Chapter 4
Fuzzy Rule-Based Systems with Polynomial Membership Functions

In order to obtain a richer class of functions to which the fuzzy rule-based system is equivalent, one can use nonlinear membership functions of fuzzy sets, to which *polynomials* of the second or higher degree belong. Such polynomials are defined by three or more parameters. It would appear that by using nonlinear membership functions, one can get a sufficiently large class of functions, to which the rule-based system is equivalent. However, if we increase the complexity of membership functions of fuzzy sets only, while preserving the number of fuzzy sets assigned for the input variables, our intuition about richness of the class of functions performed by the rule-based system can fail us. The number of fuzzy sets is important, since it determines the number of consequents of the rules; thus, it constrains the class of functions performed by the zero-order TS rule-based systems. This fact will be shown further on.

The consequents of "If-then" rules can be defined as functions depending on input variables, e.g. they can be polynomials. However, if it is not stated differently, we will consider the zero-order rule-based systems. A special attention will be paid to the TS systems which use the second degree polynomials as the membership functions of fuzzy sets. First we will show that it is not possible to obtain any second degree polynomial function, to which a TS rule-based system is equivalent, on the assumption that only two complementary membership functions as the second degree polynomials are defined for the input variables for this system. We prove however, that three quadratic membership functions suffice to model every second degree polynomial function. For such membership functions the natural requirements that guarantee a clear interpretability of fuzzy sets will be defined as well. The TS systems that use as a basis three normalized second degree polynomial membership functions, called *P2-TS systems*, will be thoroughly investigated. Similarly to the fuzzy rule-based systems with linear membership functions, we will define both a generator and a fundamental matrix for the P2-TS systems. The features of the fundamental matrix for such systems and its inverse will be given.

The curse of dimensionality problem is more serious for the P2-TS systems than the one for the P1-TS systems. Therefore we will develop the recursive procedures for the computation of the inverse of the fundamental matrix and for the crisp output of the P2-TS systems.

J. Kluska: Analytical Methods in Fuzzy Modeling and Control, STUDFUZZ 241, pp. 61–99.
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4.1 TS Systems with Two Polynomial Membership Functions for Every Input

Below we prove the following

Remark 4.1. Suppose the inputs of a zero-order TS system are $z_k \in [-\alpha_k, \beta_k]$, $(k = 1, 2, \ldots, n)$, and every input has assigned two complementary membership functions, say $N_k(z_k)$ and $P_k(z_k) = 1 - N_k(z_k)$. If all membership functions are polynomials of the degree d, then

(1) the crisp output $f(z_1, \ldots, z_n)$ of this system is the following multivariate polynomial

$$f(z_1, \ldots, z_n) = \sum_{p_1, \ldots, p_n \in \{0,1,2,\ldots,d\}^n} \theta_{p_1,\ldots,p_n} z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}, \tag{4.1}$$

where $\theta_{p_1,\ldots,p_n} \in \mathbb{R}$,

(2) every multilinear function of type (4.1), can be exactly expressed by the fuzzy "If-then" rules if, and only if the degree of polynomials is $d = 1$,

(3) not every nonlinear function of type (4.1) can be unambiguously expressed by the fuzzy "If-then" rules, when the degree $d > 1$.

Proof.

(1) First observe that the system output S is a linear combination of 2^n polynomials in the form "$\prod_{k=1}^{n} \left(a_{d,k} z_k^d + \ldots + a_{1,k} z_k + a_{0,k}\right)$". Thus, the output S is in the form (4.1), indeed.

(2) For two fuzzy sets for every input (N_k and P_k), there are 2^n consequents of the rules, which are free design parameters. The polynomial of degree d is described by $(d + 1)$ parameters. Thus, the number of functions (4.1), which are structurally different one from another, is $(d + 1)^n$, and it is equal to the number of different consequents of the rules if, and only if $(d + 1)^n = 2^n$. In this case we apply Theorem 2.4.

(3) For $d \geq 2$ we have $(d + 1)^n > 2^n$. Thus, not every nonlinear function (4.1) can be exactly expressed by TS system; this finishes the proof of Remark 4.1. □

Let us consider an example which is of twofold goal. Firstly, we will give an additional proof of Remark 4.1 for the second degree polynomial ($d = 2$). Secondly, we will show that by using some nonlinear bijection for the crisp input x of the TS system with two linear membership functions, we can obtain its nonlinear output $S(x)$, (see Fig. 4.1). Of course, the use of such bijection is not necessary to prove Remark 4.1.

Example 4.2. Let us consider the zero-order TS system with the input x and the output S, as shown in Fig. 4.1. We define a nonlinear mapping between the original input $x \in [-\alpha, \beta]$ and an ancillary variable $z \in [-\alpha, \beta]$, in the form of the second order polynomial

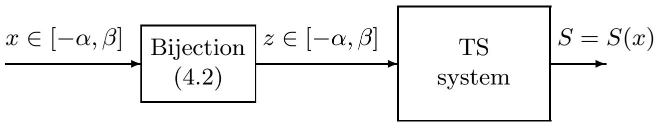


Fig. 4.1 SISO TS system from Example 4.2

$$z(x) = x + m\frac{(x+\alpha)(x-\beta)}{\alpha+\beta}, \tag{4.2}$$

where m is a parameter - see Fig. 4.2.We assume that $0 \neq |m| < 1$, since (4.2) is a bijection $z : [-\alpha, \beta] \to [-\alpha, \beta]$ if, and only if $|m| < 1$, and we omit the trivial case $z = x$. If the membership functions are linear:

$$N(z) = (\alpha+\beta)^{-1}(\beta - z), \qquad P(z) = 1 - N(z),$$

then from two fuzzy rules:

$$\left.\begin{aligned} R_1 &: \text{If } z \text{ is } N, \text{ then } S = q_1, \\ R_2 &: \text{If } z \text{ is } P, \text{ then } S = q_2, \end{aligned}\right\}$$

we obtain the system output

$$\begin{aligned} S &= \frac{q_1 N(z) + q_2 P(z)}{N(z) + P(z)} \\ &= \frac{(q_2 - q_1)x + \alpha q_2 + \beta q_1}{\alpha+\beta} + m\frac{(q_2-q_1)(x+\alpha)(x-\beta)}{(\alpha+\beta)^2}. \end{aligned} \tag{4.3}$$

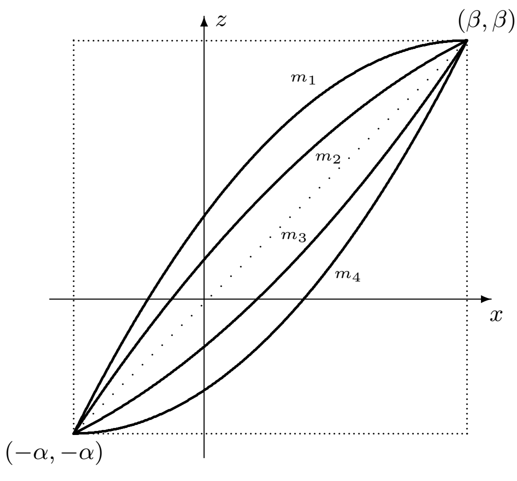


Fig. 4.2 The bijection (4.2) with parameter $m = m_i$: $m_1 = -1$, $m_2 = -0.5$, $m_3 = 0.5$ and $m_4 = 1$

It can be equivalently written as

$$S(x) = Ax^2 + Bx + C,$$

where

$$A = m\frac{q_2 - q_1}{(\alpha + \beta)^2},$$

$$B = \frac{(\alpha + \beta + m(\alpha - \beta))(q_2 - q_1)}{(\alpha + \beta)^2},$$

$$C = \frac{q_1\beta(\alpha + \beta + m\alpha) + q_2\left(\alpha\beta - m\alpha\beta + \alpha^2\right)}{(\alpha + \beta)^2}.$$

Thus, independently of the consequents of the rules (q_1 and q_2), the system output is restricted to the following class of functions as second degree polynomials

$$S(x) = Ax^2 + \left(\frac{\alpha + \beta}{m} + \alpha - \beta\right)Ax + C, \qquad x \in [-\alpha, \beta], \tag{4.4}$$

where $A, C \in \mathbb{R}$, by $1 > |m| \neq 0$. This means that there are "many", but not all second degree polynomials, which can be exactly represented by the rule-based system. For example, by the fixed interval $[-\alpha, \beta]$, we are not able to formulate such two fuzzy rules, that the rule-based system would be equivalent to the following polynomial

$$f(x) = Ax^2 + A(\alpha - \beta)x + C, \qquad x \in [-\alpha, \beta], \tag{4.5}$$

where $A, C \in \mathbb{R}$. This is because there is no m such that $0 \neq |m| < 1$ and $\left(\frac{\alpha + \beta}{m} + \alpha - \beta\right)A = (\alpha - \beta)A$ for any real α, β and A. In other words, the function (4.5) is not from the class of functions defined by (4.4). This example shows by contradiction that the second part of Remark 4.1 is true.

The zero-order rule-based TS systems in which the membership functions of input variables are polynomials of the degree d will be called *Pd-TS systems*. A special attention will be paid to *P2-TS systems* further on.

4.2 The Normalized Membership Functions for P2-TS Systems

From the preceding section we know that it is not possible to obtain any second degree polynomial by using the TS systems, in which only two complementary membership functions as second degree polynomials are defined. However, we will prove further on that three membership functions as the second degree polynomials suffice to model any second degree polynomial function. Such membership functions defining the fuzzy sets for input variables will be defined below.

In the interval $[-\alpha, \beta]$ we define three membership functions of fuzzy sets, say $N(z)$, $Z(z)$ and $P(z)$, which are the second degree polynomials and satisfy the following additional conditions:

1. $N : [-\alpha, \beta] \to [0, 1]$ is a monotonic function with *negative slope*, i.e. $dN(z)/dz < 0$ for $z \in [-\alpha, \beta]$, which satisfies two boundary conditions:

   a) $N(-\alpha) = 1$,
   b) $N(\beta) = 0$.

2. $P : [-\alpha, \beta] \to [0, 1]$ is the monotonic function with *positive slope*, i.e. $dP(z)/dz > 0$ for $z \in [-\alpha, \beta]$, symmetric to the function N with respect to the interval centre $\sigma \in [-\alpha, \beta]$:

$$\sigma = \frac{-\alpha + \beta}{2}. \tag{4.6}$$

3. $Z : [-\alpha, \beta] \to [0, 1]$ is the function which reaches *zero slope* in σ, i.e. $dZ(\sigma)/dz = 0$.
4. The functions N, Z and P satisfy the *normalization condition*

$$N(z) + Z(z) + P(z) = 1, \qquad \forall\, z \in [-\alpha, \beta]\,. \tag{4.7}$$

One can prove that the functions N, Z and P meeting the above needs can be expressed as follows

$$N(z) = \frac{(\alpha + \beta - \lambda(z + \alpha))(\beta - z)}{(\alpha + \beta)^2}, \tag{4.8}$$

$$Z(z) = 2\lambda \frac{(\beta - z)(z + \alpha)}{(\alpha + \beta)^2}, \tag{4.9}$$

$$P(z) = \frac{(\alpha + \beta + \lambda(z - \beta))(z + \alpha)}{(\alpha + \beta)^2}, \tag{4.10}$$

where the parameter λ satisfies the condition

$$0 < \lambda \leq 1. \tag{4.11}$$

We do not allow $\lambda = 0$, since in such case $Z(z) = 0$ for all z, and there would be two nonzero membership functions only: $N(z)$ and $P(z)$. In other words, by $\lambda = 0$, the class of rule-based systems reduces to the formerly considered P1-TS systems. Figures 4.3 and 4.4 show plots of functions (4.8)-(4.10) for different values of parameter λ. Observe that N and P are normal fuzzy sets but Z is not normal. The cores of the fuzzy sets N, Z and P are three *characteristic points* of the universe of discourse: "$-\alpha$", "σ" and "β", respectively.

Fig. 4.3 The basis of normalized second degree polynomial membership functions by the maximal value of parameter λ, ($\lambda = 1$)

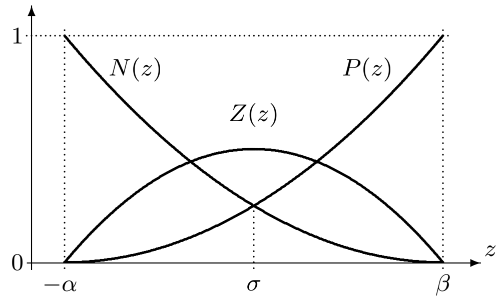
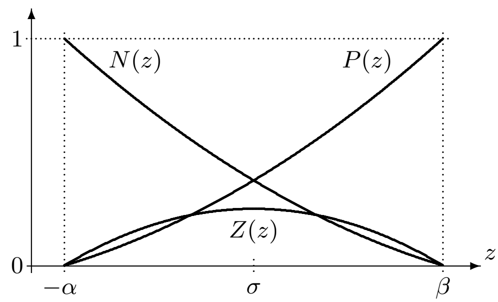


Fig. 4.4 The basis of normalized second degree polynomial membership functions by parameter $\lambda = 0.5$



The membership functions N, Z and P have a clear linguistic interpretation in any case of boundaries "$-\alpha$" and "β" as real numbers:

1. If $-\alpha < \beta < 0$, then N can be interpreted as *negative big*, Z - *negative medium* and P - *negative small*,
2. If $-\alpha < \beta = 0$, then N can be interpreted as *negative*, Z - *negative small* and P - *negative zero*,
3. If $-\alpha < 0 < \beta$, then N can be interpreted as *negative*, Z - *zero* and P - *positive*,
4. If $0 = -\alpha < \beta$, then N can be interpreted as *positive zero*, Z - *positive small* and P - *positive*,
5. If $0 < -\alpha < \beta$, then N can be interpreted as *positive small*, Z - *positive medium* and P - *positive big*.

As discussed in Section 2.2, the linguistic terms can be substituted by others depending on the context or specific application.

The rule-based TS systems with the above membership functions we will call *P2-TS systems* for short.

