Equivalences in Answer-Set Programming by Countermodels in the Logic of Here-and-There

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Abstract. In Answer-Set Programming different notions of equivalence, such as the prominent notions of strong and uniform equivalence, have been studied and characterized by various selections of models in the logic of Here-and-There (HT). For uniform equivalence however, correct characterizations in terms of HT-models can only be obtained for finite theories, respectively programs. In this paper, we show that a selection of countermodels in HT captures uniform equivalence also for infinite theories. This result is turned into coherent characterizations of the different notions of equivalence by countermodels, as well as by a mixture of HT-models and countermodels (so-called equivalence interpretations), which are lifted to first-order theories under a very general semantics given in terms of a quantified version of HT. We show that countermodels exhibit expedient properties like a simplified treatment of extended signatures, and provide further results for non-ground logic programs. In particular, uniform equivalence coincides under open and ordinary answer-set semantics, and for finite non-ground programs under these semantics, also the usual characterization of uniform equivalence in terms of maximal and total HT-models of the grounding is correct, even for infinite domains, when corresponding ground programs are infinite.

Keywords: answer-set programming, uniform equivalence, knowledge representation, program optimization.

1 Introduction

Logic programming under the answer-set semantics, called Answer-Set Programming (ASP), is a fundamental paradigm for nonmonotonic knowledge representation [1]. It is distinguished by a purely declarative semantics and efficient solvers [2,3,4,5]. Initially providing a semantics for rules with default negation in the body, the answer-set semantics [6] has been continually extended in terms of expressiveness, and recently the formalism has been lifted to a general answer-set semantics for first-order theories [7].

In a different line of research, the restriction to Herbrand domains for programs with variables, i.e., non-ground programs, has been relaxed in order to cope with open domains [8]. The open answer-set semantics has been further generalized by dropping the unique names assumption [9] for application settings where it does not apply, for instance, when combining ontologies with nonmontonic rules [10].

As for a logical characterization of the answer-set semantics, the logic of Here-and-There (HT), a nonclassical logic extending intuitionistic logic, served as a basis.

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Equilibrium Logic selects certain minimal HT-models for characterizing the answer-set semantics for propositional theories and programs. It has recently been extended to Quantified Equilibrium Logic (QEL) for first-order theories on the basis of a quantified version of Here-and-There (QHT) [11]. Equilibrium Logic serves as a viable formalism for the study of semantic comparisons of theories and programs, like different notions of equivalence [12,13,14,15,16]. The practical relevance of this research originates in program optimization tasks that rely on modifications that preserve certain properties [17,18,19].

In this paper, we contribute by tackling an open problem concerning uniform equivalence of propositional theories and programs. Intuitively, two propositional logic programs are uniformly equivalent if the have the same answer sets under the addition of an arbitrary set of atoms to both programs. As has been shown in [20], so-called UE-models, a selection of HT-models based on a maximality criterion, do not characterize uniform equivalence for infinite propositional programs. Moreover, uniform equivalence of infinite programs cannot be captured by any selection of HT-models [20], as this is the case, e.g., for strong equivalence.

While the problem might seem esoteric at a first glance, since infinite propositional programs are rarely dealt with in practice, it is relevant when turning to the non-ground setting, respectively first-order theories, where infinite domains, such as the natural numbers, are encountered in many application domains.

The main contributions can be summarized as follows:

- We show that uniform equivalence of possibly infinite propositional theories, and thus programs, can be characterized by certain countermodels in HT. However, HT is not 'dual' (wrt. the characterization of countermodels) in the following sense: The countermodels of a theory Γ cannot be characterized by the models of a theory Γ'. Therefore, we also study *equivalence interpretations*, a mixture of models and countermodels of a theory, that can be characterized by a transformation of the theory if it is finite. We characterize classical equivalence, answer-set equivalence, strong equivalence, and uniform equivalence by appropriate selections of countermodels and equivalence interpretations.
- We lift these results to first-order theories by means of QHT, essentially introducing uniform equivalence for first-order theories under the most general form of answer-set semantics currently considered. We prove that, compared to QHT-models, countermodels alow for a simplified treatment of extended signatures.
- Finally, we show that the notion generalizes uniform equivalence for logic programs, and prove that it coincides for open and ordinary answer-set semantics. For finite non-ground programs under both ordinary and open answer-set semantics, we establish that uniform equivalence can be handled by the usual characterization in terms of HT-models of the grounding also for infinite domains.

Our results provide an elegant, uniform model-theoretic characterization of the different notions of equivalence considered in ASP. They generalize to first-order theories without finiteness restrictions, and are relevant for practical ASP systems that handle finite non-ground programs over infinite domains. For the sake of presentation, the technical content is split into two parts, discussing the propositional case first (Sections 2 and 3), and addressing first order theories and nonground programs in Sections 4 and 5.

2 Preliminaries

We start with the propositional setting and briefly summarize the necessary background. Corresponding first-order formalisms will be introduced when discussing first-order theories, respectively non-ground logic programs.

2.1 Propositional Here-and-There

In the propositional case we consider formulas of a propositional signature $\mathcal{L}$, i.e., a set of propositional constants, and the connectives $\wedge$, $\vee$, $\rightarrow$, and $\bot$ for conjunction, disjunction, implication, and falsity, respectively. Furthermore we make use of the following abbreviations: $\phi \equiv \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$; $\neg\phi$ for $\phi \rightarrow \bot$; and $\top$ for $\bot \rightarrow \bot$. A formula is said to be *factual*[1] if it is built using $\wedge$, $\vee$, $\bot$, and $\neg$ (i.e., implications of the form $\phi \rightarrow \bot$), only. A theory Γ is factual if every formula of Γ has this property.

The logic of here-and-there is an intermediate logic between intuitionistic logic and classical logic. Like intuitionistic logic it can be semantically characterized by Kripke models, in particular using just two worlds, namely "*here*" and "*there*" (assuming that the *here* world is ordered before the *there* world). Accordingly, interpretations (HT-interpretations) are pairs (X, Y) of sets of atoms from $\mathcal{L}$, such that $X \subseteq Y$. An HT-interpretation is *total* if $X = Y$. The intuition is that atoms in X (the *here* part) are considered to be true, atoms not in Y (the *there* part) are considered to be false, while the remaining atoms (from $Y \setminus X$) are undefined.

