium $\mathbf{c} = (c_{v_1},\ldots,c_{v_n}) \in X^n$ of $\Gamma(G)$
Initialization: **for** $i=1$ **to** n **do** $c_{v_i} = x_i$
repeat
  find an unsatisfied vertex $v \in V$ and a color $x \in X$ such that $\lambda_v(x,\mathbf{c}_{-v}) > \lambda_v(\mathbf{c})$
  set $c_v = x$
until all vertices are satisfied

I.e., at each step, $\mathcal{A}$ allows one unsatisfied vertex to perform a selfish step, until all vertices are satisfied. Note that, at each step, there may be more than one unsatisfied vertices, and more than one colors that a vertex could choose in order to increase its payoff. So actually $\mathcal{A}$ is a whole class of algorithms, since one could define a specific ordering (e.g., some fixed or some random order) of vertices and colors, and examine vertices and colors according to this order. In any case however, the algorithm is guaranteed to terminate in $O(n\cdot\alpha(G))$ selfish steps. Furthermore, each selfish step can be implemented straightforwardly in $O(n^2)$ time, since there are n vertices and n colors that each vertex can be assigned. It might be possible to improve the $O(n^2)$ complexity of a selfish step, e.g. by using appropriate data structures; this is a matter of future research and we leave it as an open question.

Let us now give a direct application of Theorem 2 to dense random graphs, and in particular consider the $G_{n,p}$ model, i.e. the class of random graphs with n vertices where each of the possible $\frac{n(n-1)}{2}$ edges occurs with probability p (for some constant $0 < p < 1$). The independence number of these graphs is known to be $(1-o(1))\frac{\log_2 n}{\log_2(1/(1-p))}$ with high probability [2], and therefore a pure Nash equilibrium can be computed in $O(n \cdot \log_2(n))$ selfish steps, with high probability.

4 Bounds on the Total Number of Colors

Lemma 2. *In any pure Nash equilibrium of $\Gamma(G)$, the number k of total colors used satisfies $k \leq \Delta_2(G)+1$ and hence $k \leq \Delta(G)+1$.*

Proof. Consider a pure Nash equilibrium $\mathbf{c}$ of $\Gamma(G)$, and let k be the total number of distinct colors used in $\mathbf{c}$. If $k=1$ then it easy to observe that G must be totally disconnected, i.e. $\Delta(G)=\Delta_2(G)=0$ and therefore $k=\Delta_2(G)+1$. Now assume $k \geq 2$. Let $x_i, x_j \in X$ be the two colors used in $\mathbf{c}$ that are assigned to the minimum number of vertices. W.l.o.g.[1], assume that $n_{x_i}(\mathbf{c}) \leq n_{x_j}(\mathbf{c}) \leq n_x(\mathbf{c})$ for all colors $x \notin \{x_i, x_j\}$ used in $\mathbf{c}$. Let v be a vertex such that $c_v = x_i$. The payoff of vertex v is $\lambda_v(\mathbf{c}) = n_{x_i}(\mathbf{c})$. Now consider any other color $x \neq x_i$ that is used in $\mathbf{c}$. Assume that there is no edge between vertex v and any vertex u with $c_u = x$. Then, since $\mathbf{c}$ is a pure Nash equilibrium, it must hold that $n_{x_i}(\mathbf{c}) \geq n_x(\mathbf{c})+1$, a contradiction. Therefore there is an edge between vertex v and at least one vertex of every other color. Hence the degree of vertex v is at least the total number of colors used minus 1, i.e. $\deg(v) \geq k-1$. Furthermore, let u be the vertex of color $c_u = x_j$ that v is connected to. Similar arguments as above yield that u must be connected to at least one vertex of color x, for all $x \notin \{x_i, x_j\}$ used in $\mathbf{c}$. Moreover, u is also connected to v. Therefore $\deg(u) \geq k-1$. Now:

$$\begin{aligned}
\Delta_2(G) &= \max_{s \in V} \max_{\substack{t \,\in\, N(s) \\ \deg(t) \,\leq\, \deg(s)}} \deg(t) \\
&\geq \max\left\{ \max_{\substack{t \,\in\, N(v) \\ \deg(t) \,\leq\, \deg(v)}} \deg(t), \max_{\substack{t \,\in\, N(u) \\ \deg(t) \,\leq\, \deg(u)}} \deg(t) \right\} \\
&\geq \min\{\deg(u), \deg(v)\} \geq k-1
\end{aligned}$$

and therefore $k \leq \Delta_2(G)+1$ as needed. □

Lemma 3. *In a pure Nash equilibrium, all vertices that are assigned unique colors form a clique.*