4.3 SISO P2-TS System

Now we will consider P2-TS system with single input $z \in [-\alpha, \beta]$ and single output S. The rule-base structure is as follows

$$\left.\begin{array}{l} R_1 : \text{If } z \text{ is } N, \text{ then } S = q_0, \\ R_2 : \text{If } z \text{ is } Z, \text{ then } S = q_1, \\ R_3 : \text{If } z \text{ is } P, \text{ then } S = q_2. \end{array}\right\} \tag{4.12}$$

The system output as a function of the input variable z is given by

$$S(z) = N(z)\, q_0 + Z(z)\, q_1 + P(z)\, q_2 = [N(z), Z(z), P(z)]\, \mathbf{q}, \tag{4.13}$$

where $\mathbf{q} = [q_0, q_1, q_2]^T$, and N, Z and P are defined in (4.8)-(4.10). By $\mathbf{s}$ we denote the vector containing values of system output in the cores of the fuzzy sets N, Z and P, respectively

$$\mathbf{s} = [S(-\alpha), S(\sigma), S(\beta)]^T .$$

It can be expressed equivalently by

$$\mathbf{s} = \mathbf{R}\mathbf{q}, \tag{4.14}$$

where the matrix $\mathbf{R}$ contains the membership degrees in the cores of the fuzzy sets

$$\mathbf{R} = \begin{bmatrix} N(-\alpha) & Z(-\alpha) & P(-\alpha) \\ N(\sigma) & Z(\sigma) & P(\sigma) \\ N(\beta) & Z(\beta) & P(\beta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (2-\lambda)/4 & \lambda/2 & (2-\lambda)/4 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.15}$$

Observe that $S(-\alpha) = q_0$ and $S(\beta) = q_2$. However, the consequent of the fuzzy rule R_2 in (4.12) is q_1, but

$$S(\sigma) = q_1' = \frac{2-\lambda}{4} q_0 + \frac{\lambda}{2} q_1 + \frac{2-\lambda}{4} q_2 \neq q_1,$$

and there is no such $\lambda \in (0, 1]$ for which q_1' would be equal to q_1. The maximal influence of the rule consequent q_1 for the crisp output q_1' one obtains for maximal value of the parameter λ. Therefore we prefer to use $\lambda = 1$.

Corollary 4.3. *The crisp output of the SISO P2-TS system is exactly the same as the consequent of the rule, if the input is either "$-\alpha$" or "β". The interpretation of the fuzzy rules R_1 and R_3 given by (4.12) for the P2-TS system is straightforward and analogous to the P1-TS systems.*

Similar considerations concerning P2-TS systems with many inputs will be given further on (see Theorem 4.11 and Example 4.13).

Now we introduce a *generator* for the SISO P2-TS system

$$\mathbf{g}(z) = \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}. \tag{4.16}$$

According to Remark 4.1, the function $f(z)$ to which the rule-based system (4.12) is equivalent, has the form

$$f(z) = \mathbf{g}^T(z)\boldsymbol{\theta}, \tag{4.17}$$

where $\boldsymbol{\theta} = [\theta_0, \theta_1, \theta_2]^T$. The equality $S(z) = f(z)$ must be satisfied for $z \in [-\alpha, \beta]$, particularly for all three characteristic points from the set $\{-\alpha, \sigma, \beta\} \subset [-\alpha, \beta]$. Thus,

$$\mathbf{s} = \begin{bmatrix} f(-\alpha) \\ f(\sigma) \\ f(\beta) \end{bmatrix} = \begin{bmatrix} \mathbf{g}^T(-\alpha) \\ \mathbf{g}^T(\sigma) \\ \mathbf{g}^T(\beta) \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \boldsymbol{\Gamma}^T\boldsymbol{\theta}$$

must be satisfied, where the matrix $\boldsymbol{\Gamma}$ is the concatenation of the values of the generator (4.16) in the points "$-\alpha$", "σ", and "β", respectively, i.e.

$$\boldsymbol{\Gamma} = [\mathbf{g}(-\alpha), \mathbf{g}(\sigma), \mathbf{g}(\beta)].$$

Thus, we obtain the exact relationship between consequents $\mathbf{q}$ of the rules (4.12) and parameters $\boldsymbol{\theta}$ of the function (4.17) as follows

$$\mathbf{Rq} = \boldsymbol{\Gamma}^T\boldsymbol{\theta}.$$

Thus,

$$\mathbf{q} = \mathbf{R}^{-1}\boldsymbol{\Gamma}^T\boldsymbol{\theta} = \boldsymbol{\Omega}^T\boldsymbol{\theta}, \tag{4.18}$$

where the *fundamental matrix* for the SISO P2-TS system is defined by

$$\boldsymbol{\Omega} = \boldsymbol{\Gamma}\left(\mathbf{R}^T\right)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha & \sigma & \beta \\ \alpha^2 & \left(\alpha^2+\beta^2\right)/2 - (\alpha+\beta)^2/(2\lambda) & \beta^2 \end{bmatrix}, \tag{4.19}$$

where $0 < \lambda \leq 1$. The inverse of $\boldsymbol{\Omega}$ always exists and is given by

$$\begin{aligned} \boldsymbol{\Omega}^{-1} &= \mathbf{R}^T\boldsymbol{\Gamma}^{-1} \\ &= \frac{1}{(\alpha+\beta)^2}\begin{bmatrix} \beta^2+\alpha\beta(1-\lambda) & -\alpha(1-\lambda)-\beta(1+\lambda) & \lambda \\ 2\lambda\alpha\beta & 4\lambda\sigma & -2\lambda \\ \alpha^2+\alpha\beta(1-\lambda) & \alpha(1+\lambda)+\beta(1-\lambda) & \lambda \end{bmatrix}. \end{aligned} \tag{4.20}$$

All equations are valid for any parameter value λ from the interval $(0, 1]$. Assuming $\lambda = 1$ and adding the index "1" for matrices in the case of SISO P2-TS system $(n = 1)$, we obtain

- the matrix (4.15) of membership degrees in the points from the set $\{-\alpha, \sigma, \beta\}$

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.21}$$

- the fundamental matrix of the SISO P2-TS system

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_1 & \sigma_1 & \beta_1 \\ \alpha_1^2 & -\alpha_1\beta_1 & \beta_1^2 \end{bmatrix}, \tag{4.22}$$

- and the inverse of the fundamental matrix

$$\boldsymbol{\Omega}_1^{-1} = \frac{1}{\left(\alpha_1 + \beta_1\right)^2} \begin{bmatrix} \beta_1^2 & -2\beta_1 & 1 \\ 2\beta_1\alpha_1 & 4\sigma_1 & -2 \\ \alpha_1^2 & 2\alpha_1 & 1 \end{bmatrix}. \tag{4.23}$$

The above formulas will be useful further on.

4.4 P2-TS System with Two and More Inputs

In this section we will investigate P2-TS systems with the inputs $z_1, \ldots, z_n$. For such systems, in order to define three membership functions N_k, Z_k and P_k as the functions of variables z_k, ($k = 1, 2, \ldots, n$), we can choose individual parameter values $\lambda_1, \lambda_2, \ldots, \lambda_n$ for the particular inputs. The membership functions take the following general form

$$N_k\left(z_k\right) = \frac{\left(\alpha_k + \beta_k - \lambda_k\left(z_k + \alpha_k\right)\right)\left(\beta_k - z_k\right)}{\left(\alpha_k + \beta_k\right)^2}, \tag{4.24}$$

$$Z_k\left(z_k\right) = 2\lambda_k \frac{\left(\beta_k - z_k\right)\left(z_k + \alpha_k\right)}{\left(\alpha_k + \beta_k\right)^2}, \tag{4.25}$$

$$P_k\left(z_k\right) = \frac{\left(\alpha_k + \beta_k + \lambda_k\left(z_k - \beta_k\right)\right)\left(z_k + \alpha_k\right)}{\left(\alpha_k + \beta_k\right)^2}, \tag{4.26}$$

where $\lambda_k \in (0, 1]$, $(k = 1, \ldots, n)$. If there are no contraindications, we prefer to assume in practice the same value $\lambda_k = 1$ for all variables (see Section 4.3) – this corresponds to membership functions shown in Fig. 4.3.

Let M_n be a crisp set of 3^n *characteristic points* for the P2-TS system as n-dimensional vectors

$$M_n = \{-\alpha_1, \sigma_1, \beta_1\} \times \{-\alpha_2, \sigma_2, \beta_2\} \times \ldots \times \{-\alpha_n, \sigma_n, \beta_n\} \subset D^n. \tag{4.27}$$

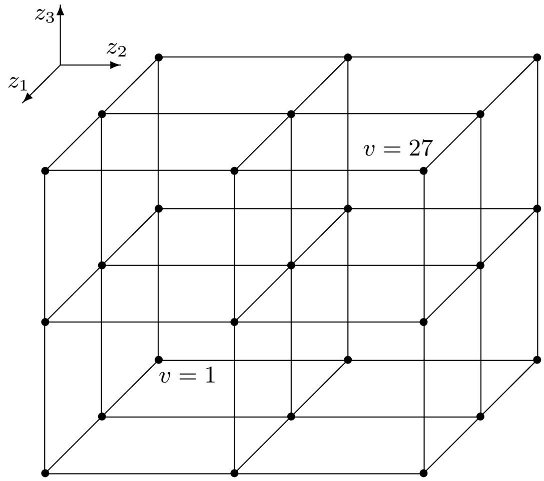


Fig. 4.5 The ordered set M_n for $n = 3$ with two depicted elements. The first one ($v = 1$) corresponds to the vector $(-\alpha_1, -\alpha_2, -\alpha_3)$ and the last one ($v = 27$) - to the vector $(\beta_1, \beta_2, \beta_3)$.

The set of characteristic points for P2-TS system includes all vertices of the hypercuboid D^n. We order M_n as follows. For every n-dimensional vector $(\gamma_1, \ldots, \gamma_n)$ as an element of the set M_n (see Fig. 4.5) we define the corresponding index v according to the following bijection

$$v = 1 + \sum_{i=1}^{n} 3^{i-1} p_i, \tag{4.28}$$

where

$$p_i = \begin{cases} 0 \Leftrightarrow \gamma_i = -\alpha_i \\ 1 \Leftrightarrow \gamma_i = \sigma_i \\ 2 \Leftrightarrow \gamma_i = \beta_i \end{cases}, \qquad i = 1, \ldots, n. \tag{4.29}$$

Thus, every element of the set M_n unambiguously corresponds to some index. For $(\gamma'_1, \ldots, \gamma'_n) \in M_n$ and $(\gamma''_1, \ldots, \gamma''_n) \in M_n$ we define an ordering relation "$\prec$" as follows

$$(\gamma'_1, \ldots, \gamma'_n) \prec (\gamma''_1, \ldots, \gamma''_n) \qquad \Leftrightarrow \qquad v_{\gamma'_1,\ldots,\gamma'_n} < v_{\gamma''_1,\ldots,\gamma''_n}. \tag{4.30}$$

- For $n = 1$ we have $v_{-\alpha} = 1 < v_\sigma = 2 < v_\beta = 3$ and therefore $-\alpha \prec \sigma \prec \beta$.
- For $n = 2$ the inequalities between indices are
  $v_{-\alpha_1,-\alpha_2} = 1 < v_{\sigma_1,-\alpha_2} = 2 < v_{\beta_1,-\alpha_2} = 3 < v_{-\alpha_1,\sigma_2} = 4 < v_{\sigma_1,\sigma_2} = 5 < v_{\beta_1,\sigma_2} = 6 < v_{-\alpha_1,\beta_2} = 7 < v_{\sigma_1,\beta_2} = 8 < v_{\beta_1,\beta_2} = 9$.

Thus, the members of M_2 are ordered as follows
$(-\alpha_1, -\alpha_2) \prec (\sigma_1, -\alpha_2) \prec (\beta_1, -\alpha_2) \prec (-\alpha_1, \sigma_2) \prec (\sigma_1, \sigma_2) \prec (\beta_1, \sigma_2) \prec (-\alpha_1, \beta_2) \prec (\sigma_1, \beta_2) \prec (\beta_1, \beta_2)$.

- For the ordered set M_3 we have
$(-\alpha_1, -\alpha_2, -\alpha_3) \prec (\sigma_1, -\alpha_2, -\alpha_3) \prec (\beta_1, -\alpha_2, -\alpha_3) \prec (-\alpha_1, \sigma_2, -\alpha_3) \prec (\sigma_1, \sigma_2, -\alpha_3) \prec (\beta_1, \sigma_2, -\alpha_3) \prec (-\alpha_1, \beta_2, -\alpha_3) \prec (\sigma_1, \beta_2, -\alpha_3) \prec (\beta_1, \beta_2, -\alpha_3) \prec (-\alpha_1, -\alpha_2, \sigma_3) \prec (\sigma_1, -\alpha_2, \sigma_3) \prec (\beta_1, -\alpha_2, \sigma_3) \prec (-\alpha_1, \sigma_2, \sigma_3) \prec (\sigma_1, \sigma_2, \sigma_3) \prec (\beta_1, \sigma_2, \sigma_3) \prec (-\alpha_1, \beta_2, \sigma_3) \prec (\sigma_1, \beta_2, \sigma_3) \prec (\beta_1, \beta_2, \sigma_3) \prec (-\alpha_1, -\alpha_2, \beta_3) \prec (\sigma_1, -\alpha_2, \beta_3) \prec (\beta_1, -\alpha_2, \beta_3) \prec (-\alpha_1, \sigma_2, \beta_3) \prec (\sigma_1, \sigma_2, \beta_3) \prec (\beta_1, \sigma_2, \beta_3) \prec (-\alpha_1, \beta_2, \beta_3) \prec (\sigma_1, \beta_2, \beta_3) \prec (\beta_1, \beta_2, \beta_3)$.

The process of ordering the set M_n is simple and unambiguous for any number of system inputs.

Finally, for the MISO P2-TS system with the inputs z_1, ..., z_k let us introduce a *generator*

$$\mathbf{g}_0 = 1,$$
$$\mathbf{g}_{k+1}(z_1, \ldots, z_{k+1}) = \begin{bmatrix} \mathbf{g}_k(z_1, \ldots, z_k) \\ z_{k+1}\mathbf{g}_k(z_1, \ldots, z_k) \\ z_{k+1}^2\mathbf{g}_k(z_1, \ldots, z_k) \end{bmatrix} \in \mathbb{R}^{3^{k+1}}, \quad k = 0, 1, 2, \ldots, n-1, \tag{4.31}$$

which is of great importance for such systems.