We denote classical satisfaction of a formula ϕ by an interpretation X, i.e., a set of atoms, as $X \models \phi$, whereas satisfaction in the logic of here-and-there (an HT-model), symbolically $(X, Y) \models \phi$, is defined recursively:

1. $(X, Y) \models a$ if $a \in X$, for any atom a,
2. $(X, Y) \not\models \bot$,
3. $(X, Y) \models \phi \wedge \psi$ if $(X, Y) \models \phi$ and $(X, Y) \models \psi$,
4. $(X, Y) \models \phi \vee \psi$ if $(X, Y) \models \phi$ or $(X, Y) \models \psi$,
5. $(X, Y) \models \phi \rightarrow \psi$ if (i) $(X, Y) \not\models \phi$ or $(X, Y) \models \psi$, and (ii) $Y \models \phi \rightarrow \psi$[2].

An HT-interpretation (X, Y) satisfies a theory Γ, iff it satisfies all formulas $\phi \in \Gamma$. For an axiomatic proof system see, e.g., [13].

A total HT-interpretation (Y, Y) is called an *equilibrium model* of a theory Γ, iff $(Y, Y) \models \Gamma$ and for all HT-interpretations (X, Y), such that $X \subset Y$, it holds that $(X, Y) \not\models \Gamma$. An interpretation Y is an *answer set* of Γ iff (Y, Y) is an equilibrium model of Γ.

We will make use of the following simple properties: if $(X, Y) \models \Gamma$ then $(Y, Y) \models \Gamma$; and $(X, Y) \models \neg\phi$ iff $Y \models \neg\phi$; as well as of the following lemma.

Lemma 1 (Lemma 5 in [21]). *Let ϕ be a factual propositional formula. If $(X, Y) \models \phi$ and $X \subseteq X' \subseteq Y$, then $(X', Y) \models \phi$.*

[1] When uniform equivalence of theories is considered, then factual theories can be considered instead of facts—hence the terminology—see also the discussion at the end of this section.

[2] That is, Y satisfies $\phi \rightarrow \psi$ classically.

2.2 Propositional Logic Programming

A (*disjunctive*) rule r is of the form

$$a_1 \vee \cdots \vee a_k \vee \neg a_{k+1} \vee \cdots \vee \neg a_l \leftarrow b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n, \qquad (1)$$

where $a_1, \ldots, a_l, b_1, \ldots, b_n$ are atoms of a propositional signature $\mathcal{L}$, such that $l \geq k \geq 0$, $n \geq m \geq 0$, and $l + n > 0$. We refer to "$\neg$" as *default negation*. The *head* of r is the set $H(r) = \{a_1, \ldots, a_k, \neg a_{k+1}, \ldots, \neg a_l\}$, and the *body* of r is denoted by $B(r) = \{b_1, \ldots, b_m, \neg b_{m+1}, \ldots, \neg b_n\}$. Furthermore, we define the sets $H^+(r) = \{a_1, \ldots, a_k\}$, $H^-(r) = \{a_{k+1}, \ldots, a_l\}$, $B^+(r) = \{b_1, \ldots, b_m\}$, and eventually $B^-(r) = \{b_{m+1}, \ldots, b_n\}$. A *program* Π (over $\mathcal{L}$) is a set of rules (over $\mathcal{L}$).

An interpretation I, i.e., a set of atoms, satisfies a rule r, symbolically $I \models r$, iff $I \cap H^+(r) \neq \emptyset$ or $H^-(r) \not\subseteq I$ if $B^+(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$. Adapted from [6], the *reduct* of a program Π with respect to an interpretation I, symbolically Π^I, is given by the set of rules

$$a_1 \vee \cdots \vee a_k \leftarrow b_1, \ldots, b_m,$$

obtained from rules in Π, such that $H^-(r) \subseteq I$ and $B^-(r) \cap I = \emptyset$.

An interpretation I is called an *answer set* of Π iff $I \models \Pi^I$ and it is subset minimal among the interpretations of $\mathcal{L}$ with this property.

2.3 Notions of Equivalence

For any two theories, respectively programs, and a potential extension by Γ, we consider the following notions of equivalence which have been shown to be the only forms of equivalence obtained by varying the logical form of extensions in the propositional case in [21].

Definition 1. *Two theories Γ_1, Γ_2 over $\mathcal{L}$ are called*

- classically equivalent, $\Gamma_1 \equiv_c \Gamma_2$, *if they have the same classical models;*
- answer-set equivalent, $\Gamma_1 \equiv_a \Gamma_2$, *if they have the same answer sets, i.e., equilibrium models;*
- strongly equivalent, $\Gamma_1 \equiv_s \Gamma_2$, *if, for any theory Γ over $\mathcal{L}' \supseteq \mathcal{L}$, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are answer-set equivalent;*
- uniformly equivalent, $\Gamma_1 \equiv_u \Gamma_2$, *if, for any factual theory Γ over $\mathcal{L}' \supseteq \mathcal{L}$, $\Gamma_1 \cup \Gamma$ and $\Gamma_2 \cup \Gamma$ are answer-set equivalent.*

Emanating from a logic programming setting, uniform equivalence is usually understood wrt. sets of *facts* (i.e., atoms). Obviously, uniform equivalence wrt. factual theories implies uniform equivalence wrt. sets of facts. The converse direction has been shown as well for general propositional theories in [21](cf. Theorem 2). Therefore, in general there is no difference whether uniform equivalence is considered wrt. sets of facts or factual theories. The latter may be regarded as facts, i.e., rules with an empty body, of so-called nested logic program rules. One might also consider sets of disjunctions of atomic formulas and their negations (i.e., clauses), accounting for facts according to the definition of program rules in this paper. Note that clauses constitute factual formulas and the classical transformation of clauses into implications is not valid under answer set semantics (respectively in HT).

3 Equivalence of Propositional Theories by HT-Countermodels

Uniform equivalence is usually characterized by so-called UE-models, i.e., total and maximal non-total HT-models, which fail to capture uniform equivalence for infinite propositional theories.

Example 1 ([20]). Let Γ_1 and Γ_2 over $\mathcal{L} = \{a_i \mid i \geq 1\}$ be the following propositional theories

$$\Gamma_1 = \{a_i \mid i \geq 1\}, \text{ and } \Gamma_2 = \{\neg a_i \rightarrow a_i,\ a_{i+1} \rightarrow a_i \mid i \geq 1\}.$$

Both, Γ_1 and Γ_2, have the single total HT-model $(\mathcal{L}, \mathcal{L})$. Furthermore, Γ_1 has no non-total HT-model $(X, \mathcal{L})$, i.e, such that $X \subset \mathcal{L}$, while Γ_2 has the non-total HT-models $(X_i, \mathcal{L})$, where $X_i = \{a_1, \ldots, a_i\}$ for $i \geq 0$. Both theories have the same total and maximal non-total (namely none) HT-models. But they are not uniformly equivalent as witnessed by the fact that $(\mathcal{L}, \mathcal{L})$ is an equilibrium model of Γ_1 but not of Γ_2. □

The reason for this failure is the inability of the concept of maximality to capture differences exhibited by an infinite number of HT-models.