Proof. Consider a pure Nash equilibrium $\mathbf{c}$. Assume that the colors c_v and c_u chosen by vertices v and u are unique, i.e. $n_{c_v}(\mathbf{c}) = n_{c_u}(\mathbf{c}) = 1$. Then the payoff for both vertices is 1. If there is no edge between u and v then, since $\mathbf{c}$ is an equilibrium, it must hold that $1 = \lambda_v(\mathbf{c}) \geq \lambda_v(c_u, \mathbf{c}_{-v}) = 2$, a contradiction. □

[1] Without loss of generality.

Lemma 4. *In any pure Nash equilibrium of $\Gamma(G)$, the number k of total colors used satisfies $k \leq \frac{n+\omega(G)}{2}$.*

Proof. Consider a pure Nash equilibrium $\mathbf{c}$ of $\Gamma(G)$. Assume there are $t \geq 0$ vertices that are each assigned a unique color. These t vertices form a clique (Lemma 3), hence $t \leq \omega(G)$. The remaining $n-t$ vertices are assigned non-unique colors, so the number of colors in $\mathbf{c}$ is $k \leq t + \frac{n-t}{2} = \frac{n+t}{2} \leq \frac{n+\omega(G)}{2}$. □

Lemma 5. *In any pure Nash equilibrium of $\Gamma(G)$, the number k of total colors used satisfies $k \leq \frac{1+\sqrt{1+8m}}{2}$.*

Proof. Consider a pure Nash equilibrium $\mathbf{c}$ of $\Gamma(G)$. W.l.o.g., assume that the k colors used in $\mathbf{c}$ are $x_1, \ldots, x_k$. Let V_i, $1 \leq i \leq k$, denote the subset of all vertices $v \in V$ such that $c_v = x_i$. W.l.o.g., assume that $|V_1| \leq |V_2| \leq \cdots \leq |V_k|$. Observe that, for each vertex $v_i \in V_i$, there is an edge between v_i and some $v_j \in V_j$, for all $j > i$. If not, then v_i could improve its payoff by choosing color x_j, since $|V_j| + 1 \geq |V_i| + 1 > |V_i|$. This implies that $m \geq \sum_{i=1}^{k-1} |V_i|(k-i)$ and, since $|V_i| \geq 1$ for all $i \in \{1, \ldots, k\}$, $m \geq \sum_{i=1}^{k-1}(k-i)$ or equivalently $m \geq \frac{k(k-1)}{2}$ or equivalently $k^2 - k - 2m \leq 0$, which implies $k \leq \frac{1+\sqrt{1+8m}}{2}$. □

Theorem 3. *In any pure Nash equilibrium of $\Gamma(G)$, the number k of total colors used satisfies $k \leq n - \alpha(G) + 1$.*

Proof. Consider any pure Nash equilibrium $\mathbf{c}$ of $\Gamma(G)$. Let t be the maximum, over all vertices, payoff in $\mathbf{c}$, i.e. $t = \max_{x \in X} n_x(\mathbf{c})$. Partition the set of vertices into t sets $V_1, \ldots, V_t$ so that $v \in V_i$ if and only if $\lambda_v(\mathbf{c}) = i$ (note that each vertex appears in exactly one such set, however not all sets have to be nonempty). Let k_i denote the total number of colors that appear in V_i. Clearly, $|V_i| = i \cdot k_i$ and the total number of colors used in $\mathbf{c}$ is $k = \sum_{i=1}^{t} k_i$. Now consider a maximum independent set I of G. The vertices in V_1 have payoff equal to 1, therefore they are assigned unique colors, so, by Lemma 3, the vertices in V_1 form a clique. Therefore I can only contain at most one vertex among the vertices in V_1. Our goal is to upper bound the size of I. First we prove the following:

Claim 1. If there exists some $i > 1$ such that $k_i = 1$ and I contains all the vertices in V_i, then $k \leq n - \alpha(G) + 1$.