4.4.1 Rule-Base Structure for Two-Inputs-One-Output P2-TS System

For $n = 2$ the rule-base structure is as follows

$$\left.\begin{aligned}
&R_1: \text{If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = q_{00},\\
&R_2: \text{If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = q_{10},\\
&R_3: \text{If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = q_{20},\\
&R_4: \text{If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = q_{01},\\
&R_5: \text{If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = q_{11},\\
&R_6: \text{If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = q_{21},\\
&R_7: \text{If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = q_{02},\\
&R_8: \text{If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = q_{12},\\
&R_9: \text{If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = q_{22},
\end{aligned}\right\} \tag{4.32}$$

and, in accordance with (4.31), the generator is given by

$$\mathbf{g}_2(z_1, z_2) = \begin{bmatrix} \mathbf{g}_1(z_1) \\ z_2\mathbf{g}_1(z_1) \\ z_2^2\mathbf{g}_1(z_1) \end{bmatrix} = \left[1, z_1, z_1^2, z_2, z_1 z_2, z_1^2 z_2, z_2^2, z_1 z_2^2, z_1^2 z_2^2\right]^T . \quad (4.33)$$

The crisp output of the system can be expressed as a scalar product of two vectors

$$S(z_1, z_2) = [N_1 N_2, Z_1 N_2, P_1 N_2, N_1 Z_2, Z_1 Z_2, P_1 Z_2, N_1 P_2, Z_1 P_2, P_1 P_2]\,\mathbf{q}, \quad (4.34)$$

where $N_k = N_k(z_k)$, $Z_k = Z_k(z_k)$ and $P_k = P_k(z_k)$ for $k = 1, 2$ are the membership functions defined by (4.24)-(4.26), and the vector $\mathbf{q}$ consists of the conclusions of the rules (4.32)

$$\mathbf{q} = [q_{00}, q_{10}, q_{20}, q_{01}, q_{11}, q_{21}, q_{02}, q_{12}, q_{22}]^T . \quad (4.35)$$

On the other hand, according to Remark 4.1 we have

$$S(\mathbf{z}) = \mathbf{g}_2^T(\mathbf{z})\,\boldsymbol{\theta}, \qquad \mathbf{z} \in D^2,$$

where $\boldsymbol{\theta} = [\theta_{00}, \theta_{10}, \theta_{20}, \theta_{01}, \theta_{11}, \theta_{21}, \theta_{02}, \theta_{12}, \theta_{22}]^T$ and $\mathbf{g}_2(\mathbf{z})$ is given by (4.33).

4.4.2 Rule-Base Structure for Three-Inputs-One-Output P2-TS System

For $n = 3$ the rule base consists of 27 rules. Its abbreviated structure is as follows

R_1 : If z_1 is N_1 and z_2 is N_2 and z_3 is N_3, then $S = q_{000}$,
R_2 : If z_1 is Z_1 and z_2 is N_2 and z_3 is N_3, then $S = q_{100}$,
R_3 : If z_1 is P_1 and z_2 is N_2 and z_3 is N_3, then $S = q_{200}$,
R_4 : If z_1 is N_1 and z_2 is Z_2 and z_3 is N_3, then $S = q_{010}$,
R_5 : If z_1 is Z_1 and z_2 is Z_2 and z_3 is N_3, then $S = q_{110}$,
R_6 : If z_1 is P_1 and z_2 is Z_2 and z_3 is N_3, then $S = q_{210}$,
R_7 : If z_1 is N_1 and z_2 is P_2 and z_3 is N_3, then $S = q_{020}$,
R_8 : If z_1 is Z_1 and z_2 is P_2 and z_3 is N_3, then $S = q_{120,}$
R_9 : If z_1 is P_1 and z_2 is P_2 and z_3 is N_3, then $S = q_{220}$,
R_{10} : If z_1 is N_1 and z_2 is N_2 and z_3 is Z_3, then $S = q_{001}$,

⋮

R_{27} : If z_1 is P_1 and z_2 is P_2 and z_3 is P_3, then $S = q_{222}$,

and the generator

$$\mathbf{g}_3(z_1,z_2,z_3) = \begin{bmatrix} \mathbf{g}_2(z_1,z_2) \\ z_3\mathbf{g}_2(z_1,z_2) \\ z_3^2\mathbf{g}_2(z_1,z_2) \end{bmatrix}$$
$$= [1, z_1, z_1^2, z_2, z_1z_2, z_1^2z_2, z_2^2, z_1z_2^2, z_1^2z_2^2,$$
$$z_3, z_1z_3, z_1^2z_3, z_2z_3, z_1z_2z_3, z_1^2z_2z_3,$$
$$z_2^2z_3, z_1z_2^2z_3, z_1^2z_2^2z_3, z_3^2, z_1z_3^2, z_1^2z_3^2, z_2z_3^2,$$
$$z_1z_2z_3^2, z_1^2z_2z_3^2, z_2^2z_3^2, z_1z_2^2z_3^2, z_1^2z_2^2z_3^2]^T. \tag{4.36}$$

The output of a three-input P2-TS system can be expressed and computed in the same way as for a two-input system - this is rather a simple task, but the equations are large for the number of inputs $n \geq 3$. For MISO P2-TS systems with $n \geq 3$ inputs we prefer to use the methods based on recurrence, which will be presented in the next sections.

4.5 The Fundamental Matrix for MISO P2-TS System

Similarly to SISO P2-TS systems, for the MISO P2-TS systems, the same equations as in (4.18) hold, namely

$$\mathbf{q} = \mathbf{R}^{-1}\boldsymbol{\Gamma}^T\boldsymbol{\theta} = \boldsymbol{\Omega}^T\boldsymbol{\theta}, \tag{4.37}$$

where

- the vector $\mathbf{q}$ contains the consequents of the "If-then" rules,
- $\boldsymbol{\theta}$ is the vector of parameters of the crisp function (4.1) to which the MISO P2-TS system is equivalent,
- the meaning of matrices $\mathbf{R}$ and $\boldsymbol{\Gamma}$ is the same as in Section 4.3, after some generalization for MISO systems,
- the matrix
$$\boldsymbol{\Omega} = \boldsymbol{\Gamma}\left(\mathbf{R}^{-1}\right)^T \tag{4.38}$$
we will call the *fundamental matrix* for P2-TS system.

Both $\boldsymbol{\Omega}$ and its inverse are important, since they enable one to establish an exact relationship between the consequents $\mathbf{q}$ of the "If-then" rules and the parameters $\boldsymbol{\theta}$ of the crisp function (4.1), to which the rule-based system is equivalent. Therefore our goal in this section is to give a procedure of how to compute the fundamental matrix and its inverse in the general case.

First we prove the following

Lemma 4.4. *For the MISO P2-TS system with the inputs* $[z_1,\ldots,z_k]^T \in D^k$*, we define the matrix*

$$\boldsymbol{\Gamma}_k = [\mathbf{g}_k(-\alpha_1,\ldots,-\alpha_k), \cdots, \ \mathbf{g}_k(\beta_1,\beta_2,\ldots,\beta_k)], \tag{4.39}$$

for $k = 1, 2, \ldots, n$, where the values of the generator $\mathbf{g}_k$ defined in (4.31) are computed for the subsequent elements of the totally ordered set M_k defined by (4.27). The matrix $\boldsymbol{\Gamma}_k$ can be computed recursively as follows

$$\boldsymbol{\Gamma}_0 = 1,$$

$$\boldsymbol{\Gamma}_{k+1} = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_{k+1} & \sigma_{k+1} & \beta_{k+1} \\ \alpha_{k+1}^2 & \sigma_{k+1}^2 & \beta_{k+1}^2 \end{bmatrix} \otimes \boldsymbol{\Gamma}_k, \tag{4.40}$$

for $k = 0, 1, 2, \ldots, n-1$.

Proof. From (4.39) by $\mathbf{g}_1(z_1) = \mathbf{g}(z)$ defined in (4.16) we obtain

$$\boldsymbol{\Gamma}_1 = [\mathbf{g}_1(-\alpha_1), \mathbf{g}_1(\sigma_1), \mathbf{g}_1(\beta_1)] = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_1 & \sigma_1 & \beta_1 \\ \alpha_1^2 & \sigma_1^2 & \beta_1^2 \end{bmatrix}.$$

On the other hand from (4.40) for $k = 0$ we have

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_1 \cdot 1 & \sigma_1 \cdot 1 & \beta_1 \cdot 1 \\ (-\alpha_1)^2 \cdot 1 & \sigma_1^2 \cdot 1 & \beta_1^2 \cdot 1 \end{bmatrix}.$$

Thus, for $k = 0$ the recurrence (4.40) is true.

For $k \geq 1$ let us rewrite the equation (4.39), taking into account (4.31)

$$\boldsymbol{\Gamma}_{k+1} = \left[\mathbf{g}_{k+1}(-\alpha_1, \ldots, -\alpha_k, -\alpha_{k+1}), \ \cdots, \ \mathbf{g}_{k+1}(\beta_1, \ldots, \beta_k, \beta_{k+1})\right]$$

$$= \left[\begin{bmatrix} \mathbf{g}_k(-\alpha_1, \ldots, -\alpha_k) \\ -\alpha_{k+1}\mathbf{g}_k(-\alpha_1, \ldots, -\alpha_k) \\ (-\alpha_{k+1})^2 \mathbf{g}_k(-\alpha_1, \ldots, -\alpha_k) \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{g}_k(\beta_1, \ldots, \beta_k) \\ \beta_{k+1}\mathbf{g}_k(\beta_1, \ldots, \beta_k) \\ \beta_{k+1}^2\mathbf{g}_k(\beta_1, \ldots, \beta_k) \end{bmatrix}\right]. \tag{4.41}$$

For example

$$\boldsymbol{\Gamma}_2 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3],$$

where the column vectors $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are

$$\mathbf{a}_1 = \begin{bmatrix} \mathbf{g}_1(-\alpha_1) \\ (-\alpha_2)\mathbf{g}_1(-\alpha_1) \\ \alpha_2^2\mathbf{g}_1(-\alpha_1) \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} \mathbf{g}_1(\sigma_1) \\ (-\alpha_2)\mathbf{g}_1(\sigma_1) \\ \alpha_2^2\mathbf{g}_1(\sigma_1) \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} \mathbf{g}_1(\beta_1) \\ (-\alpha_2)\mathbf{g}_1(\beta_1) \\ \alpha_2^2\mathbf{g}_1(\beta_1) \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} \mathbf{g}_1(-\alpha_1) \\ \sigma_2 \mathbf{g}_1(-\alpha_1) \\ \sigma_2^2 \mathbf{g}_1(-\alpha_1) \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} \mathbf{g}_1(\sigma_1) \\ \sigma_2 \mathbf{g}_1(\sigma_1) \\ \sigma_2^2 \mathbf{g}_1(\sigma_1) \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} \mathbf{g}_1(\beta_1) \\ \sigma_2 \mathbf{g}_1(\beta_1) \\ \sigma_2^2 \mathbf{g}_1(\beta_1) \end{bmatrix},$$

$$\mathbf{c}_1 = \begin{bmatrix} \mathbf{g}_1(-\alpha_1) \\ \beta_2 \mathbf{g}_1(-\alpha_1) \\ \beta_2^2 \mathbf{g}_1(-\alpha_1) \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} \mathbf{g}_1(\sigma_1) \\ \beta_2 \mathbf{g}_1(\sigma_1) \\ \beta_2^2 \mathbf{g}_1(\sigma_1) \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} \mathbf{g}_1(\beta_1) \\ \beta_2 \mathbf{g}_1(\beta_1) \\ \beta_2^2 \mathbf{g}_1(\beta_1) \end{bmatrix},$$

what results in

$$\boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -\alpha_1 & \sigma_1 & \beta_1 & -\alpha_1 & \sigma_1 & \beta_1 & -\alpha_1 & \sigma_1 & \beta_1 \\ \alpha_1^2 & \sigma_1^2 & \beta_1^2 & \alpha_1^2 & \sigma_1^2 & \beta_1^2 & \alpha_1^2 & \sigma_1^2 & \beta_1^2 \\ -\alpha_2 & -\alpha_2 & -\alpha_2 & \sigma_2 & \sigma_2 & \sigma_2 & \beta_2 & \beta_2 & \beta_2 \\ \alpha_1\alpha_2 & -\sigma_1\alpha_2 & -\alpha_2\beta_1 & -\alpha_1\sigma_2 & \sigma_1\sigma_2 & \beta_1\sigma_2 & -\alpha_1\beta_2 & \sigma_1\beta_2 & \beta_1\beta_2 \\ -\alpha_1^2\alpha_2 & -\sigma_1^2\alpha_2 & -\alpha_2\beta_1^2 & \alpha_1^2\sigma_2 & \sigma_1^2\sigma_2 & \beta_1^2\sigma_2 & \alpha_1^2\beta_2 & \sigma_1^2\beta_2 & \beta_1^2\beta_2 \\ \alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \sigma_2^2 & \sigma_2^2 & \sigma_2^2 & \beta_2^2 & \beta_2^2 & \beta_2^2 \\ -\alpha_1\alpha_2^2 & \sigma_1\alpha_2^2 & \alpha_2^2\beta_1 & -\alpha_1\sigma_2^2 & \sigma_1\sigma_2^2 & \beta_1\sigma_2^2 & -\alpha_1\beta_2^2 & \sigma_1\beta_2^2 & \beta_1\beta_2^2 \\ \alpha_1^2\alpha_2^2 & \sigma_1^2\alpha_2^2 & \alpha_2^2\beta_1^2 & \alpha_1^2\sigma_2^2 & \sigma_1^2\sigma_2^2 & \beta_1^2\sigma_2^2 & \alpha_1^2\beta_2^2 & \sigma_1^2\beta_2^2 & \beta_1^2\beta_2^2 \end{bmatrix},$$

or equivalently

$$\boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_2 & \sigma_2 & \beta_2 \\ (-\alpha_2)^2 & \sigma_2^2 & \beta_2^2 \end{bmatrix} \otimes \boldsymbol{\Gamma}_1.$$

One can observe that in the general case, because of the generator structure (4.31) and the sequence of the characteristic points from the set M_k, the structure of the matrix $\boldsymbol{\Gamma}_{k+1}$ is as follows

(a) The first 3^k columns of $\boldsymbol{\Gamma}_{k+1}$ constitute the submatrix $\begin{bmatrix} \boldsymbol{\Gamma}_k \\ -\alpha_{k+1}\boldsymbol{\Gamma}_k \\ (-\alpha_{k+1})^2 \boldsymbol{\Gamma}_k \end{bmatrix}$.

(b) The next 3^k columns of $\boldsymbol{\Gamma}_{k+1}$ constitute the submatrix $\begin{bmatrix} \boldsymbol{\Gamma}_k \\ \sigma_{k+1}\boldsymbol{\Gamma}_k \\ \sigma_{k+1}^2 \boldsymbol{\Gamma}_k \end{bmatrix}$.

(c) The last 3^k columns of $\boldsymbol{\Gamma}_{k+1}$ constitute the submatrix $\begin{bmatrix} \boldsymbol{\Gamma}_k \\ \beta_{k+1}\boldsymbol{\Gamma}_k \\ \beta_{k+1}^2 \boldsymbol{\Gamma}_k \end{bmatrix}$.