3.1 HT-Countermodels

The above problem can be avoided by taking HT-countermodels that satisfy a closure condition instead of the maximality criterion.

Definition 2. *An HT-interpretation (X, Y) is an* HT-countermodel *of a theory Γ if $(X, Y) \not\models \Gamma$. The set of HT-countermodels of a theory Γ is denoted by $C_s(\Gamma)$.*

Intuitively, an HT-interpretation fails to be an HT-model of a theory Γ when the theory is not satisfied at one of the worlds (*here* or *there*). Note that satisfaction at the *there* world amounts to classical satisfaction of the theory by Y. A simple consequence is that if $Y \not\models \Gamma$, then (X, Y) is an HT countermodel of Γ for any $X \subseteq Y$. At the *here* world, classical satisfaction is a sufficient condition but not necessary. For logic programs, satisfaction at the *here* world is precisely captured by the reduct of the program Π wrt. the interpretation at the *there* world, i.e., if $X \models \Pi^Y$.

Definition 3. *A total HT-interpretation (Y, Y) is*

- total-open *in a set S of HT-interpretations if $(Y, Y) \in S$ and $(X, Y) \notin S$ for every $X \subset Y$.*
- total-closed *in a set S of HT-interpretations if $(X, Y) \in S$ for every $X \subseteq Y$.*

We say that an HT-interpretation (X, Y) is there-closed *in a set S of HT-interpretations if $(X', Y) \in S$ for every $X \subseteq X' \subset Y$.*

A set S of HT-interpretations is total-closed, respectively total-open, if every total HT-interpretation $(Y, Y) \in S$ is total-closed in S, respectively total-open in S. By the remarks on the satisfaction at the *there* world above, it is obvious that every total HT-countermodel of a theory is also total-closed in $C_s(\Gamma)$. Consequently, $C_s(\Gamma)$ is a total-closed set for any theory Γ. By the same argument, if (X, Y) is an HT-countermodel such that $X \subset Y$ and $Y \not\models \Gamma$, then (X, Y) is there-closed in $C_s(\Gamma)$. The more relevant cases concerning the characterization of equivalence are HT-countermodels (X, Y) such that $Y \models \Gamma$.

Example 2. Consider the theory Γ_1 in Example 1 and a non-total HT-interpretation $(X, \mathcal{L})$. Since $(X, \mathcal{L})$ is non-total, $X \subset \mathcal{L}$ holds, and therefore $(X, \mathcal{L}) \not\models a_i$, for some $a_i \in \mathcal{L}$. Thus, we have identified a HT-countermodel of Γ_1. Moreover the same argument holds for any non-total HT-interpretation of the from $(X', \mathcal{L})$ (in particular such that $X \subseteq X' \subset Y$). Therefore, $(X, \mathcal{L})$ is there-closed in $C_s(\Gamma_1)$.

The intuition that, essentially, there-closed countermodels can be used instead of maximal non-total HT-models for characterizing uniform equivalence draws from the following observation. If (X, Y) is a maximal non-total HT-model, then every (X', Y), such that $X \subset X' \subset Y$, is a there-closed HT-countermodel. However, there-closed HT-countermodels are not sensitive to the problems that infinite chains cause for maximality.

Given a theory Γ, let $C_u(\Gamma)$ denote the set of there-closed HT-interpretations in $C_s(\Gamma)$.

Theorem 1. *Two propositional theories Γ_1, Γ_2 are uniformly equivalent iff they have the same sets of there-closed HT-countermodels, in symbols $\Gamma_1 \equiv_u \Gamma_2$ iff $C_u(\Gamma_1) = C_u(\Gamma_2)$.*

Proof. For the only-if direction, assume that two theories, Γ_1 and Γ_2, are uniformly equivalent. Then they are classically equivalent, i.e., they coincide on total HT-models, and therefore also on total HT-countermodels. Moreover, since every theory has a total-closed set of countermodels [22], we conclude that Γ_1 and Γ_2 coincide on all HT-models (X, Y) such that (Y, Y) is a (common) total HT-countermodel. Note that all these models are there-closed.

To prove our claim, it remains to show that Γ_1 and Γ_2 coincide on there-closed HT-countermodels (X, Y) such that (Y, Y) is an HT-model of both theories. Consider such a there-closed HT-countermodel of Γ_1. Then, (Y, Y) is a total HT-model of $\Gamma_1 \cup X$ and no $X' \subset Y$ exists such that $(X', Y) \models \Gamma_1 \cup X$, either because it is an HT-countermodel of Γ_1 (in case $X \subseteq X' \subset Y$) or of X (in case $X' \subset X$). Thus, Y is an answer set of $\Gamma_1 \cup X$ and, by hypothesis since X is factual, it is also an answer set of $\Gamma_2 \cup X$. The latter implies for all $X \subseteq X' \subset Y$ that $(X', Y) \not\models \Gamma_2 \cup X$. All these HT-interpretations are HT-models of X. Therefore we conclude that they all are HT-countermodels of Γ_2 and hence (X, Y) is a there-closed HT-countermodel of Γ_2. Again by symmetric arguments, we establish the same for any there-closed HT countermodel (X, Y) of Γ_2 such that (Y, Y) is a common total HT-model. This proves that Γ_1 and Γ_2 have the same sets of there-closed HT countermodels.

For the if direction, assume that two theories, Γ_1 and Γ_2, have the same sets of there-closed HT-countermodels. This implies that they have the same total HT-countermodels (since these are total-closed and thus there-closed) and hence the same total HT-models. Consider any factual theory Γ' such that Y is an answer set of $\Gamma_1 \cup \Gamma'$. We show that Y is an answer set of $\Gamma_2 \cup \Gamma'$ as well. Clearly, $(Y, Y) \models \Gamma_1 \cup \Gamma'$ implies $(Y, Y) \models \Gamma'$ and therefore $(Y, Y) \models \Gamma_2 \cup \Gamma'$. Consider any $X \subset Y$. Since Y is an answer set of $\Gamma_1 \cup \Gamma'$, it holds that $(X, Y) \not\models \Gamma_1 \cup \Gamma'$. We show that $(X, Y) \not\models \Gamma_2 \cup \Gamma'$. If $(X, Y) \not\models \Gamma'$ this is trivial, and in particular the case if $(X, Y) \models \Gamma_1$. So let us consider the case where $(X, Y) \not\models \Gamma_1$ and $(X, Y) \models \Gamma'$. By Lemma 1 we conclude from the latter that, for any $X \subseteq X' \subset Y$, $(X', Y) \models \Gamma'$. Therefore, $(X', Y) \not\models \Gamma_1$, as well. This implies that (X, Y) is a there-closed HT-countermodel of Γ_1. By hypothesis, (X, Y) is a there-closed HT-countermodel of Γ_2, i.e., $(X, Y) \not\models \Gamma_2$. Consequently, $(X, Y) \not\models \Gamma_2 \cup \Gamma'$. Since this argument applies to any $X \subset Y$, (Y, Y) is an equilibrium model of $\Gamma_2 \cup \Gamma'$, i.e., Y is an answer set of $\Gamma_2 \cup \Gamma'$. The same argument with Γ_1 and Γ_2 interchanged, proves that Y is an answer set of $\Gamma_1 \cup \Gamma'$ if it is an answer set of $\Gamma_2 \cup \Gamma'$. Therefore, the answer sets of $\Gamma_1 \cup \Gamma'$ and $\Gamma_2 \cup \Gamma'$ coincide for any factual theory Γ', i.e., Γ_1 and Γ_2 are uniformly equivalent. □