Proof of Claim 1. Let x denote the unique color that appears in V_i. Since I contains all the vertices in V_i, then it cannot contain any vertex in $V_1 \cup \cdots \cup V_{i-1}$. This is so because each vertex $v \in V_j$, $j < i$, is connected by an edge with at least one vertex of color x (otherwise v could increase its payoff by selecting x, which contradicts the equilibrium). Furthermore, each vertex in V_i has at least one neighbor of each color that appears in $V_{i+1} \cup \cdots \cup V_t$. Therefore

$$|I| = \alpha(G) \leq |V_i| + \sum_{j=i+1}^{t} |V_j| - \sum_{j=i+1}^{t} k_j = n - \sum_{j=1}^{i-1} |V_j| - k + \sum_{j=1}^{i} k_j$$

which gives $k \leq n - \alpha(G) + \sum_{j=1}^{i-1}(k_j - |V_j|) + k_i \leq n - \alpha(G) + k_i = n - \alpha(G) + 1$. □

So now it suffices to consider the case where, for all $i > 1$ such that $k_i = 1$, I does not contain all the vertices in V_i. So I contains at most $|V_i| - 1 = |V_i| - k_i$ vertices that belong to V_i. In order to complete the proof we need the following:

Claim 2. For all $i > 1$ with $k_i \neq 1$, I cannot contain more than $|V_i| - k_i$ vertices among the vertices in V_i.

Proof of Claim 2. This is clearly true for $k_i = 0$ (and hence $|V_i| = 0$). Now assume that $k_i \geq 2$. Observe that, for all vertices $v_i \in V_i$ there must exist an edge between v_i and a vertex of each one of the remaining $k_i - 1$ colors that appear in V_i (otherwise, v_i could change its color and increase its payoff by 1, which contradicts the equilibrium). Fix a color x of the k_i colors that appear in V_i. If I contains all vertices of color x, then it cannot contain any vertex of any color other than x that appears in V_i. Therefore I can contain at most $i \leq (i-1)k_i = |V_i| - k_i$ vertices among the vertices in V_i. On the other hand, if I contains at most $i - 1$ vertices of each color x that appears in V_i, then I contains again at most $(i-1)k_i = |V_i| - k_i$ vertices among the vertices in V_i. □

Therefore I cannot contain more than $|V_i| - k_i$ vertices among the vertices of V_i, for all $i > 1$, plus one vertex from V_1. Therefore:

$$|I| = \alpha(G) \leq 1 + \sum_{i=2}^{t}(|V_i| - k_i) = 1 + n - |V_1| - (k - |V_1|) = n - k + 1.$$

So, in any case, $k \leq n - \alpha(G) + 1$ as needed. □

The bounds given by Lemmata 2, 4, 5 and Theorem 3 imply the following:

Theorem 4. *For any graph coloring game $\Gamma(G)$ and any pure Nash equilibrium* $\mathbf{c}$ *of $\Gamma(G)$,* $\mathsf{SC}(G, \mathbf{c}) \leq \min\left\{\Delta_2(G) + 1,\ \frac{n+\omega(G)}{2},\ \frac{1+\sqrt{1+8m}}{2},\ n - \alpha(G) + 1\right\}$.

Furthermore, since any Nash equilibrium is a proper coloring (Lemma 1) and a Nash equilibrium can be computed in polynomial time (Theorem 2):

Corollary 2. *For any graph G, a proper coloring that uses at most* $k \leq \min\left\{\Delta_2(G) + 1,\ \frac{n+\omega(G)}{2},\ \frac{1+\sqrt{1+8m}}{2},\ n - \alpha(G) + 1\right\}$ *colors can be computed in* $O(n^4)$ *time.*

5 The Approximation Ratio

Lemma 6. *For any graph G with n vertices and m edges,*

$$\mathsf{R}(G) \leq \frac{\min\left\{\Delta_2(G) + 1,\ \frac{n+\omega(G)}{2},\ \frac{1+\sqrt{1+8m}}{2},\ n - \alpha(G) + 1\right\}}{\max\left\{\omega(G),\ \frac{n}{\alpha(G)}\right\}}.$$

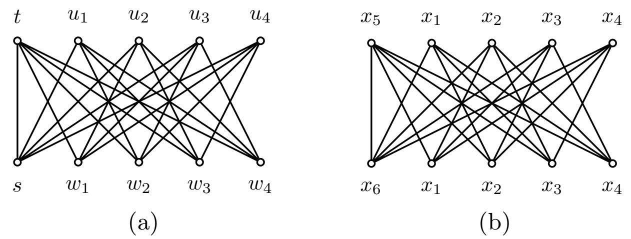

Fig. 1. (a) A graph with chromatic number 2 and (b) a Nash equilibrium using 6 colors