This finishes the proof of Lemma 4.4. □

Lemma 4.5. *For the MISO P2-TS system with the inputs* $z_1, \ldots, z_k$*, let us denote by* $\mathbf{s}_k$ *the vector of its outputs in the consecutive points of the ordered set* M_k *defined by (4.27), and the vector* $\mathbf{q}_k$ *containing the consequents of the rules*

$$\mathbf{s}_k = \begin{bmatrix} S\left(-\alpha_1, -\alpha_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\sigma_1, -\alpha_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\beta_1, -\alpha_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(-\alpha_1, \sigma_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\sigma_1, \sigma_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\beta_1, \sigma_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(-\alpha_1, \beta_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\sigma_1, \beta_2, -\alpha_3, \ldots, -\alpha_k\right) \\ S\left(\beta_1, \beta_2, -\alpha_3, \ldots, -\alpha_k\right) \\ \vdots \\ S\left(\beta_1, \beta_2, \beta_3, \ldots, \beta_k\right) \end{bmatrix}, \qquad \mathbf{q}_k = \begin{bmatrix} q_{000\ldots0} \\ q_{100\ldots0} \\ q_{200\ldots0} \\ q_{010\ldots0} \\ q_{110\ldots0} \\ q_{210\ldots0} \\ q_{020\ldots0} \\ q_{120\ldots0} \\ q_{220\ldots0} \\ \vdots \\ q_{222\ldots2} \end{bmatrix}. \tag{4.42}$$

There exists a matrix $\mathbf{R}_k \in \mathbb{R}^{3^k \times 3^k}$ *such that*

$$\mathbf{s}_k = \mathbf{R}_k \mathbf{q}_k, \tag{4.43}$$

and $\mathbf{R}_k$ *can be recursively computed as follows*

$$\begin{aligned} \mathbf{R}_0 &= 1, \\ \mathbf{R}_{k+1} &= \begin{bmatrix} 1 & 0 & 0 \\ \left(2-\lambda_{k+1}\right)/4 & \lambda_{k+1}/2 & \left(2-\lambda_{k+1}\right)/4 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbf{R}_k, \end{aligned} \tag{4.44}$$

for $k = 0, 1, 2, \ldots, n-1$*, where* $\lambda_k \in (0, 1]$ *is the parameter of membership functions (4.24)-(4.26).*

Proof. Let us consider the system with one input $z_1 \in [-\alpha_1, \beta_1]$. From the results in Section 4.3 we have

$$\begin{bmatrix} S\left(-\alpha_1\right) \\ S\left(\sigma_1\right) \\ S\left(\beta_1\right) \end{bmatrix} = \mathbf{R}_1 \mathbf{q}_1 = \begin{bmatrix} N_1\left(-\alpha_1\right) & Z_1\left(-\alpha_1\right) & P_1\left(-\alpha_1\right) \\ N_1\left(\sigma_1\right) & Z_1\left(\sigma_1\right) & P_1\left(\sigma_1\right) \\ N_1\left(\beta_1\right) & Z_1\left(\beta_1\right) & P_1\left(\beta_1\right) \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix}.$$

Thus,

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \left(2-\lambda_1\right)/4 & \lambda_1/2 & \left(2-\lambda_1\right)/4 \\ 0 & 0 & 1 \end{bmatrix} \otimes 1,$$

i.e. the result is the same as in (4.44).

For the system with two inputs the equality $\mathbf{s}_2 = \mathbf{R}_2\mathbf{q}_2$ holds, where

$$\mathbf{s}_2 = \begin{bmatrix} S(-\alpha_1,-\alpha_2) \\ S(\sigma_1,-\alpha_2) \\ S(\beta_1,-\alpha_2) \\ S(-\alpha_1,\sigma_2) \\ S(\sigma_1,\sigma_2) \\ S(\beta_1,\sigma_2) \\ S(-\alpha_1,\beta_2) \\ S(\sigma_1,\beta_2) \\ S(\beta_1,\beta_2) \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} \mathbf{R}_{2,11} & \mathbf{R}_{2,12} & \mathbf{R}_{2,13} \\ \mathbf{R}_{2,21} & \mathbf{R}_{2,22} & \mathbf{R}_{2,23} \\ \mathbf{R}_{2,31} & \mathbf{R}_{2,32} & \mathbf{R}_{2,33} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} q_{00} \\ q_{10} \\ q_{20} \\ q_{01} \\ q_{11} \\ q_{21} \\ q_{02} \\ q_{12} \\ q_{22} \end{bmatrix},$$

and

$$\mathbf{R}_{2,11} = \begin{bmatrix} N_1(-\alpha_1)N_2(-\alpha_2) & Z_1(-\alpha_1)N_2(-\alpha_2) & P_1(-\alpha_1)N_2(-\alpha_2) \\ N_1(\sigma_1)N_2(-\alpha_2) & Z_1(\sigma_1)N_2(-\alpha_2) & P_1(\sigma_1)N_2(-\alpha_2) \\ N_1(\beta_1)N_2(-\alpha_2) & Z_1(\beta_1)N_2(-\alpha_2) & P_1(\beta_1)N_2(-\alpha_2) \end{bmatrix}$$
$$= \mathbf{R}_1 \cdot N_2(-\alpha_2),$$

$$\mathbf{R}_{2,12} = \begin{bmatrix} N_1(-\alpha_1)Z_2(-\alpha_2) & Z_1(-\alpha_1)Z_2(-\alpha_2) & P_1(-\alpha_1)Z_2(-\alpha_2) \\ N_1(\sigma_1)Z_2(-\alpha_2) & Z_1(\sigma_1)Z_2(-\alpha_2) & P_1(\sigma_1)Z_2(-\alpha_2) \\ N_1(\beta_1)Z_2(-\alpha_2) & Z_1(\beta_1)Z_2(-\alpha_2) & P_1(\beta_1)Z_2(-\alpha_2) \end{bmatrix}$$
$$= \mathbf{R}_1 \cdot Z_2(-\alpha_2),$$

$$\mathbf{R}_{2,13} = \begin{bmatrix} N_1(-\alpha_1)P_2(-\alpha_2) & Z_1(-\alpha_1)P_2(-\alpha_2) & P_1(-\alpha_1)P_2(-\alpha_2) \\ N_1(\sigma_1)P_2(-\alpha_2) & Z_1(\sigma_1)P_2(-\alpha_2) & P_1(\sigma_1)P_2(-\alpha_2) \\ N_1(\beta_1)P_2(-\alpha_2) & Z_1(\beta_1)P_2(-\alpha_2) & P_1(\beta_1)P_2(-\alpha_2) \end{bmatrix}$$
$$= \mathbf{R}_1 \cdot P_2(-\alpha_2),$$

$$\mathbf{R}_{2,21} = \begin{bmatrix} N_1(-\alpha_1)N_2(\sigma_2) & Z_1(-\alpha_1)N_2(\sigma_2) & P_1(-\alpha_1)N_2(\sigma_2) \\ N_1(\sigma_1)N_2(\sigma_2) & Z_1(\sigma_1)N_2(\sigma_2) & P_1(\sigma_1)N_2(\sigma_2) \\ N_1(\beta_1)N_2(\sigma_2) & Z_1(\beta_1)N_2(\sigma_2) & P_1(\beta_1)N_2(\sigma_2) \end{bmatrix}$$
$$= \mathbf{R}_1 \cdot N_2(\sigma_2),$$

$$\mathbf{R}_{2,22} = \begin{bmatrix} N_1(-\alpha_1)Z_2(\sigma_2) & Z_1(-\alpha_1)Z_2(\sigma_2) & P_1(-\alpha_1)Z_2(\sigma_2) \\ N_1(\sigma_1)Z_2(\sigma_2) & Z_1(\sigma_1)Z_2(\sigma_2) & P_1(\sigma_1)Z_2(\sigma_2) \\ N_1(\beta_1)Z_2(\sigma_2) & Z_1(\beta_1)Z_2(\sigma_2) & P_1(\beta_1)Z_2(\sigma_2) \end{bmatrix}$$
$$= \mathbf{R}_1 \cdot Z_2(\sigma_2),$$

$$\mathbf{R}_{2,23} = \begin{bmatrix} N_1(-\alpha_1)\,P_2(\sigma_2) & Z_1(-\alpha_1)\,P_2(\sigma_2) & P_1(-\alpha_1)\,P_2(\sigma_2) \\ N_1(\sigma_1)\,P_2(\sigma_2) & Z_1(\sigma_1)\,P_2(\sigma_2) & P_1(\sigma_1)\,P_2(\sigma_2) \\ N_1(\beta_1)\,P_2(\sigma_2) & Z_1(\beta_1)\,P_2(\sigma_2) & P_1(\beta_1)\,P_2(\sigma_2) \end{bmatrix}$$

$$= \mathbf{R}_1 \cdot P_2(\sigma_2)\,,$$

$$\mathbf{R}_{2,31} = \begin{bmatrix} N_1(-\alpha_1)\,N_2(\beta_2) & Z_1(-\alpha_1)\,N_2(\beta_2) & P_1(-\alpha_1)\,N_2(\beta_2) \\ N_1(\sigma_1)\,N_2(\beta_2) & Z_1(\sigma_1)\,N_2(\beta_2) & P_1(\sigma_1)\,N_2(\beta_2) \\ N_1(\beta_1)\,N_2(\beta_2) & Z_1(\beta_1)\,N_2(\beta_2) & P_1(\beta_1)\,N_2(\beta_2) \end{bmatrix}$$

$$= \mathbf{R}_1 \cdot N_2(\beta_2)\,,$$

$$\mathbf{R}_{2,32} = \begin{bmatrix} N_1(-\alpha_1)\,Z_2(\beta_2) & Z_1(-\alpha_1)\,Z_2(\beta_2) & P_1(-\alpha_1)\,Z_2(\beta_2) \\ N_1(\sigma_1)\,Z_2(\beta_2) & Z_1(\sigma_1)\,Z_2(\beta_2) & P_1(\sigma_1)\,Z_2(\beta_2) \\ N_1(\beta_1)\,Z_2(\beta_2) & Z_1(\beta_1)\,Z_2(\beta_2) & P_1(\beta_1)\,Z_2(\beta_2) \end{bmatrix}$$

$$= \mathbf{R}_1 \cdot Z_2(\beta_2)\,,$$

and

$$\mathbf{R}_{2,33} = \begin{bmatrix} N_1(-\alpha_1)\,P_2(\beta_2) & Z_1(-\alpha_1)\,P_2(\beta_2) & P_1(-\alpha_1)\,P_2(\beta_2) \\ N_1(\sigma_1)\,P_2(\beta_2) & Z_1(\sigma_1)\,P_2(\beta_2) & P_1(\sigma_1)\,P_2(\beta_2) \\ N_1(\beta_1)\,P_2(\beta_2) & Z_1(\beta_1)\,P_2(\beta_2) & P_1(\beta_1)\,P_2(\beta_2) \end{bmatrix}$$

$$= \mathbf{R}_1 \cdot P_2(\beta_2)\,.$$

In a more compact form we can write

$$\mathbf{R}_2 = \begin{bmatrix} N_2(-\alpha_2) & Z_2(-\alpha_2) & P_2(-\alpha_2) \\ N_2(\sigma_2) & Z_2(\sigma_2) & P_2(\sigma_2) \\ N_2(\beta_2) & Z_2(\beta_2) & P_2(\beta_2) \end{bmatrix} \otimes \mathbf{R}_1.$$

The same procedure must be applied for the construction of the matrix $\mathbf{R}_k$ in (4.43), remembering the order of the set M_k. Finally, we conclude that the following recurrence

$$\mathbf{R}_{k+1} = \begin{bmatrix} N_{k+1}(-\alpha_{k+1}) & Z_{k+1}(-\alpha_{k+1}) & P_{k+1}(-\alpha_{k+1}) \\ N_{k+1}(\sigma_{k+1}) & Z_{k+1}(\sigma_{k+1}) & P_{k+1}(\sigma_{k+1}) \\ N_{k+1}(\beta_{k+1}) & Z_{k+1}(\beta_{k+1}) & P_{k+1}(\beta_{k+1}) \end{bmatrix} \otimes \mathbf{R}_k$$

holds for every natural k. After computing the membership degrees according to (4.24)-(4.26) we obtain the recursive formula (4.44). This ends the proof of Lemma 4.5. □

Observe that for $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ we have

$$\mathbf{R}_{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbf{R}_k, \qquad k = 0, 1, 2, \ldots, n-1.$$

For example

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \otimes \mathbf{R}_1$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/2 & 0 & 0 & 1/4 & 0 & 0 \\ 1/16 & 1/8 & 1/16 & 1/8 & 1/4 & 1/8 & 1/16 & 1/8 & 1/16 \\ 0 & 0 & 1/4 & 0 & 0 & 1/2 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix $\mathbf{R}_3$ contains $3^3 \times 3^3 = 729$ elements and because of its large size it will not be presented here.

Now we prove the following

Theorem 4.6. *The fundamental matrix of the MISO P2-TS system with the inputs $[z_1, \ldots, z_k]^T \in D^k$ and the membership functions of fuzzy sets for the inputs defined by (4.24)-(4.26), can be computed recursively as follows*

$$\mathbf{\Omega}_0 = 1,$$
$$\mathbf{\Omega}_k = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_k & \sigma_k & \beta_k \\ \alpha_k^2 & \frac{1}{2}\left(\alpha_k^2 + \beta_k^2 - \frac{(\alpha_k + \beta_k)^2}{\lambda_k}\right) & \beta_k^2 \end{bmatrix} \otimes \mathbf{\Omega}_{k-1}, \qquad (4.45)$$

for $k = 1, \ldots, n$, where $\lambda_k \in (0, 1]$ is the parameter of membership functions.