Example 3. Reconsider the theories in Example 1. Every non-total HT-interpretation $(X, \mathcal{L})$ is an HT-countermodel of Γ_1, and thus, each of them is there-closed. On the other hand, none of these HT-interpretations is an HT countermodel of Γ_2. Therefore, Γ_1 and Γ_2 are not uniformly equivalent. □

Countermodels have the drawback however, that they cannot be characterized directly in HT itself, i.e., as the HT-models of a 'dual' theory. The usage of "dual" here is non-standard compared to its application to particular calculi or consequence relations, but it likewise conveys the idea of a dual concept. In this sense HT therefore is non-dual:

Proposition 1. *Given a theory Γ, in general there is no theory Γ' such that (X, Y) is an HT-countermodel of Γ iff it is a HT-model of Γ', for any HT-interpretation (X, Y).*

3.2 Characterizing Equivalence by means of Equivalence Interpretations

The characterization of countermodels by a theory in HT essentially fails due to total HT-countermodels. However, total HT-countermodels of a theory are not necessary for characterizing equivalence, in the sense that they can be replaced by total HT-models of the theory for this purpose.

Definition 4. *An HT-countermodel (X, Y) of a theory Γ is called a* here-countermodel *of Γ if $Y \models \Gamma$.*

Definition 5. *An HT-interpretation is an* equivalence interpretation *of a theory Γ if it is a total HT-model of Γ or a here-countermodel of Γ. The set of equivalence interpretations of a theory Γ is denoted by $E_s(\Gamma)$.*

Theorem 2. *Two theories Γ_1 and Γ_2 coincide on their HT-countermodels iff they have the same equivalence interpretations, symbolically $C_s(\Gamma_1) = C_s(\Gamma_2)$ iff $E_s(\Gamma_1) = E_s(\Gamma_2)$.*

As a consequence of this result, and the usual relationships on HT-models, we can characterize equivalences of propositional theories also by selections of equivalence interpretations, i.e., a mixture of non-total HT countermodels and total HT-models, such that the characterizations, in particular for uniform equivalence, are also correct for infinite theories.

Given a theory Γ, let $C_c(\Gamma)$, resp. $E_c(\Gamma)$, denote the restriction to total HT-interpretations in $C_s(\Gamma)$, resp. in $E_s(\Gamma)$. $C_a(\Gamma)$ is the set of there-closed HT-interpretations of the form $(\emptyset, Y)$ in $C_s(\Gamma)$ such that $(Y, Y) \notin C_s(\Gamma)$, and $E_a(\Gamma)$ is the set of total-open HT-interpretations in $E_s(\Gamma)$ (i.e., equilibrium models). Finally, $E_u(\Gamma)$ denotes the set of there-closed HT-interpretations in $E_s(\Gamma)$.

Corollary 1. *Given two propositional theories Γ_1 and Γ_2, the following propositions are equivalent for $e \in \{c, a, s, u\}$:*

$$\textit{(1)}\ \Gamma_1 \equiv_e \Gamma_2; \quad \textit{(2)}\ C_e(\Gamma_1) = C_e(\Gamma_2); \quad \textit{(3)}\ E_e(\Gamma_1) = E_e(\Gamma_2).$$

Example 4. In our running example, $C_u(\Gamma_1) \neq C_u(\Gamma_2)$, as well as $E_u(\Gamma_1) \neq E_u(\Gamma_2)$, by the remarks on non-total HT-interpretations in Example 3. □

Since equivalence interpretations do not encompass total HT-countermodels, we attempt a direct characterization in HT.

Proposition 2. *Let M be an HT-interpretation over $\mathcal{L}$. Then, $M \in E_s(\Gamma)$ for a theory Γ iff $M \models \Gamma_\phi$ for some $\phi \in \Gamma$, where $\Gamma_\phi = \{\neg\neg\psi \mid \psi \in \Gamma\} \cup \{\phi \rightarrow (\neg\neg a \rightarrow a) \mid a \in \mathcal{L}\}$.*

For infinite propositional theories, we thus end up with a characterization of equivalence interpretations as the union of the HT-models of an infinite number of (infinite) theories. At least for finite theories, however, a characterization in terms of a (finite) theory is obtained (even for a potentially extended infinite signature).

If $\mathcal{L}' \supset \mathcal{L}$ and $M = (X, Y)$ is an HT-interpretation over $\mathcal{L}'$, then $M|_\mathcal{L}$ denotes the restriction of M to $\mathcal{L}$: $M|_\mathcal{L} = (X|_\mathcal{L}, Y|_\mathcal{L})$. The restriction is *totality preserving*, if $X \subset Y$ implies $X|_\mathcal{L} \subset Y|_\mathcal{L}$.

Proposition 3. *Let Γ be a theory over $\mathcal{L}$, let $\mathcal{L}' \supset \mathcal{L}$, and let M an HT-interpretation over $\mathcal{L}'$ such that $M|_\mathcal{L}$ is totality preserving. Then, $M \in C_s(\Gamma)$ implies $M|_\mathcal{L} \in C_s(\Gamma)$.*

Theorem 3. *Let Γ be a finite theory over $\mathcal{L}$, and let M be an HT-interpretation. Then, $M \in E_s(\Gamma)$ iff $M|_\mathcal{L} \models \bigvee_{\phi\in\Gamma} \bigwedge_{\psi\in\Gamma_\phi} \psi$, and $M|_\mathcal{L}$ is totality preserving.*

4 Generalization to First-Order Theories

Since the characterizations, in particular of uniform equivalence, presented in the previous section capture also infinite theories, they pave the way for generalizing this notion of equivalence to non-ground settings without any finiteness restrictions. In this section we study first-order theories.