Proof. Follows from Theorem 4 and the fact that $\chi(G) \geq \max\{\omega(G), \frac{n}{\alpha(G)}\}$. □

Lemma 7. *For any constant* $\epsilon > 0$*, there exists a graph* $G(\epsilon)$ *such that* $\mathsf{R}(G(\epsilon)) \geq n^{1-\epsilon}$ *unless* $\mathsf{NP} \subseteq \mathsf{co\text{-}RP}$.

Proof. Assume the contrary. Then there exists some constant $\epsilon > 0$ such that, for all graphs G, $\mathsf{R}(G) < n^{1-\epsilon}$. But then our selfish improvements algorithm $\mathcal{A}$ of Theorem 2 achieves, in $O(n^4)$ time, a proper coloring of G with a number of colors $k \leq \mathsf{R}(G) \cdot \chi(G)$, i.e., $k \leq n^{1-\epsilon}\chi(G)$. Thus, for all G, algorithm $\mathcal{A}$ approximates $\chi(G)$ in polynomial time with an approximation ratio $R \leq n^{1-\epsilon}$ for some constant $\epsilon > 0$. This cannot happen unless $\mathsf{NP} \subseteq \mathsf{co\text{-}RP}$ [7]. □

However, can we construct a graph certificate G with unconditionally high $\mathsf{R}(G)$? The answer is yes:

Lemma 8. *We can construct a graph certificate* G *such that* $\mathsf{R}(G) = \frac{n}{4} + \frac{1}{2}$.

Proof. Consider a bipartite graph $G = (V, E)$ with $n = 2\kappa + 2$ vertices, $\kappa \geq 1$. Let $V = U \cup W \cup \{s, t\}$ where $U = \{u_1, \ldots, u_\kappa\}$ and $W = \{w_1, \ldots, w_\kappa\}$. The set of edges E is defined as

$$E = \{\{u_i, w_j\} \in U \times W \mid i \neq j\} \cup \bigcup_{i=1}^{\kappa}\{s, u_i\} \cup \bigcup_{i=1}^{\kappa}\{t, w_i\} \cup \{s, t\}.$$

(Figure 1(a) shows such a graph with $n = 10$ vertices.) There exists a pure Nash equilibrium that uses $\kappa + 2$ colors: vertices u_1, w_1 are colored x_1, vertices u_2, w_2 are colored x_2 e.t.c., while vertex t is colored $x_{\kappa+1}$ and vertex s is colored $x_{\kappa+2}$ (see Fig. 1(b)). This coloring is a pure Nash equilibrium since each vertex $v \in U \cup W$ receives payoff equal to 2 and the set of vertices $N(v) \cup \{v\}$ uses all colors $x_1, \ldots, x_\kappa$. Vertices s and t get payoff 1, but each of them is connected to a vertex of each of the remaining colors. The optimum coloring would use 2 colors, one to color the vertices in $U \cup \{t\}$ and another to color the vertices in $W \cup \{s\}$. Therefore $\mathsf{R}(G) \geq \frac{\kappa+2}{2} = \frac{n}{4} + \frac{1}{2}$. But $\omega(G) = 2$, so from Lemma 6 we can easily get $\mathsf{R}(G) \leq \frac{n}{4} + \frac{1}{2}$, which completes the proof. □

6 On Mechanisms to Improve the Approximation Ratio

6.1 Refinements of the Selfish Steps Sequence: Randomness

The existence of the potential function $\Phi(\mathbf{c})$ assures that if we start with a proper coloring and allow at each step any single unsatisfied vertex to perform a selfish step, then a pure Nash equilibrium will be reached in polynomial time, no matter in which order the vertices are examined or which is the initial configuration. In this section we study whether there exists a sequence of selfish steps, i.e. a specific ordering of the vertices according to which the vertices are allowed to perform a selfish step, such that the Social Cost of the equilibrium reached is even less than the general bounds presented before.