Proof. From (4.38) for MISO P2-TS system with the inputs $[z_1, \ldots, z_k]^T \in D^k$ we have

$$\mathbf{\Omega}_k = \boldsymbol{\Gamma}_k \left(\mathbf{R}_k^{-1}\right)^T. \qquad (4.46)$$

Next we apply Lemma 4.4 and Lemma 4.5

$$\boldsymbol{\Omega}_k = (\mathbf{A}_k \otimes \boldsymbol{\Gamma}_{k-1}) \left((\mathbf{B}_k \otimes \mathbf{R}_{k-1})^{-1} \right)^T ,$$

where according to (4.40) and (4.44) the matrices $\mathbf{A}_k$ and $\mathbf{B}_k$ are

$$\mathbf{A}_k = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_k & \sigma_k & \beta_k \\ \alpha_k^2 & \sigma_k^2 & \beta_k^2 \end{bmatrix}, \qquad \mathbf{B}_k = \begin{bmatrix} 1 & 0 & 0 \\ (2-\lambda_k)/4 & \lambda_k/2 & (2-\lambda_k)/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

From (A.4) and (A.5) given in Appendix A we obtain

$$\left((\mathbf{B}_k \otimes \mathbf{R}_{k-1})^{-1} \right)^T = \left(\mathbf{B}_k^{-1} \otimes \mathbf{R}_{k-1}^{-1} \right)^T = \left(\mathbf{B}_k^{-1} \right)^T \otimes \left(\mathbf{R}_{k-1}^{-1} \right)^T .$$

Thus,

$$\boldsymbol{\Omega}_k = (\mathbf{A}_k \otimes \boldsymbol{\Gamma}_{k-1}) \left(\mathbf{B}_k^{-1} \right)^T \otimes \left(\mathbf{R}_{k-1}^{-1} \right)^T = \left(\mathbf{A}_k \left(\mathbf{B}_k^{-1} \right)^T \right) \otimes \left(\boldsymbol{\Gamma}_{k-1} \left(\mathbf{R}_{k-1}^{-1} \right)^T \right).$$

One can check that

$$\mathbf{A}_k \left(\mathbf{B}_k^{-1} \right)^T = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_k & \sigma_k & \beta_k \\ \alpha_k^2 & \frac{1}{2}\left(\alpha_k^2 + \beta_k^2 - \frac{(\alpha_k + \beta_k)^2}{\lambda_k} \right) & \beta_k^2 \end{bmatrix}.$$

Now we apply the Kronecker product property (A.3) from Appendix A:

$$\boldsymbol{\Omega}_k = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_k & \sigma_k & \beta_k \\ \alpha_k^2 & \frac{1}{2}\left(\alpha_k^2 + \beta_k^2 - \frac{(\alpha_k + \beta_k)^2}{\lambda_k} \right) & \beta_k^2 \end{bmatrix} \otimes \left(\boldsymbol{\Gamma}_{k-1} \left(\mathbf{R}_{k-1}^{-1} \right)^T \right). \tag{4.47}$$

According to (4.46) the equality $\boldsymbol{\Gamma}_{k-1} \left(\mathbf{R}_{k-1}^{-1} \right)^T = \boldsymbol{\Omega}_{k-1}$ holds. Thus, the equation (4.47) is the same as (4.45) and this finishes the proof of Theorem 4.6. □

For $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ we obtain a much simpler recurrence

$$\begin{aligned} \boldsymbol{\Omega}_0 &= 1, \\ \boldsymbol{\Omega}_k &= \begin{bmatrix} 1 & 1 & 1 \\ -\alpha_k & \sigma_k & \beta_k \\ \alpha_k^2 & -\beta_k \alpha_k & \beta_k^2 \end{bmatrix} \otimes \boldsymbol{\Omega}_{k-1}, \qquad k = 1, \ldots, n, \end{aligned} \tag{4.48}$$

which we prefer to use in practice.

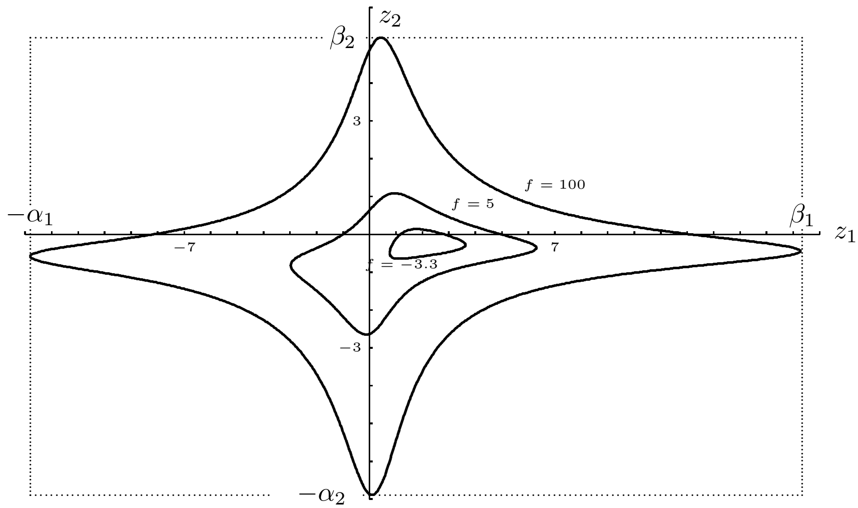


Fig. 4.6 Contour lines of the function (4.49)

Example 4.7. Our goal is to obtain the fuzzy rules for P2-TS system which exactly model the following nonlinear function

$$f(z_1, z_2) = 2z_1^2 z_2^2 + 2z_1^2 z_2 - z_1 z_2^2 + z_1^2 - 5z_1 z_2 + 3z_2^2 - 4z_1 + 6z_2 \tag{4.49}$$

for $(z_1, z_2) \in D^2 = [-12.8040, 16.2860] \times [-6.8844, 5.2029]$. Three contour lines of the above function as the set of points

$$\bigcup_{c \in \{-3.3,\ 5,\ 100\}} \left\{ (z_1, z_2) \in D^2 : f(z_1, z_2) = c \right\}$$

are shown in Fig. 4.6. We assume that the first input z_1 of the TS system has assigned the fuzzy sets N_1, Z_1 and P_1, whereas the second one - the fuzzy sets N_2, Z_2 and P_2. The membership functions are defined by (4.24)-(4.26), with the parameters $\lambda_1 = \lambda_2 = 1$, and boundaries of the intervals $\alpha_1 = 12.8040$, $\beta_1 = 16.2860$, $\alpha_2 = 6.8844$, and $\beta_2 = 5.2029$. The cores of fuzzy sets Z_1 and Z_2 are $\sigma_1 = 1.7410$ and $\sigma_2 = -0.8407$, respectively. Observe that the function (4.49) can be written equivalently as

$$f(z_1, z_2) = \boldsymbol{\theta}^T \mathbf{g}_2(z_1, z_2) = [0, -4, 1, 6, -5, 2, 3, -1, 2]\, \mathbf{g}_2(z_1, z_2),$$

where the generator $\mathbf{g}_2(z_1, z_2)$ is given by (4.33). Taking from (4.22) the fundamental matrix $\mathbf{\Omega}_1$ for one-input P2-TS system, we compute the fundamental matrix $\mathbf{\Omega}_2$ for two-inputs P2-TS system, according to Theorem 4.6 (for $\lambda_1 = \lambda_2 = 1$). After computations we obtain

$$\mathbf{\Omega}_2^T = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2 & -\alpha_2 & \alpha_1\alpha_2 & -\alpha_1^2\alpha_2 & \alpha_2^2 & -\alpha_1\alpha_2^2 & \alpha_1^2\alpha_2^2 \\ 1 & \sigma_1 & -\alpha_1\beta_1 & -\alpha_2 & -\sigma_1\alpha_2 & \alpha_1\alpha_2\beta_1 & \alpha_2^2 & \sigma_1\alpha_2^2 & -\alpha_1\alpha_2^2\beta_1 \\ 1 & \beta_1 & \beta_1^2 & -\alpha_2 & -\alpha_2\beta_1 & -\alpha_2\beta_1^2 & \alpha_2^2 & \alpha_2^2\beta_1 & \alpha_2^2\beta_1^2 \\ 1 & -\alpha_1 & \alpha_1^2 & \sigma_2 & -\alpha_1\sigma_2 & \alpha_1^2\sigma_2 & -\alpha_2\beta_2 & \alpha_1\alpha_2\beta_2 & -\alpha_1^2\alpha_2\beta_2 \\ 1 & \sigma_1 & -\alpha_1\beta_1 & \sigma_2 & \sigma_1\sigma_2 & -\alpha_1\beta_1\sigma_2 & -\alpha_2\beta_2 & -\sigma_1\alpha_2\beta_2 & \alpha_1\alpha_2\beta_1\beta_2 \\ 1 & \beta_1 & \beta_1^2 & \sigma_2 & \beta_1\sigma_2 & \beta_1^2\sigma_2 & -\alpha_2\beta_2 & -\alpha_2\beta_1\beta_2 & -\alpha_2\beta_1^2\beta_2 \\ 1 & -\alpha_1 & \alpha_1^2 & \beta_2 & -\alpha_1\beta_2 & \alpha_1^2\beta_2 & \beta_2^2 & -\alpha_1\beta_2^2 & \alpha_1^2\beta_2^2 \\ 1 & \sigma_1 & -\alpha_1\beta_1 & \beta_2 & \sigma_1\beta_2 & -\alpha_1\beta_1\beta_2 & \beta_2^2 & \sigma_1\beta_2^2 & -\alpha_1\beta_1\beta_2^2 \\ 1 & \beta_1 & \beta_1^2 & \beta_2 & \beta_1\beta_2 & \beta_1^2\beta_2 & \beta_2^2 & \beta_1\beta_2^2 & \beta_1^2\beta_2^2 \end{bmatrix}, \tag{4.50}$$

and numerically

$$\mathbf{\Omega}_2^T = \begin{bmatrix} 1 & -12.80 & 163.94 & -6.884 & 88.15 & -1128.6 & 47.39 & -606.85 & 7770.0 \\ 1 & 1.74 & -208.53 & -6.884 & -11.99 & 1435.6 & 47.39 & 82.515 & -9883.1 \\ 1 & 16.29 & 265.23 & -6.884 & -112.1 & -1826.0 & 47.39 & 771.87 & 12571. \\ 1 & -12.80 & 163.94 & -0.841 & 10.76 & -137.83 & -35.82 & 458.62 & -5872.2 \\ 1 & 1.74 & -208.53 & -0.841 & -1.464 & 175.31 & -35.82 & -62.361 & 7469.2 \\ 1 & 16.29 & 265.23 & -0.841 & -13.69 & -222.98 & -35.82 & -583.35 & -9500.4 \\ 1 & -12.80 & 163.94 & 5.203 & -66.62 & 852.98 & 27.07 & -346.61 & 4437.9 \\ 1 & 1.74 & -208.53 & 5.203 & 9.058 & -1084.9 & 27.07 & 47.129 & -5644.8 \\ 1 & 16.29 & 265.23 & 5.203 & 84.73 & 1380.0 & 27.07 & 440.86 & 7179.9 \end{bmatrix}.$$

For the P2-TS systems we have

$$S_2 = \mathbf{g}_2^T(z_1, z_2)\left(\mathbf{\Omega}_2^T\right)^{-1}\mathbf{q}_2 = f(z_1, z_2) = \boldsymbol{\theta}^T\mathbf{g}_2(z_1, z_2).$$

Thus, the vector of conclusions of the fuzzy rules is given by

$$\begin{aligned} \mathbf{q}_2 &= \mathbf{\Omega}_2^T\boldsymbol{\theta} \\ &= [13764.9420,\ -17032.2043,\ 21579.2316,\ -12429.8971,\ 15030.6206, \\ &\quad -18707.3075,\ 11589.1320,\ -13655.0266,\ 16567.7977]^T. \end{aligned}$$

Finally, the system of fuzzy rules for the 2-iputs-1-output P2-TS system is as follows

$$\left.\begin{array}{l}
R_1: \text{ If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = 13764.9420,\\
R_2: \text{ If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = -17032.2043,\\
R_3: \text{ If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } N_2, \text{ then } S = 21579.2316,\\
R_4: \text{ If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = -12429.8971,\\
R_5: \text{ If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = 15030.6206,\\
R_6: \text{ If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } Z_2, \text{ then } S = -18707.3075,\\
R_7: \text{ If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = 11589.1320,\\
R_8: \text{ If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = -13655.0266,\\
R_9: \text{ If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } P_2, \text{ then } S = 16567.7977.
\end{array}\right\} \quad (4.51)$$

One can check that the above rule-based system exactly models the function (4.49), since the expression $\mathbf{g}_2^T(\mathbf{z})\left(\boldsymbol{\Omega}_2^T\right)^{-1}\mathbf{q}_2$ results in the same polynomial as in (4.49) for all points $\mathbf{z}$ from the rectangle D^2.

Example 4.8. Let us consider the system of fuzzy rules (4.51) for 2-inputs-one-output P2-TS system from Example 4.7. Assume the same data $\alpha_1 = 12.8040$, $\beta_1 = 16.2860$, $\alpha_2 = 6.8844$, $\beta_2 = 5.2029$, $\lambda_1 = \lambda_2 = 1$ and the consequents of the rules (4.51): $q_{00} = 13764.9420$, $q_{10} = -17032.2043$, $q_{20} = 21579.2316$, $q_{01} = -12429.8971$, $q_{11} = 15030.6206$, $q_{21} = -18707.3075$, $q_{02} = 11589.1320$, $q_{12} = -13655.0266$ and $q_{22} = 16567.7977$. From (4.24)-(4.26) and (4.34)-(4.35) we obtain the system output $S = S_2(z_1, z_2 \mid q_{00}, \ldots, q_{22})$ which can be expressed by

$$\begin{aligned}
S = {} & N_2(z_2)\left(N_1(z_1)\, q_{00} + Z_1(z_1)\, q_{10} + P_1(z_1)\, q_{20}\right)\\
& + Z_2(z_2)\left(N_1(z_1)\, q_{01} + Z_1(z_1)\, q_{11} + P_1(z_1)\, q_{21}\right)\\
& + P_2(z_2)\left(N_1(z_1)\, q_{02} + Z_1(z_1)\, q_{12} + P_1(z_1)\, q_{22}\right).
\end{aligned}$$

The above expression gives the same function as in (4.49) exact to numerical errors.

4.6 Recursion in MISO P2-TS Systems

In order to obtain the crisp output of a MISO P2-TS system, we need to obtain an inverse of the fundamental matrix. Our first goal is to give a procedure for computing this inverse. We prove the following

Theorem 4.9. *Let $\boldsymbol{\Omega}_0 = 1$ and $\boldsymbol{\Omega}_n$ be the fundamental matrix of the P2-TS system with n inputs, ($n \geq 1$). The inverse of the fundamental matrix can be computed as follows*

$$\boldsymbol{\Omega}_n^{-1} = \frac{1}{L_n^2}\begin{bmatrix} \beta_n\left(L_n - \alpha_n\lambda_n\right) & -L_n + \left(\alpha_n - \beta_n\right)\lambda_n & \lambda_n \\ 2\alpha_n\beta_n\lambda_n & 4\sigma_n\lambda_n & -2\lambda_n \\ \alpha_n\left(L_n - \beta_n\lambda_n\right) & L_n + \left(\alpha_n - \beta_n\right)\lambda_n & \lambda_n \end{bmatrix} \otimes \boldsymbol{\Omega}_{n-1}^{-1}, \tag{4.52}$$

where $L_n = \alpha_n + \beta_n$.