As first-order theories we consider sets of sentences (closed formulas) of a first-order signature $\mathcal{L} = \langle \mathcal{F}, \mathcal{P} \rangle$ in the sense of classical first-order logic. Hence, $\mathcal{F}$ and $\mathcal{P}$ are pairwise disjoint sets of function symbols and predicate symbols with an associated arity, respectively. Elements of $\mathcal{F}$ with arity 0 are called object constants. A 0-ary predicate symbol is a propositional constant. Formulas are constructed as usual and variable-free formulas or theories are called *ground*. A sentence is said to be *factual* if it is built using connectives $\wedge$, $\vee$, and $\neg$ (i.e., implications of the form $\phi \rightarrow \bot$), only. A theory Γ is factual if every sentence of Γ has this property. The abbreviations introduced for propositional formulas carry over: $\phi \equiv \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$; $\neg\phi$ for $\phi \rightarrow \bot$; and $\top$ for $\bot \rightarrow \bot$.

4.1 Static Quantified Logic of Here-and-There

Semantically we refer to the static quantified version of here-and-there with decidable equality as captured axiomatically by the system $\mathbf{QHT}^{\mathbf{s}}_{=}$ [13]. It is characterized by Kripke models of two worlds with a common universe (hence static) that interpret function symbols in the same way.

More formally, consider a first-order interpretation I of a first-order signature $\mathcal{L}$ on a universe $\mathcal{U}$. We denote by $\mathcal{L}^I$ the extension of $\mathcal{L}$ obtained by adding pairwise distinct names c_ε as object constants for the objects in the universe, i.e., for each $\varepsilon \in \mathcal{U}$. We write $\mathcal{C}_\mathcal{U}$ for the set $\{c_\varepsilon \mid \varepsilon \in \mathcal{U}\}$ and identify I with its extension to $\mathcal{L}^I$ given by $I(c_\varepsilon) = \varepsilon$. Furthermore, let t^I denote the value assigned by I to a ground term t (of signature $\mathcal{L}^I$), and let $\mathcal{L}_\mathcal{F}$ denote the restriction of $\mathcal{L}$ to function symbols (thus including object constants). By $\mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$ we denote the set of atomic formulas built using predicates from $\mathcal{P}$ and constants $\mathcal{C}_\mathcal{U}$.

We represent a first-order interpretation I of $\mathcal{L}$ on $\mathcal{U}$ as a pair $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle$,[3] where $I|_{\mathcal{L}_\mathcal{F}}$ is the restriction of I on function symbols, and $I|_{\mathcal{C}_\mathcal{U}}$ is the set of atomic formulas from $\mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$ which are satisfied in I. Correspondingly, classical satisfaction of a sentence ϕ by a first-order interpretation $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle$ is denoted by $\langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}}\rangle \models \phi$. We also define a subset relation for first-order interpretations I_1, I_2 of $\mathcal{L}$ on $\mathcal{U}$ (ie., over the same domain) by $I_1 \subseteq I_2$ if $I_1|_{\mathcal{C}_\mathcal{U}} \subseteq I_2|_{\mathcal{C}_\mathcal{U}}$.[4]

A QHT-interpretation of $\mathcal{L}$ is a triple $\langle I, J, K\rangle$, such that (i) I is an interpretation of $\mathcal{L}_\mathcal{F}$ on $\mathcal{U}$, and (ii) $J \subseteq K \subseteq \mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}$.

The satisfaction of a sentence ϕ of signature $\mathcal{L}^I$ by a QHT-interpretation $M = \langle I, J, K\rangle$ (a QHT-model) is defined as:

1. $M \models p(t_1, \ldots, t_n)$ if $p(c_{t_1^I}, \ldots, c_{t_n^I}) \in J$;
2. $M \models t_1 = t_2$ if $t_1^I = t_2^I$;
3. $M \not\models \bot$;
4. $M \models \phi \wedge \psi$ if $M \models \phi$ and $M \models \psi$,
5. $M \models \phi \vee \psi$, if $M \models \phi$ or $M \models \psi$,
6. $M \models \phi \rightarrow \psi$ if (i) $M \not\models \phi$ or $M \models \psi$, and (ii) $\langle I, K\rangle \models \phi \rightarrow \psi$[5];
7. $M \models \forall x \phi(x)$ if $M \models \phi(c_\varepsilon)$ for all $\varepsilon \in \mathcal{U}$;
8. $M \models \exists x \phi(x)$ if $M \models \phi(c_\varepsilon)$ for some $\varepsilon \in \mathcal{U}$;.

A QHT-interpretation $M = \langle I, J, K\rangle$ is called a *QHT-countermodel* of a theory Γ iff $M \not\models \Gamma$; it is called *total* if $J = K$. A total QHT-interpretation $M = \langle I, K, K\rangle$ is called a *quantified equilibrium model* (*QEL-model*) of a theory Γ, iff $M \models \Gamma$ and $M' \not\models \Gamma$, for all QHT-interpretations $M' = \langle I, J, K\rangle$ such that $J \subset K$. A first-order interpretation $\langle I, K\rangle$ is an *answer set* of Γ iff $M = \langle I, K, K\rangle$ is a QEL-model of a theory Γ.

In analogy to the propositional case, we will use the following simple properties.

Lemma 2. *If* $\langle I, J, K\rangle \models \phi$ *then* $\langle I, K, K\rangle \models \phi$.

Lemma 3. $\langle I, J, K\rangle \models \neg\phi$ *iff* $\langle I, K\rangle \models \neg\phi$.

[3] We use angle brackets to distinguish from HT-interpretations.

[4] Note that one could additionally require that $I_1|_{\mathcal{L}_\mathcal{F}} = I_2|_{\mathcal{L}_\mathcal{F}}$, which is not necessary for our purpose, however.

[5] That is, $\langle I, K\rangle$ satisfies $\phi \rightarrow \psi$ classically.

4.2 Characterizing Equivalence by QHT-countermodels

We aim at generalizing uniform equivalence for first-order theories, in its most liberal form, which means wrt. factual theories. For this purpose, we first lift Lemma 1.

Lemma 4. *Let ϕ be a factual sentence. If $\langle I, J, K\rangle \models \phi$ and $J \subseteq J' \subseteq K$, then $\langle I, J', K\rangle \models \phi$.*

The different notions of closure naturally extend to (sets of) QHT-interpretations. In particular, a total QHT-interpretation $M = \langle I, K, K\rangle$ is called *total-open* in a set S of QHT-interpretations, if $M \in S$ and $\langle I, J, K\rangle \notin S$ for every $J \subset K$. It is called *total-closed* if $\langle I, J, K\rangle \in S$ for every $J \subset K$. A QHT-interpretation $\langle I, J, K\rangle$ is *there-closed* in a set S of QHT-interpretations if $\langle I, J', K\rangle \in S$ for every $J \subseteq J' \subset K$.