Assume that, at each step, the vertex that is allowed to perform a selfish step is chosen independently and uniformly at random, among all vertices that are unsatisfied. Moreover, assume that the vertex chosen to perform a selfish step chooses a color independently and uniformly at random among the colors that can increase its payoff. Then, we can prove the following (the proof is omitted):

Proposition 1. *The random selfish steps sequence applied to the graph of Lemma 8 terminates in polynomial time at a pure Nash equilibrium that, with high probability, corresponds to an optimum coloring.*

Although Proposition 1 is rather restrictive, since it only applies to the graph of Lemma 8, we believe that the random selfish steps sequence can color other classes of graphs with a number of colors much smaller than the bounds presented previously. We expect that randomization can help in avoiding equilibria that are too far from an optimum coloring. However, we have not yet been able to prove this; this is a matter of future research and we leave it as an open problem.

6.2 Stackelberg Strategies

Consider a graph coloring game $\Gamma(G)$. Assume that there is a central authority (a *Leader*) that controls a portion $V^L \subset V$ of the vertices of $G = (V, E)$, i.e. the Leader colors the vertices in V^L and, after that, the rest of the vertices in $V \setminus V^L$ (the *followers*) are colored selfishly. The goal of the Leader is to find an assignment of colors to V^L (a *Leader's strategy*) so as to induce the followers to a pure Nash equilibrium where the total number of colors used in V is as close to the chromatic number of G as possible.

Definition 2. *For a constant $k \in \mathbb{N}$, a random balanced k-partite graph, denoted $G_{n,k,p}$, is a k-partite graph with n vertices, where the size of each vertex class is either $\lceil \frac{n}{k} \rceil$ or $\lfloor \frac{n}{k} \rfloor$, and each edge $\{u, v\}$ (such that u and v belong to different vertex classes) exists in G independently at random with probability p.*

Lemma 9. *The chromatic number of $G_{n,k,\frac{1}{2}}$ is k, with high probability.*

Proof (Sketch). Clearly, $\omega(G_{n,k,\frac{1}{2}}) \leq \chi(G_{n,k,\frac{1}{2}}) \leq k$. The proof follows by showing that, with high probability, there exists a clique of size k in $G_{n,k,\frac{1}{2}}$. □

Theorem 5. *Consider the graph coloring game $\Gamma(G_{n,k,\frac{1}{2}})$. There exists a polynomial time computable Leader's strategy, such that with high probability the total number of colors used in the resulting pure Nash equilibrium is k.*

Proof. Let $P_1, \ldots, P_k$ denote the k vertex classes of $G_{n,k,\frac{1}{2}}$. Assume that the Leader chooses uniformly at random a subset $S \subset V$ of $|S| = c \log n$ vertices, for some constant $c > 10k$. The Leader can exhaustively search among all possible k-colorings of S in time polynomial in n, since $|S| = c \log n$. Among these possible colorings, there exists one proper coloring $\mathbf{c}^L$ that colors each vertex $s \in S \cap P_1$ with the same color x_1, each vertex $s \in S \cap P_2$ with the same color $x_2 \neq x_1$ e.t.c. In the following, assume that the Leader's strategy is $\mathbf{c}^L$.

Our next step is to show that, with high probability, each follower $v_i \in P_i \setminus S$ is connected to at least one vertex in S of color x_j, for all $j \neq i$. To do so, we use Hoeffding bounds [11] and obtain

$$\Pr\{\exists i,\ \exists v_i \in P_i,\ \exists j\ :\ \{v_i, v_j\} \notin E \quad \forall v_j \in S \cap P_j\} \leq \frac{2k}{n}.$$

So with probability at least $1 - \frac{2k}{n}$, each follower $v_i \in P_i \backslash S$ (for all $i \in \{1, \ldots, k\}$) has all the colors x_j $(j \neq i)$ in its neighborhood. But if this is the case, then the pure Nash equilibrium that will be reached by any selfish steps sequence will use the same color x_i for all $v_i \in P_i \setminus S$, for each $i = \{1, \ldots, k\}$. Therefore, with probability at least $1 - \frac{2k}{n}$, there will be k colors in the resulting pure Nash equilibrium. However, we assumed that the Leader's strategy is $\mathbf{c}^L$. This is not restrictive, since the Leader can repeatedly choose one of the possible k-colorings of S (their number is $k^{c \log n}$, i.e. polynomial in n) and then leave the followers converge to a pure Nash equilibrium. The precedent analysis shows that there exists a proper coloring $\mathbf{c}^L$ of S such that there will be k colors in the equilibrium reached by the followers, with high probability. □

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