Proof. Taking into account Theorem 4.6, the Kronecker product property (A.4) from Appendix A, the equalities $\sigma_n = \left(-\alpha_n + \beta_n\right)/2$ and $L_n = \alpha_n + \beta_n$, we have

$$\boldsymbol{\Omega}_n^{-1} = \left(\begin{bmatrix} 1 & 1 & 1 \\ -\alpha_n & \sigma_n & \beta_n \\ \left(-\alpha_n\right)^2 & \frac{1}{2}\left(\alpha_n^2 + \beta_n^2 - \frac{\left(\alpha_n + \beta_n\right)^2}{\lambda_n}\right) & \beta_n^2 \end{bmatrix} \otimes \boldsymbol{\Omega}_{n-1}\right)^{-1}.$$

Thus,

$$\boldsymbol{\Omega}_n^{-1} = \frac{1}{L_n^2}\begin{bmatrix} \beta_n\left(L_n - \alpha_n\lambda_n\right) & -L_n + \left(\alpha_n - \beta_n\right)\lambda_n & \lambda_n \\ 2\alpha_n\beta_n\lambda_n & 2\lambda_n\left(\beta_n - \alpha_n\right) & -2\lambda_n \\ \alpha_n\left(L_n - \beta_n\lambda_n\right) & L_n + \left(\alpha_n - \beta_n\right)\lambda_n & \lambda_n \end{bmatrix} \otimes \boldsymbol{\Omega}_{n-1}^{-1}.$$

The last matrix is the same as in (4.52), because $2\lambda_n\left(\beta_n - \alpha_n\right) = 4\sigma_n\lambda_n$. This finishes the proof of Theorem 4.9. □

For $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ we obtain a much simpler recurrence

$$\boldsymbol{\Omega}_0 = 1,$$
$$\boldsymbol{\Omega}_n^{-1} = \frac{1}{L_n^2}\begin{bmatrix} \beta_n^2 & -2\beta_n & 1 \\ 2\alpha_n\beta_n & 4\sigma_n & -2 \\ \alpha_n^2 & 2\alpha_n & 1 \end{bmatrix} \otimes \boldsymbol{\Omega}_{n-1}^{-1}, \qquad n = 1, 2, \ldots, \tag{4.53}$$

which can be used in practice.

4.6.1 Rule-Base Decomposition

Without loss of generality we will consider a zero-order TS system with one output. The inputs are components of the vector $\mathbf{z} = \left[z_1, \ldots, z_n\right]^T \in D^n$, $(n = 2, 3, \ldots)$. We assume that three polynomial membership functions $N_k\left(z_k\right)$, $Z_k\left(z_k\right)$ and $P_k\left(z_k\right)$ defined by (4.24)-(4.26), are assigned for every input z_k, $(k = 1, \ldots, n)$.

The complete and noncontradictory rule-base is defined by the following 3^n "If-then" fuzzy rules:

$$\left.\begin{array}{l}
R_1 : \text{If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } N_2 \text{ and } \ldots \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{0,0,\ldots,0,0}, \\
R_2 : \text{If } z_1 \text{ is } Z_1 \text{ and } z_2 \text{ is } N_2 \text{ and } \ldots \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{1,0,\ldots,0,0}, \\
R_3 : \text{If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } N_2 \text{ and } \ldots \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{2,0,\ldots,0,0}, \\
R_4 : \text{If } z_1 \text{ is } N_1 \text{ and } z_2 \text{ is } Z_2 \text{ and } \ldots \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{0,1,\ldots,0,0}, \\
\vdots \\
R_{3^n} : \text{If } z_1 \text{ is } P_1 \text{ and } z_2 \text{ is } P_2 \text{ and } \ldots \text{ and } z_n \text{ is } P_n, \text{ then } S = q_{2,2,\ldots,2,2}.
\end{array}\right\} \tag{4.54}$$

One can decompose this system into the following three subsystems:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
R_1 : \quad \text{If } \mathcal{P}_1 \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{0,\ldots,0,0}, \\
\vdots \\
R_{3^{n-1}} : \quad \text{If } \mathcal{P}_{3^{n-1}} \text{ and } z_n \text{ is } N_n, \text{ then } S = q_{2,\ldots,2,0},
\end{array}\right\} \\
\left.\begin{array}{l}
R_{3^{n-1}+1} : \text{If } \mathcal{P}_1 \text{ and } z_n \text{ is } Z_n, \text{ then } S = q_{0,\ldots,0,1}, \\
\vdots \\
R_{2\cdot 3^{n-1}} : \text{If } \mathcal{P}_{3^{n-1}} \text{ and } z_n \text{ is } Z_n, \text{ then } S = q_{2,\ldots,2,1},
\end{array}\right\} \\
\left.\begin{array}{l}
R_{2\cdot 3^{n-1}+1} : \text{If } \mathcal{P}_1 \text{ and } z_n \text{ is } P_n, \text{ then } S = q_{0,\ldots,0,2}, \\
\vdots \\
R_{3^n} : \text{If } \mathcal{P}_{3^{n-1}} \text{ and } z_n \text{ is } P_n, \text{ then } S = q_{2,\ldots,2,2},
\end{array}\right\}
\end{array}\right\} \tag{4.55}$$

where $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3^{n-1}}$ are "If" parts in the system with $(n-1)$ inputs $[z_1, \ldots, z_{n-1}]^T \in D^{n-1}$, $(n = 2, 3, \ldots)$:

$$\left.\begin{array}{l}
R'_1 : \underbrace{\text{If } z_1 \text{ is } N_1 \text{ and } \ldots \text{ and } z_{n-1} \text{ is } N_{n-1}}_{\mathcal{P}_1}, \text{ then } S = q_{0,0,\ldots,0}, \\
\vdots \\
R'_{3^{n-1}} : \underbrace{\text{If } z_1 \text{ is } P_1 \text{ and } \ldots \text{ and } z_{n-1} \text{ is } P_{n-1}}_{\mathcal{P}_{3^{n-1}}}, \text{ then } S = q_{2,2,\ldots,2}.
\end{array}\right\} \tag{4.56}$$

The decomposition (4.55) of the original P2-TS system (4.54) will be used for proving the most important recurrence for such systems.

4.6.2 Crisp Output Calculation for P2-TS System Using Recursion

Now we prove the following

Theorem 4.10. *(**on recursion in systems with membership functions as second degree polynomials**) The recursive formula that enables one to compute the crisp output for any P2-TS system with inputs $z_1 \in [-\alpha_1, \beta_1]$, ..., $z_n \in [-\alpha_n, \beta_n]$, for $n = 2, 3, \ldots$, is as follows*

$$\begin{aligned} S_n(\mathbf{z} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}) &= N_n(z_n) S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}) \\ &+ Z_n(z_n) S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}) \\ &+ P_n(z_n) S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}), \end{aligned} \tag{4.57}$$

where

- $\mathbf{z}_{n-1} = [z_1, \ldots, z_{n-1}]^T \in D^{n-1}$ *and* $\mathbf{z} = \begin{bmatrix} \mathbf{z}_{n-1} \\ z_n \end{bmatrix} \in D^n$ *are the input vectors,*
- $S_n(\mathbf{z} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2})$ *is the crisp output of the system (4.54) with input vector* $\mathbf{z} \in D^n$ *and the consequents of the rules constituting the vector* $[q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}]^T$,
- $N_n(z_n)$, $Z_n(z_n)$ *and* $P_n(z_n)$ *are membership functions for the input* $z_n \in [-\alpha_n, \beta_n]$ *defined by (4.24)-(4.26),*
- $S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0})$ *is the crisp output of the first subsystem in (4.55) with input vector* $\mathbf{z}_{n-1} \in D^{n-1}$ *and the consequents of the rules constituting the vector* $[q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}]^T$,
- $S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1})$ *is the crisp output of the second subsystem in (4.55) with input vector* $\mathbf{z}_{n-1} \in D^{n-1}$ *and the consequents of the rules constituting the vector* $[q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}]^T$,
- $S_{n-1}(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2})$ *is the crisp output of the third subsystem in (4.55) with input vector* $\mathbf{z}_{n-1} \in D^{n-1}$ *and the consequents of the rules constituting the vector* $[q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}]^T$.

Proof. We will use notation of Theorem 4.10. The rules for SISO P2-TS system are given by (4.12). According to (4.13) the system output is as follows

$$\begin{aligned} S_1(z_1 \mid a, b, c) &= N_1(z_1) a + Z_1(z_1) b + P_1(z_1) c \\ &= [N_1(z_1), Z_1(z_1), P_1(z_1)] [a, b, c]^T \end{aligned}$$

First we prove theorem for $n = 2$. The rules for P2-TS system are given by (4.32), where the consequents of the fuzzy rules constitute the vector $\mathbf{q} = [q_{00}, q_{10}, q_{20}, q_{01}, q_{11}, q_{21}, q_{02}, q_{12}, q_{22}]^T$. According to (4.57) and (4.35) the system output is as follows

$$\begin{aligned} S_2(z_1, z_2 \mid q_{00}, \ldots, q_{22}) &= N_2(z_2) S_1(z_1 \mid q_{00}, q_{10}, q_{20}) \\ &+ Z_2(z_2) S_1(z_1 \mid q_{01}, q_{11}, q_{21}) \\ &+ P_2(z_2) S_1(z_1 \mid q_{02}, q_{12}, q_{22}) \\ &= N_2(z_2)[N_1(z_1), Z_1(z_1), P_1(z_1)][q_{00}, q_{10}, q_{20}]^T \\ &+ Z_2(z_2)[N_1(z_1), Z_1(z_1), P_1(z_1)][q_{01}, q_{11}, q_{21}]^T \\ &+ P_2(z_2)[N_1(z_1), Z_1(z_1), P_1(z_1)][q_{02}, q_{12}, q_{22}]^T . \end{aligned}$$

The last formula gives the same result as the scalar product (4.34). This implies that Theorem 4.10 is true for P2-TS systems with $n = 2$ inputs.

The output of the MISO P2-TS system defined by the rules (4.54), can be expressed as follows

$$S_n = S_n(\mathbf{z} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}) = \mathbf{q}_n^T \mathbf{\Omega}_n^{-1} \mathbf{g}_n(\mathbf{z}), \tag{4.58}$$

where $\mathbf{q}_n^T = [q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}]$ is the vector of consequents of the rules (4.54), $\mathbf{\Omega}_n$ is the fundamental matrix, and $\mathbf{g}_n(\mathbf{z})$ is the generator of the system with n-inputs $z_1, \ldots, z_n$. Taking into account (4.58), the Kronecker product properties (A.4) and (A.2c) from Appendix A, the equalities $\sigma_n = (-\alpha_n + \beta_n)/2$ and $L_n = \alpha_n + \beta_n$, we obtain

$$\begin{aligned} S_n &= \mathbf{q}_n^T \left(\begin{bmatrix} 1 & 1 & 1 \\ -\alpha_n & \sigma_n & \beta_n \\ (-\alpha_n)^2 & \frac{1}{2}\left(\alpha_n^2 + \beta_n^2 - \frac{(\alpha_n + \beta_n)^2}{\lambda_n}\right) & \beta_n^2 \end{bmatrix} \otimes \mathbf{\Omega}_{n-1} \right)^{-1} \mathbf{g}_n(\mathbf{z}) \\ &= \mathbf{q}_n^T \left(\begin{bmatrix} 1 & 1 & 1 \\ -\alpha_n & \sigma_n & \beta_n \\ (-\alpha_n)^2 & \frac{1}{2}\left(\alpha_n^2 + \beta_n^2 - \frac{(\alpha_n + \beta_n)^2}{\lambda_n}\right) & \beta_n^2 \end{bmatrix}^{-1} \otimes \mathbf{\Omega}_{n-1}^{-1} \right) \mathbf{g}_n(\mathbf{z}) \\ &= \frac{\mathbf{q}_n^T}{L_n^2} \left(\begin{bmatrix} \beta_n(L_n - \alpha_n\lambda_n) & (\alpha_n - \beta_n)\lambda_n - L_n & \lambda_n \\ 2\alpha_n\beta_n\lambda_n & 2\lambda_n(\beta_n - \alpha_n) & -2\lambda_n \\ \alpha_n(L_n - \beta_n\lambda_n) & (\alpha_n - \beta_n)\lambda_n + L_n & \lambda_n \end{bmatrix} \otimes \mathbf{\Omega}_{n-1}^{-1} \right) \mathbf{g}_n(\mathbf{z}). \end{aligned}$$

According to the definition (4.31) of the generator for P2-TS system, we have

$$\begin{aligned} S_n = \frac{\mathbf{q}_n^T}{L_n^2} &\begin{bmatrix} \beta_n(L_n - \alpha_n\lambda_n)\mathbf{\Omega}_{n-1}^{-1} & ((\alpha_n - \beta_n)\lambda_n - L_n)\mathbf{\Omega}_{n-1}^{-1} & \lambda_n\mathbf{\Omega}_{n-1}^{-1} \\ 2\alpha_n\beta_n\lambda_n\mathbf{\Omega}_{n-1}^{-1} & 2\lambda_n(\beta_n - \alpha_n)\mathbf{\Omega}_{n-1}^{-1} & -2\lambda_n\mathbf{\Omega}_{n-1}^{-1} \\ \alpha_n(L_n - \beta_n\lambda_n)\mathbf{\Omega}_{n-1}^{-1} & ((\alpha_n - \beta_n)\lambda_n + L_n)\mathbf{\Omega}_{n-1}^{-1} & \lambda_n\mathbf{\Omega}_{n-1}^{-1} \end{bmatrix} \\ \times &\begin{bmatrix} \mathbf{g}_{n-1}(z_1, \ldots, z_{n-1}) \\ z_n\mathbf{g}_{n-1}(z_1, \ldots, z_{n-1}) \\ z_n^2\mathbf{g}_{n-1}(z_1, \ldots, z_{n-1}) \end{bmatrix} . \end{aligned}$$

Let us denote

$$\mathbf{q}_n^T = \left[\mathbf{a}^T, \mathbf{b}^T, \mathbf{c}^T\right],$$

where **a**, **b**, and **c** correspond to the three consecutive parts of the conclusions in the decomposed system (4.55), i.e.

$$\mathbf{a} = \begin{bmatrix} q_{0,\ldots,0,0} \\ \vdots \\ q_{2,\ldots,2,0} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} q_{0,\ldots,0,1} \\ \vdots \\ q_{2,\ldots,2,1} \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} q_{0,\ldots,0,2} \\ \vdots \\ q_{2,\ldots,2,2} \end{bmatrix}.$$

According to (4.58) for the MISO P2-TS system with the inputs $\mathbf{z}_{n-1} \in D^{n-1}$, the crisp outputs S_{n-1} can be expressed as follows

$$\begin{aligned} S_{n-1}\left(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}\right) &= \left[q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}\right] \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ &= \mathbf{a}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right), \end{aligned} \tag{4.59}$$

$$\begin{aligned} S_{n-1}\left(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}\right) &= \left[q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}\right] \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ &= \mathbf{b}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right), \end{aligned} \tag{4.60}$$

$$\begin{aligned} S_{n-1}\left(\mathbf{z}_{n-1} \mid q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}\right) &= \left[q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}\right] \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ &= \mathbf{c}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right). \end{aligned} \tag{4.61}$$

Thus,

$$\begin{aligned} S_n\left(\mathbf{z} \mid \mathbf{q}_n\right) = {} & \frac{\beta_n\left(L_n - \alpha_n\lambda_n\right) + \left(-L_n + \left(\alpha_n - \beta_n\right)\lambda_n\right) z_n + \lambda_n z_n^2}{L_n^2} \\ & \times \mathbf{a}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ & + \frac{2\alpha_n\beta_n\lambda_n + 2\lambda_n\left(\beta_n - \alpha_n\right) z_n - 2\lambda_n z_n^2}{L_n^2} \mathbf{b}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ & + \frac{\alpha_n\left(L_n - \beta_n\lambda_n\right) + \left(L_n + \left(\alpha_n - \beta_n\right)\lambda_n\right) z_n + \lambda_n z_n^2}{L_n^2} \\ & \times \mathbf{c}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right). \end{aligned}$$

Taking into consideration (4.24)-(4.26) we obtain

$$\begin{aligned} S_n\left(\mathbf{z} \mid \mathbf{q}_n\right) = {} & N_n\left(z_n\right) \mathbf{a}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ & + Z_n\left(z_n\right) \mathbf{b}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right) \\ & + P_n\left(z_n\right) \mathbf{c}^T \boldsymbol{\Omega}_{n-1}^{-1} \mathbf{g}_{n-1}\left(\mathbf{z}_{n-1}\right). \end{aligned}$$

Finally, according to equations (4.59)-(4.61) we obtain (4.57) and this ends the proof of Theorem 4.10. □

The above theorem is important, because it says that we do not need to inverse large matrices to obtain the crisp output of the P2-TS systems. As a result of this theorem the curse of dimensionality in P2-TS systems is going to disappear. A generalization of Theorem 4.10 for MIMO systems is straightforward and will be omitted.