The first main result lifts the characterization of uniform equivalence for theories by HT-countermodels to the first-order case.

Theorem 4. *Two first-order theories are uniformly equivalent iff they have the same sets of there-closed QHT-countermodels.*

We next turn to an alternative characterization by a mixture of QHT-models and QHT-countermodels as in the propositional case. A QHT-countermodel $\langle I, J, K\rangle$ of a theory Γ is called QHT here-countermodel of Γ if $\langle I, K\rangle \models \Gamma$. A QHT-interpretation $\langle I, J, K\rangle$ is an QHT equivalence-interpretation of a theory Γ, if it is a total QHT-model of Γ or a QHT here-countermodel of Γ. In slight abuse of notation, we reuse the notation S_e, $S \in \{C, E\}$ and $e \in \{c, a, s, u\}$, for respective sets of QHT-interpretations, and arrive at the following formal result:

Theorem 5. *Two theories coincide on their QHT-countermodels iff they have the same QHT equivalence-interpretations, in symbols $C_s(\Gamma_1) = C_s(\Gamma_2)$ iff $E_s(\Gamma_1) = E_s(\Gamma_2)$.*

As a consequence of these two main results, we obtain an elegant, unified formal characterization of the different notions of equivalence for first-order theories under generalized answer-set semantics.

Corollary 2. *Given two first-order theories Γ_1 and Γ_2, the following propositions are equivalent for $e \in \{c, a, s, u\}$: $\Gamma_1 \equiv_e \Gamma_2$; $C_e(\Gamma_1) = C_e(\Gamma_2)$; $E_e(\Gamma_1) = E_e(\Gamma_2)$.*

Moreover, QHT-countermodels allow for a simplified treatment of extended signatures, which is not the case for QHT-models. For QHT-models it is known that $M \models \Gamma$ implies $M|_{\mathcal{L}} \models \Gamma$ (cf. e.g., Prop. 3 in [10]), hence $M|_{\mathcal{L}} \not\models \Gamma$ implies $M \not\models \Gamma$, i.e., $M|_{\mathcal{L}} \in C_s(\Gamma)$ implies $M \in C_s(\Gamma)$. The converse direction holds for totality preserving restrictions.

Proposition 4. *Let M be a QHT-interpretation over $\mathcal{L}$ on $\mathcal{U}$. Then, $M \in E_s(\Gamma)$ for a theory Γ iff $M \models \Gamma_\phi(M)$ for some $\phi \in \Gamma$, where $\Gamma_\phi(M) = \{\neg\neg\psi \mid \psi \in \Gamma\} \cup \{\phi \to (\neg\neg a \to a) \mid a \in \mathcal{B}_{\mathcal{P},\mathcal{C}_\mathcal{U}}\}$.*

Theorem 6. *Let Γ be a theory over $\mathcal{L}$, let $\mathcal{L}' \supset \mathcal{L}$, and let M an HT-interpretation over $\mathcal{L}'$ such that $M|_{\mathcal{L}}$ is totality preserving. Then, $M \in C_s(\Gamma)$ implies $M|_{\mathcal{L}} \in C_s(\Gamma)$.*

Proof. Let $M = \langle I', J', K'\rangle$, $M|_{\mathcal{L}} = \langle I, J, K\rangle$, and assume $M \not\models \Gamma$. First, suppose $\langle I', K', K'\rangle \not\models \Gamma$, i.e., there exists a sentence $\phi \in \Gamma$, such that $\langle I', K', K'\rangle \not\models \phi$. We show that $\langle I, K, K\rangle \not\models \phi$ by induction on the formula structure of ϕ.

Let us denote $\langle I, K, K\rangle$ by N and $\langle I', K', K'\rangle$ by N'. For the base case, consider an atomic sentence ϕ. If ϕ is of the form $p(t_1, \ldots, t_n)$, then $p(c_{t_1^I}, \ldots, c_{t_n^I}) \notin K$ because $N' \not\models \phi$. By the fact that $K \subseteq K'$ we conclude that $p(c_{t_1^I}, \ldots, c_{t_n^I}) \notin K$ and hence $N \not\models \phi$. If ϕ is of the form $t_1 = t_2$ then $N' \not\models \phi$ implies $t_1^I \neq t_2^I$, and thus $N \not\models \phi$. If ϕ is $\bot$ then $N' \not\models \phi$ and $N \not\models \phi$. This proves the claim for atomic formulas.

For the induction step, assume that $N' \not\models \phi$ implies $N \not\models \phi$, for any sentence of depth $n-1$, and let ϕ be a sentence of depth n. We show that $M|_{\mathcal{L}} \models \phi$ implies $M \models \phi$. Suppose ϕ is the conjunction or disjunction of two sentences ϕ_1 and ϕ_2. Then ϕ_1 and ϕ_2 are sentences of depth $n-1$. Hence, $N' \not\models \phi_1$ implies $N \not\models \phi_1$, and the same for ϕ_2. Therefore, if N' is a QHT-countermodel of one or both of the sentences then so is N, which implies $N' \not\models \phi$ implies $N \not\models \phi$ if ϕ is the conjunction or disjunction of two sentences. As for implication, let ϕ be of the form $\phi_1 \rightarrow \phi_2$. In this case, $N' \not\models \phi$ implies $N' \models \phi_1$ and $N' \not\models \phi_2$. Therefore, $N \models \phi_1$ by the usual sub-model property for QHT-models, and $N \not\models \phi_2$ by assumption. Hence, $N \not\models \phi$. Eventually, consider a quantified sentence ϕ, i.e., ϕ is of the form $\forall x\phi_1(x)$ or $\exists x\phi_1(x)$. In this case, $N' \not\models \phi$ implies $N' \not\models \phi_1(c_\varepsilon)$ for some, respectively all, $\varepsilon \in \mathcal{U}$. Since each of the sentences $\phi_1(c_\varepsilon)$ is of depth $n-1$, the same is true for N by assumption. It follows that $N' \not\models \phi$ implies $N \not\models \phi$ also for quantified sentences ϕ of depth n, and therefore, for any sentence ϕ of depth n. This concludes the inductive argument and proves the claim for total QHT-countermodels.

Moreover, because QHT-countermodels are total-closed, this proves the claim for any QHT-countermodel $M = \langle I', J', K'\rangle$, such that $\langle I', K', K'\rangle \not\models \Gamma$.