Now we generalize Corollary 4.3 for MISO P2-TS systems.

Theorem 4.11. *The crisp output of the MISO P2-TS system in the vertex of the hypercuboid D^n is exactly the same as the appropriate conclusion of the fuzzy rule contained in the rule-base.*

Proof. The crisp output of the MISO P2-TS system with the input vector $\mathbf{z} = [z_1, \ldots, z_n]^T$, for which consequents of the rules constitute the vector $\mathbf{q} = [q_{0,\ldots,0}, \ldots, q_{p_1,\ldots,p_n}, \ldots, q_{2,\ldots,2}]^T$, can be expressed as follows

$$S(\mathbf{z} \mid \mathbf{q}) = \sum_{(p_1,\ldots,p_n)\in\{0,1,2\}^n} q_{p_1,\ldots,p_n} \prod_{k=1}^{n} A_{p_k}(z_k), \tag{4.62}$$

where $q_{p_1,\ldots,p_n}$ is a consequent of the fuzzy rule and $A_{p_k}(z_k)$ is the membership degree to which input z_k belongs to A_{p_k}. The name of the membership function A_{p_k} in (4.62) depends on the index $p_k \in \{0, 1, 2\}$ as follows

$$A_{p_k} = \begin{cases} N_k & \text{for } p_k = 0 \\ Z_k & \text{for } p_k = 1 \\ P_k & \text{for } p_k = 2 \end{cases}, \qquad k = 1, \ldots, n. \tag{4.63}$$

If the input vector is a fixed vertex $\boldsymbol{\gamma}_v$ of the hypercuboid D^n, i.e.

$$\mathbf{z} = \boldsymbol{\gamma}_v = [\gamma_1, \ldots, \gamma_n]^T \in \{-\alpha_1, \beta_1\} \times \ldots \times \{-\alpha_n, \beta_n\},$$

then the equation (4.62) reduces to

$$S(\gamma_1, \ldots, \gamma_n \mid \mathbf{q}) = \sum_{(p_1,\ldots,p_n)\in\{0,2\}^n} q_{p_1,\ldots,p_n} \prod_{k=1}^{n} A_{p_k}(\gamma_k), \tag{4.64}$$

since $\prod_{k=1}^{n} A_{p_k}(\gamma_k) = 0$ by $\gamma_k \in \{-\alpha_k, \beta_k\}$ if among indices at least one index $p_k = 1$, $(k = 1, \ldots, n)$. This follows from (4.63) and (4.25). In the summation (4.64) if $\gamma_k = -\alpha_k$, then $p_k = 0$, and if $\gamma_k = \beta_k$, then $p_k = 2$, $(k = 1, \ldots, n)$, but in both cases $\prod_{k=1}^{n} A_{p_k}(\gamma_k) = 1$ according to (4.24) and (4.26). Finally, taking into account the bijection (4.28) we obtain the complete proof of Theorem 4.11. □

It should be noticed that we are able to choose the consequents of the rules so that, the crisp output of a given P2-TS system will be exactly the same as the appropriate conclusions of its fuzzy rules, not only in 2^n vertices of the hypercuboid D^n, but also in all 3^n characteristic points of the set M_n defined in (4.27). However, the class of crisp functions to which such P2-TS system is equivalent becomes much simpler than expected for systems with membership functions as the second degree polynomials.

Example 4.12. Let us consider the P2-TS system with 2 inputs $z_1 \in [-\alpha_1, \beta_1]$ and $z_2 \in [-\alpha_2, \beta_2]$ with quadratic membership functions of fuzzy sets as in (4.24)-(4.26) by $\lambda_k \in (0,1]$, $(k = 1,2)$. If this system is defined by the following fuzzy rules:

R_1 : If z_1 is N_1 and z_2 is N_2, then $S = q_{00}$,
R_2 : If z_1 is Z_1 and z_2 is N_2, then $S = q_{10} = (q_{00} + q_{20})/2$,
R_3 : If z_1 is P_1 and z_2 is N_2, then $S = q_{20}$,
R_4 : If z_1 is N_1 and z_2 is Z_2, then $S = q_{01} = (q_{00} + q_{02})/2$,
R_5 : If z_1 is Z_1 and z_2 is Z_2, then $S = q_{11} = (q_{00} + q_{02} + q_{20} + q_{22})/4$,
R_6 : If z_1 is P_1 and z_2 is Z_2, then $S = q_{21} = (q_{20} + q_{22})/2$,
R_7 : If z_1 is N_1 and z_2 is P_2, then $S = q_{02}$,
R_8 : If z_1 is Z_1 and z_2 is P_2, then $S = q_{12} = (q_{02} + q_{22})/2$,
R_9 : If z_1 is P_1 and z_2 is P_2, then $S = q_{22}$,

then

(i) The crisp output of this system as a function of the inputs $S(z_1, z_2)$ takes the same values in all points of the set $M_2 = \{-\alpha_1, \sigma_1, \beta_1\} \times \{-\alpha_2, \sigma_2, \beta_2\}$, as appear in the appropriate conclusions of the fuzzy rules, i.e.

$$S(-\alpha_1, -\alpha_2) = q_{00}, \qquad S(\sigma_1, -\alpha_2) = q_{10}, \qquad S(\beta_1, -\alpha_2) = q_{20},$$

$$S(-\alpha_1, \sigma_2) = q_{01}, \qquad S(\sigma_1, \sigma_2) = q_{11}, \qquad S(\beta_1, \sigma_2) = q_{21},$$

$$S(-\alpha_1, \beta_2) = q_{02}, \qquad S(\sigma_1, \beta_2) = q_{12}, \qquad S(\beta_1, \beta_2) = q_{22},$$

where $\sigma_k = (-\alpha_k + \beta_k)/2$, $k = 1,2$.

(ii) The crisp output of this system is equivalent to a simple bilinear function

$$S(z_1, z_2) = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \theta_{12} z_1 z_2,$$

where

$$\theta_0 = (\alpha_1 + \beta_1)^{-1} (\alpha_2 + \beta_2)^{-1} (q_{00}\beta_1\beta_2 + q_{02}\alpha_2\beta_1 + q_{20}\alpha_1\beta_2 + q_{22}\alpha_1\alpha_2),$$

$$\theta_1 = (\alpha_1 + \beta_1)^{-1} (\alpha_2 + \beta_2)^{-1} (q_{20}\beta_2 - q_{02}\alpha_2 - q_{00}\beta_2 + q_{22}\alpha_2),$$

$$\theta_2 = (\alpha_1 + \beta_1)^{-1} (\alpha_2 + \beta_2)^{-1} (q_{02}\beta_1 - q_{20}\alpha_1 - q_{00}\beta_1 + q_{22}\alpha_1),$$

$$\theta_{12} = (\alpha_1 + \beta_1)^{-1} (\alpha_2 + \beta_2)^{-1} (q_{00} - q_{02} - q_{20} + q_{22}) .$$

Taking into account e.g. the equation (4.34) the proof of the above facts is simple and will be omitted.

Example 4.13. Let us consider a P2-TS system with four inputs which constitute the vector $\mathbf{z} = [z_1, z_2, z_3, z_4]^T \in D^4$, where D^4 is the hypercube $[-1, 1]^4$. The output of the system is S. For every input z_k we assume three membership functions of fuzzy sets: N_k, Z_k and P_k, defined by (4.24)-(4.26) with the parameter $\lambda_k = 1$ for $k = 1, 2, 3, 4$. The system is defined by the following metarules and ordinary rules:

M_1 : If z_2 is N_2 and z_3 is N_3 and z_4 is N_4, then $S = 1$,
M_2 : If z_2 is Z_2 and z_3 is N_3 and z_4 is N_4, then $S = 2$,
M_3 : If z_2 is P_2 and z_3 is N_3 and z_4 is N_4, then $S = 3$,
M_4 : If z_1 is N_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is N_4, then $S = 4$,
M_5 : If z_1 is Z_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is N_4, then $S = 5$,
M_6 : If z_1 is P_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is N_4, then $S = 6$,
M_7 : If z_2 is Z_2 and z_3 is Z_3 and z_4 is N_4, then $S = 7$,
M_8 : If z_2 is P_2 and z_3 is Z_3 and z_4 is N_4, then $S = 8$,
M_9 : If z_3 is P_3 and z_4 is N_4, then $S = 9$,
M_{10} : If z_2 is N_2 and z_3 is N_3 and z_4 is Z_4, then $S = -1$,
M_{11} : If z_2 is Z_2 and z_3 is N_3 and z_4 is Z_4, then $S = -2$,
M_{12} : If z_2 is P_2 and z_3 is N_3 and z_4 is Z_4, then $S = -3$,
M_{13} : If z_1 is N_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is Z_4, then $S = -4$,
M_{14} : If z_1 is Z_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is Z_4, then $S = -5$,
M_{15} : If z_1 is P_1 and z_2 is N_2 and z_3 is Z_3 and z_4 is Z_4, then $S = -6$,
M_{16} : If z_2 is Z_2 and z_3 is Z_3 and z_4 is Z_4, then $S = -7$,
M_{17} : If z_2 is P_2 and z_3 is Z_3 and z_4 is Z_4, then $S = -8$,
M_{18} : If z_3 is P_3 and z_4 is Z_4, then $S = -9$,
M_{19} : If z_3 is N_3 and z_4 is P_4, then $S = 1$,
M_{20} : If z_3 is $(Z_3$ or $P_3)$ and z_4 is P_4, then $S = 0$,

We assume that the fragment " z_3 is (Z_3 or P_3)" in the "If" part of the metarule M_{20} is equivalent to "z_3 is not N_3" and generates two metarules.

The above 20 metarules are equivalent to 81 complete and noncontradictory fuzzy rules with consequents given symbolically in Table 4.1 and numerically in Table 4.2.

Formally the system output $S = S_4 (z_1, z_2, z_3, z_4 \mid q_{0000}, \ldots, q_{2222})$. According to Theorem 4.10 a general form of the crisp system output is given by (4.57) for $n = 4$, i.e.

Table 4.1 Look-up-table for the P2-TS system with $n = 4$ input variables in the general case

$z_1 z_2 \setminus z_3 z_4 \rightarrow$ $\downarrow$	$N_3 N_4$	$Z_3 N_4$	$P_3 N_4$	$N_3 Z_4$	$Z_3 Z_4$	$P_3 Z_4$	$N_3 P_4$	$Z_3 P_4$	$P_3 P_4$
$N_1 N_2$	q_{0000}	q_{0010}	q_{0020}	q_{0001}	q_{0011}	q_{0021}	q_{0002}	q_{0012}	q_{0022}
$Z_1 N_2$	q_{1000}	q_{1010}	q_{1020}	q_{1001}	q_{1011}	q_{1021}	q_{1002}	q_{1012}	q_{1022}
$P_1 N_2$	q_{2000}	q_{2010}	q_{2020}	q_{2001}	q_{2011}	q_{2021}	q_{2002}	q_{2012}	q_{2022}
$N_1 Z_2$	q_{0100}	q_{0110}	q_{0120}	q_{0101}	q_{0111}	q_{0121}	q_{0102}	q_{0112}	q_{0122}
$Z_1 Z_2$	q_{1100}	q_{1110}	q_{1120}	q_{1101}	q_{1111}	q_{1121}	q_{1102}	q_{1112}	q_{1122}
$P_1 Z_2$	q_{2100}	q_{2110}	q_{2120}	q_{2101}	q_{2111}	q_{2121}	q_{2102}	q_{2112}	q_{2122}
$N_1 P_2$	q_{0200}	q_{0210}	q_{0220}	q_{0201}	q_{0211}	q_{0221}	q_{0202}	q_{0212}	q_{0222}
$Z_1 P_2$	q_{1200}	q_{1210}	q_{1220}	q_{1201}	q_{1211}	q_{1221}	q_{1202}	q_{1212}	q_{1222}
$P_1 P_2$	q_{2200}	q_{2210}	q_{2220}	q_{2201}	q_{2211}	q_{2221}	q_{2202}	q_{2212}	q_{2222}