We continue with the case that $\langle I', K', K'\rangle \models \Gamma$. Then $J' \subset K'$ holds, which means that M is a QHT equivalence-interpretation of Γ. Therefore, $M \not\models \phi$ for some $\phi \in \Gamma$. Additionally, $M \models \neg\neg\psi$ for all $\psi \in \Gamma$ (recall that $\langle I', K', K'\rangle \models \Gamma$, thus $\langle I', K'\rangle \models \Gamma$). By construction this implies $M \models \Gamma_\phi(M|_{\mathcal{L}})$. Therefore, $M|_{\mathcal{L}} \models \Gamma_\phi(M|_{\mathcal{L}})$, i.e., $M|_{\mathcal{L}}$ is a QHT equivalence-interpretation of Γ. Since the restriction is totality preserving, $M|_{\mathcal{L}}$ is non-total. This proves $M|_{\mathcal{L}} \not\models \Gamma$. □

Since QHT equivalence-interpretations consist of non-total QHT-countermodels and total QHT-models, the result carries over to QHT equivalence-interpretations. However, QHT-models do not satisfy such an extended property:

Example 5. Consider the theory $\Gamma = \{q(X) \rightarrow p(X)\}$ over $\mathcal{L} = \langle\{c_1\}, \{p, q\}\rangle$, and let $\mathcal{L}' = \langle\{c_1, c_2\}, \{p, q\}\rangle$. Then, $M = \langle id, \{p(c_1)\}, \{p(c_1), q(c_1)\}\rangle$ is a QHT-model of Γ over $\mathcal{L}$ on $\mathcal{U} = \{c_1, c_2\}$. However, $M' = \langle id, \{p(c_1), q(c_2)\}, \{p(c_1), p(c_2), q(c_1), q(c_2)\}\rangle$ is not a QHT-model of Γ on $\mathcal{U}$, although $M'|_{\mathcal{L}} = M$ is totality preserving.

Moreover, it is indeed necessary that the reduction is totality preserving. For instance, $M = \langle id, \{p(c_1), q(c_2)\}, \{p(c_1), p(c_2), q(c_2)\}\rangle$ is a non-total QHT-countermodel, but $M|_{\mathcal{L}} = \langle id, \{p(c_1)\}, \{p(c_1)\}\rangle$ is a QHT-model of Γ. □

5 Non-ground Logic Programs

In this section we apply the characterizations obtained for first-order theories to non-ground logic programs under various extended semantics—compared to the traditional semantics in terms of Herbrand interpretations. For a proper treatment of these issues, further background is required and introduced (succinctly, but at sufficient detail) below.

In non-ground logic programming, we restrict to a function-free first-order signature $\mathcal{L} = \langle \mathcal{F}, \mathcal{P} \rangle$ (i.e., $\mathcal{F}$ contains object constants only) without equality. A *program* Π (over $\mathcal{L}$) is a set of rules (over $\mathcal{L}$) of the form (1). A rule r is *safe* if each variable occurring in $H(r) \cup B^-(r)$ also occurs in $B^+(r)$; a rule r is *ground*, if all atoms occurring in it are ground. A program is safe, respectively ground, if all of its rules enjoy this property.

Given Π over $\mathcal{L}$ and a universe $\mathcal{U}$, let $\mathcal{L}^{\mathcal{U}}$ be the extension of $\mathcal{L}$ as before. The *grounding of* Π wrt. $\mathcal{U}$ and an interpretation $I|_{\mathcal{L}_\mathcal{F}}$ of $\mathcal{L}_\mathcal{F}$ on $\mathcal{U}$ is defined as the set $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})$ of ground rules obtained from $r \in \Pi$ by (i) replacing any constant c in r by c_ε such that $I|_{\mathcal{L}_\mathcal{F}}(c) = \varepsilon$, and (ii) all possible substitutions of elements in $\mathcal{C}_\mathcal{U}$ for the variables in r.

Adapted from [6], the *reduct* of a program Π with respect to a first-order interpretation $I = \langle I|_{\mathcal{L}_\mathcal{F}}, I|_{\mathcal{C}_\mathcal{U}} \rangle$ on universe $\mathcal{U}$, in symbols $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})^I$, is given by the set of rules

$$a_1 \vee \dots \vee a_k \ \leftarrow \ b_1, \dots, b_m,$$

obtained from rules in $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})$ of the form (1), such that $I \models a_i$ for all $k < i \leq l$ and $I \not\models b_j$ for all $m < j \leq n$.

A first-order interpretation I satisfies a rule r, $I \models r$, iff $I \models \Gamma_r$, where $\Gamma_r = \forall \boldsymbol{x}(\beta_r \rightarrow \alpha_r)$, $\boldsymbol{x}$ are the free variables in r, α_r is the disjunction of $H(r)$, and β_r is the conjunction of $B(r)$. It satisfies a program Π, symbolically $I \models \Pi$, iff it satisfies every $r \in \Pi$, i.e., if $I \models \Gamma_\Pi$, where $\Gamma_\Pi = \bigcup_{r \in \Pi} \Gamma_r$.

A first-order interpretation I is called a *generalized answer set* of Π iff it satisfies $grd_\mathcal{U}(\Pi, I|_{\mathcal{L}_\mathcal{F}})^I$ and it is subset minimal among the interpretations of $\mathcal{L}$ on $\mathcal{U}$ with this property.

Traditionally, only *Herbrand interpretations* are considered as the answer sets of a logic program. The set of all (object) constants occurring in Π is called the *Herbrand universe* of Π, symbolically $\mathcal{H}$. If no constant appears in Π, then $\mathcal{H} = \{c\}$, for an arbitrary constant c. A Herbrand interpretation is any interpretation I of $\mathcal{L}_\mathcal{H} = \langle \mathcal{H}, \mathcal{P} \rangle$ on $\mathcal{H}$ interpreting object constants by identity, id, i.e., $I(c) = id(c) = c$ for all $c \in \mathcal{H}$. A Herbrand interpretation I is an *ordinary answer set* of Π iff it is subset minimal among the interpretations of $\mathcal{L}_\mathcal{H}$ on $\mathcal{H}$ satisfying $grd_\mathcal{H}(\Pi, id)^I$.

Furthermore, an *extended Herbrand interpretation* is an interpretation of $\mathcal{L}$ on $U \supseteq \mathcal{F}$ interpreting object constants by identity. An extended Herbrand interpretation I is an *open answer set* [8] of Π iff it is subset minimal among the interpretations of $\mathcal{L}$ on $\mathcal{U}$ satisfying $grd_\mathcal{U}(\Pi, id)^I$.

Note that since we consider programs without equality, we semantically resort to the logic $\mathbf{QHT^s}$, which results from $\mathbf{QHT^s_=}$ by dropping the axioms for equality. Concerning Kripke models, however, in slight abuse of notation, we reuse QHT-models as defined for the general case. A QHT-interpretation $M = \langle I, J, K \rangle$ is called an (extended) QHT Herbrand interpretation, if $\langle I, K \rangle$ is an (extended) Herbrand interpretation. Given a program Π, $\langle I, K \rangle$ is a generalized answer set of Π iff $\langle I, K, K \rangle$ is a QEL-model of Γ_Π, and $\langle I, K \rangle$ is an open, respectively ordinary, answer set of Π iff $\langle I, K, K \rangle$ is an extended Herbrand, respectively Herbrand, QEL-model of Γ_Π. Notice that the static interpretation of constants introduced by Item (i) of the grounding process is essential for this correspondences in terms of $\mathbf{QHT^s}$. In slight abuse of notation, we further on identify Π and Γ_Π.

As already mentioned for propositional programs, uniform equivalence is usually understood wrt. sets of *ground facts* (i.e., ground atoms). Obviously, uniform equivalence wrt. factual theories implies uniform equivalence wrt. ground atoms. We show the converse direction (lifting Theorem 2 in [21]).

Proposition 5. *Given two programs Π_1, Π_2, then $\Pi_1 \equiv_u \Pi_2$ iff $(\Pi_1 \cup A) \equiv_a (\Pi_2 \cup A)$, for any set of ground atoms A.*

Thus, there is no difference whether we consider uniform equivalence wrt. sets of ground facts or factual theories. Since one can also consider sets of clauses, i.e. disjunctions of atomic formulas and their negations, which is a more suitable representation of facts according to the definition of program rules in this paper, we adopt the following terminology. A rule r is called a *fact* if $B(r) = \emptyset$, and a *factual program* is a set of facts. Then, by our result $\Pi_1 \equiv_u \Pi_2$ holds for programs Π_1, Π_2 iff $(\Pi_1 \cup \Pi) \equiv_a (\Pi_2 \cup \Pi)$, for any factual program Π.

5.1 Uniform Equivalence under Herbrand Interpretations

The results in the previous section generalize the notion of uniform equivalence to programs under generalized open answer-set semantics and provide alternative characterizations for other notions of equivalence. They apply to programs under open answer-set semantics and ordinary answer-set semantics, when QHT-interpretations are restricted to extended Herbrand interpretations and Herbrand interpretations, respectively. For programs Π_1 and Π_2 and $e \in \{c, a, s, u\}$, we use $\Pi_1 \equiv_e^{\mathcal{E}} \Pi_2$ and $\Pi_1 \equiv_e^{\mathcal{H}} \Pi_2$ to denote (classical, answer-set, strong, or uniform) equivalence under open answer-set semantics and ordinary answer-set semantics, respectively.

Corollary 3. *Given two programs Π_1 and Π_2, it holds that*
- *$\Pi_1 \equiv_e^{\mathcal{E}} \Pi_2$, $C_e^{\mathcal{E}}(\Pi_1) = C_e^{\mathcal{E}}(\Pi_2)$, and $E_e^{\mathcal{E}}(\Pi_1) = E_e^{\mathcal{E}}(\Pi_2)$ are equivalent; and*
- *$\Pi_1 \equiv_e^{\mathcal{H}} \Pi_2$, $C_e^{\mathcal{H}}(\Pi_1) = C_e^{\mathcal{H}}(\Pi_2)$, and $E_e^{\mathcal{H}}(\Pi_1) = E_e^{\mathcal{H}}(\Pi_2)$ are equivalent;*

where $e \in \{c, a, s, u\}$, and superscript $\mathcal{H}$ ($\mathcal{E}$) denotes the restriction to (extended) Herbrand interpretations.

For safe programs the notions of open answer set and ordinary answer set coincide [10]. Note that a fact is safe if it is ground. We obtain that uniform equivalence coincides under the two semantics even for programs that are not safe. Intuitively, the potential addition of arbitrary facts accounts for the difference in the semantics since it requires to consider larger domains than the Herbrand universe.

Theorem 7. *Let Π_1, Π_2 be programs over $\mathcal{L}$. Then, $\Pi_1 \equiv_u^{\mathcal{E}} \Pi_2$ iff $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$.*

Finally, we turn to the practically relevant setting of finite, possibly unsafe, programs under Herbrand interpretations, i.e., ordinary (and open) answer-set semantics. For finite programs, uniform equivalence can be characterized by HT-models of the grounding, also for infinite domains. In other words, the problems of "infinite chains" as in Example 1 cannot be generated by the process of grounding.

Theorem 8. *Let Π_1, Π_2 be finite programs over $\mathcal{L}$. Then, $\Pi_1 \equiv_u^{\mathcal{H}} \Pi_2$ iff Π_1 and Π_2 have the same (i) total and (ii) maximal, non-total extended Herbrand QHT-models.*

6 Conclusion

Countermodels in equilibrium logic have recently been used in [22] to show that propositional disjunctive logic programs with negation in the head are strongly equivalent to propositional theories, and in [23] to generate a minimal logic program for a given propositional theory.

By means of Quantified Equilibrium Logic, in [13], the notion of strong equivalence has been extended to first-order theories with equality, under the generalized notion of answer set we have adopted. QEL has also been shown to capture open answer-sets [8] and generalized open answer-sets [9], and is a promising framework to study hybrid knowledge bases providing a unified semantics, since it encompasses classical logic as well as disjunctive logic programs under the answer-set semantics [10].

Our results complete the picture by making uniform equivalence, which so far has only been dealt with for finite programs under ordinary answer-set semantics, amenable to these generalized settings without any finiteness restrictions, in particular on the domain. Thus, we arrived at a uniform model-theoretic characterization of the notions of equivalence studied in ASP. We have also shown that for finite programs, i.e., those programs solvers are able to deal with, infinite domains do not cause the problems observed for infinite propositional programs, when dealing with uniform equivalence in terms of HT-models of the grounding.

The combination of ontologies and nonmonotonic rules is an important issue in knowledge representation and reasoning for the Semantic Web. Therefore, the study of optimizations and correspondences under an appropriate semantics, such as the generalizations of answer-set semantics characterized by QEL, constitute an interesting topic for further research of relevance in this application domain. Like for Datalog, uniform equivalence may serve investigations on query equivalence and query containment in these hybrid settings. The simplified treatment of extended signatures for countermodels and equivalence interpretations is expected to be of avail, in particular for the study of relativized notions of equivalence and correspondence [24].

On the foundational level, our results raise the interesting question whether extensions of intuitionistic logics that allow for a direct characterization of countermodels, would provide a more suitable formal apparatus for the study of (at least uniform) equivalence in ASP.

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