Table 4.2 Look-up-table for the P2-TS system from Example 4.13

$z_1 z_2 \setminus z_3 z_4 \rightarrow$ $\downarrow$	$N_3 N_4$	$Z_3 N_4$	$P_3 N_4$	$N_3 Z_4$	$Z_3 Z_4$	$P_3 Z_4$	$N_3 P_4$	$Z_3 P_4$	$P_3 P_4$
$N_1 N_2$	1	4	9	−1	−4	−9	1	0	0
$Z_1 N_2$	1	5	9	−1	−5	−9	1	0	0
$P_1 N_2$	1	6	9	−1	−6	−9	1	0	0
$N_1 Z_2$	2	7	9	−2	−7	−9	1	0	0
$Z_1 Z_2$	2	7	9	−2	−7	−9	1	0	0
$P_1 Z_2$	2	7	9	−2	−7	−9	1	0	0
$N_1 P_2$	3	8	9	−3	−8	−9	1	0	0
$Z_1 P_2$	3	8	9	−3	−8	−9	1	0	0
$P_1 P_2$	3	8	9	−3	−8	−9	1	0	0

$$\begin{aligned} S = {} & N_4 (z_4) S_3 (z_1, z_2, z_3 \mid q_{0000}, q_{1000}, q_{2000}, \dots, q_{0220}, q_{1220}, q_{2220}) \\ & + Z_4 (z_4) S_3 (z_1, z_2, z_3 \mid q_{0001}, q_{1001}, q_{2001}, \dots, q_{0221}, q_{1221}, q_{2221}) \\ & + P_4 (z_4) S_3 (z_1, z_2, z_3 \mid q_{0002}, q_{1002}, q_{2002}, \dots, q_{0222}, q_{1222}, q_{2222}) , \end{aligned} \tag{4.65}$$

where for $S_3 = S_3 (z_1, z_2, z_3 \mid q_{000}, \dots, q_{222})$ we have

$$\begin{aligned} S_3 = {} & N_3(z_3)\, S_2(z_1, z_2 \mid q_{000}, q_{100}, q_{200}, q_{010}, q_{110}, q_{210}, q_{020}, q_{120}, q_{220}) \\ & + Z_3(z_3)\, S_2(z_1, z_2 \mid q_{001}, q_{101}, q_{201}, q_{011}, q_{111}, q_{211}, q_{021}, q_{121}, q_{221}) \\ & + P_3(z_3)\, S_2(z_1, z_2 \mid q_{002}, q_{102}, q_{202}, q_{012}, q_{112}, q_{212}, q_{022}, q_{122}, q_{222}), \end{aligned} \tag{4.66}$$

$$\begin{aligned} S_2(z_1, z_2 \mid q_{00}, q_{10}, q_{20}, q_{01}, q_{11}, q_{21}, q_{02}, q_{12}, q_{22}) = {} & N_2(z_2)\, S_1(z_1 \mid q_{00}, q_{10}, q_{20}) \\ & + Z_2(z_2)\, S_1(z_1 \mid q_{01}, q_{11}, q_{21}) \\ & + P_2(z_2)\, S_1(z_1 \mid q_{02}, q_{12}, q_{22}), \end{aligned} \tag{4.67}$$

$$S_1(z_1 \mid q_0, q_1, q_2) = N_1(z_1)\, q_0 + Z_1(z_1)\, q_1 + P_1(z_1)\, q_2. \tag{4.68}$$

Assume that the membership functions of the fuzzy sets are

$$N_k(z_k) = \frac{(1-z_k)^2}{4}, \quad Z_k(z_k) = \frac{1-z_k^2}{2}, \quad P_k(z_k) = \frac{(1+z_k)^2}{4},$$

for $\alpha_k = \beta_k = 1$ and $\lambda_k = 1$, $(k = 1, 2, 3, 4)$. After computations we obtain

$$\begin{aligned} S = {} & \frac{1}{32} z_1 z_3^2 - \frac{1}{4} z_2 - z_3 - \frac{47}{16} z_4 - \frac{1}{32} z_1 z_2^2 - \frac{1}{32} z_1 + \frac{3}{32} z_1 z_4^2 + \frac{1}{8} z_2 z_3^2 \\ & + \frac{3}{4} z_2 z_4^2 + \frac{1}{16} z_2^2 z_4 + \frac{5}{2} z_3 z_4^2 + \frac{7}{16} z_3^2 z_4 + \frac{1}{32} z_2^2 + \frac{7}{32} z_3^2 \\ & + \frac{149}{32} z_4^2 - \frac{1}{32} z_2^2 z_3^2 - \frac{3}{32} z_2^2 z_4^2 - \frac{13}{32} z_3^2 z_4^2 + \frac{1}{16} z_1 z_2 - \frac{1}{16} z_1 z_4 \\ & + \frac{1}{8} z_2 z_3 - \frac{1}{2} z_2 z_4 - 2 z_3 z_4 + \frac{3}{32} z_2^2 z_3^2 z_4^2 - \frac{1}{16} z_1 z_2 z_3^2 - \frac{3}{16} z_1 z_2 z_4^2 \\ & - \frac{1}{16} z_1 z_2^2 z_4 + \frac{1}{16} z_1 z_3^2 z_4 - \frac{3}{8} z_2 z_3 z_4^2 + \frac{1}{4} z_2 z_3^2 z_4 + \frac{1}{32} z_1 z_2^2 z_3^2 \\ & + \frac{3}{32} z_1 z_2^2 z_4^2 - \frac{3}{32} z_1 z_3^2 z_4^2 - \frac{3}{8} z_2 z_3^2 z_4^2 - \frac{1}{16} z_2^2 z_3^2 z_4 + \frac{1}{8} z_1 z_2 z_4 + \frac{1}{4} z_2 z_3 z_4 \\ & + \frac{3}{16} z_1 z_2 z_3^2 z_4^2 + \frac{1}{16} z_1 z_2^2 z_3^2 z_4 - \frac{3}{32} z_1 z_2^2 z_3^2 z_4^2 - \frac{1}{8} z_1 z_2 z_3^2 z_4 - \frac{47}{32}. \end{aligned}$$

If we consider the output S as a function of four independent variables, i.e. $S = S(z_1, z_2, z_3, z_4)$, we have

$S(-1,-1,-1,-1) = 1,\ S(1,-1,-1,-1) = 1,\ S(-1,1,-1,-1) = 3,$

$S(1,1,-1,-1) = 3,\ S(-1,-1,1,-1) = 9,\ S(1,-1,1,-1) = 9,$

$S(-1,1,1,-1) = 9,\ S(1,1,1,-1) = 9,\ S(-1,-1,-1,1) = 1,$

$S(1,-1,-1,1) = 1,\ S(-1,1,-1,1) = 1,\ S(1,1,-1,1) = 1,$

$S(-1,-1,1,1) = 0,\ S(1,-1,1,1) = 0,\ S(-1,1,1,1) = 0,\ S(1,1,1,1) = 0.$

This means that in all 2^n points from the set $\times_{k=1}^{n} \{-\alpha_k, \beta_k\}$, $(n = 4)$, the values of the output of the P2-TS system are exactly the same as the

Table 4.3 The metarules M_1, M_2, M_3 and all fuzzy rules (M_1&M_2&M_3&R_1) for the first system in Example 4.14 in the form of look-up-tables

M_1:

$z_1z_2 \setminus z_3 \rightarrow$ ↓	N_3	Z_3	P_3
N_1N_2	0	0	0
Z_1N_2	*	*	*
P_1N_2	0	0	0
N_1Z_2	0	0	0
Z_1Z_2	*	*	*
P_1Z_2	0	0	0
N_1P_2	0	0	0
Z_1P_2	*	*	*
P_1P_2	0	0	0

M_2:

$z_1z_2 \setminus z_3 \rightarrow$ ↓	N_3	Z_3	P_3
N_1N_2	0	0	0
Z_1N_2	0	0	0
P_1N_2	0	0	0
N_1Z_2	*	*	*
Z_1Z_2	*	*	*
P_1Z_2	*	*	*
N_1P_2	0	0	0
Z_1P_2	0	0	0
P_1P_2	0	0	0

M_3:

$z_1z_2 \setminus z_3 \rightarrow$ ↓	N_3	Z_3	P_3
N_1N_2	0	*	0
Z_1N_2	0	*	0
P_1N_2	0	*	0
N_1Z_2	0	*	0
Z_1Z_2	0	*	0
P_1Z_2	0	*	0
N_1P_2	0	*	0
Z_1P_2	0	*	0
P_1P_2	0	*	0

all rules:

$z_1z_2 \setminus z_3 \rightarrow$ ↓	N_3	Z_3	P_3
N_1N_2	0	0	0
Z_1N_2	0	0	0
P_1N_2	0	0	0
N_1Z_2	0	0	0
Z_1Z_2	0	a	0
P_1Z_2	0	0	0
N_1P_2	0	0	0
Z_1P_2	0	0	0
P_1P_2	0	0	0

appropriate conclusions of the fuzzy rules (see Table 4.2). However, the value $S(z_1, z_2, z_3, z_4)$ in the other points (z_1, z_2, z_3, z_4) from the set M_n defined by (4.27) for $n = 4$, does not satisfy this condition, e.g. $S(-1, -1, 0, -1) = 4.5 \neq 4$. The result confirms the correctness of Theorem 4.11.

Example 4.14. Let us consider two simple P2-TS systems with 3 inputs $z_k \in [-\alpha_k, \beta_k]$ and quadratic membership functions (4.24)-(4.26), for $k = 1, 2, 3$. The first system is given by three metarules M_1-M_3 and one rule R_1:

M_1 : If z_1 is not Z_1, then $S = 0$,
M_2 : If z_2 is not Z_2, then $S = 0$,
M_3 : If z_3 is not Z_3, then $S = 0$,
R_1 : If z_1 is Z_1 and z_2 is Z_2 and z_3 is Z_3, then $S = a$,

and the second one by three metarules M_1'-M_3' and one rule R_1':

M_1' : If z_1 is not N_1, then $S' = 0$,
M_2' : If z_2 is not N_2, then $S' = 0$,
M_3' : If z_3 is not N_3, then $S' = 0$,
R_1' : If z_1 is N_1 and z_2 is N_2 and z_3 is N_3, then $S' = b$.

The meaning of all logical operators "and", "or", "not" used in the "If" parts of the metarules is natural and explained by the look-up-tables (see Tables 4.3 and 4.4). They describe the metarules and all the fuzzy rules. Zero in a table

Table 4.4 The metarules M_1', M_2', M_3' and all fuzzy rules $(M_1' \& M_2' \& M_3' \& R_1')$ for the first system in Example 4.14 in the form of look-up-tables

M_1'

$z_1z_2 \setminus z_3$ → ↓	N_3	Z_3	P_3
N_1N_2	*	*	*
Z_1N_2	0	0	0
P_1N_2	0	0	0
N_1Z_2	*	*	*
Z_1Z_2	0	0	0
P_1Z_2	0	0	0
N_1P_2	*	*	*
Z_1P_2	0	0	0
P_1P_2	0	0	0

M_2'

$z_1z_2 \setminus z_3$ → ↓	N_3	Z_3	P_3
N_1N_2	*	*	*
Z_1N_2	*	*	*
P_1N_2	*	*	*
N_1Z_2	0	0	0
Z_1Z_2	0	0	0
P_1Z_2	0	0	0
N_1P_2	0	0	0
Z_1P_2	0	0	0
P_1P_2	0	0	0

M_3'

$z_1z_2 \setminus z_3$ → ↓	N_3	Z_3	P_3
N_1N_2	*	0	0
Z_1N_2	*	0	0
P_1N_2	*	0	0
N_1Z_2	*	0	0
Z_1Z_2	*	0	0
P_1Z_2	*	0	0
N_1P_2	*	0	0
Z_1P_2	*	0	0
P_1P_2	*	0	0

all rules

$z_1z_2 \setminus z_3$ → ↓	N_3	Z_3	P_3
N_1N_2	b	0	0
Z_1N_2	0	0	0
P_1N_2	0	0	0
N_1Z_2	0	0	0
Z_1Z_2	0	0	0
P_1Z_2	0	0	0
N_1P_2	0	0	0
Z_1P_2	0	0	0
P_1P_2	0	0	0

denotes the consequent "0" expressed by some metarule and a star denotes any number (including 0). Observe that the metarules define a complete and noncontradictory system of rules.

One can check that the crisp output of the first system is given by

$$S\left(z_1, z_2, z_3\right) = 8a \prod_{k=1}^{3} \frac{\lambda_k}{\left(\alpha_k + \beta_k\right)^2} \prod_{k=1}^{3} \left(\beta_k - z_k\right)\left(z_k + \alpha_k\right).$$

The sign of S is the same as the sign of the consequent of the rule R_1. Furthermore, $S = 0$ if there is some $k \in \{1, 2, 3\}$ for which $z_k = -\alpha_k$ or $z_k = \beta_k$.

The crisp output of the second system is given by

$$S'\left(z_1, z_2, z_3\right) = b \prod_{k=1}^{3} \frac{\lambda_k}{\left(\alpha_k + \beta_k\right)^2} \prod_{k=1}^{3} \left(\beta_k - z_k\right)\left(\frac{\beta_k + \alpha_k\left(1 - \lambda_k\right)}{\lambda_k} - z_k\right).$$

The sign of the crisp output S' in the second system is the same as the sign of b, since $\frac{1}{\lambda_k}\left(\beta_k + \alpha_k\left(1 - \lambda_k\right)\right) - z_k \geq 0$ and $\left(\beta_k - z_k\right) \geq 0$ for $z_k \in [-\alpha_k, \beta_k]$, $k = 1, 2, 3$. Furthermore, $S' = 0$ for all points where $z_1 = \beta_1$ or $z_2 = \beta_2$ or $z_3 = \beta_3$.

As one can see, the interpretation of the fuzzy rules in both P2-TS systems is natural and simple. The crisp functions $S\left(z_1, z_2, z_3\right)$ and $S'\left(z_1, z_2, z_3\right)$ intuitively correspond to the systems of rules in any case.

4.7 Recursion in More General TS Systems with Three Fuzzy Sets for Every Input

Theorem 4.10 has been proved using the idea of the fundamental matrix for P2-TS systems, since this matrix is important for many applications. However, we will show below that the same theorem is valid for a more general class of the fuzzy rule-based TS systems, i.e. the systems with three fuzzy sets for every input, where the assumptions 1, 2 and 3 for the membership functions from Section 4.2 are not necessary. We will prove the following generalization of Theorem 4.10.

Theorem 4.15. *Theorem 4.10 is valid for any TS system described by the fuzzy rules (4.54), with the inputs* $z_1 \in [-\alpha_1, \beta_1]$, $\ldots$, $z_n \in [-\alpha_n, \beta_n]$, *where for any input* z_k *there are assigned three fuzzy sets with the normalized membership functions, i.e.* $N_k : [-\alpha_k, \beta_k] \to [0,1]$, $Z_k : [-\alpha_k, \beta_k] \to [0,1]$, *and* $P_k : [-\alpha_k, \beta_k] \to [0,1]$ *and* $N_k(z_k) + Z_k(z_k) + P_k(z_k) = 1$ *for* $k = 1, \ldots, n$. *This means that if* $S_n(\mathbf{z} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2})$ *denotes the crisp output of the system (4.54) with input vector* $\mathbf{z} \in D^n$ *and the consequents of the rules constituting the vector* $[q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}]^T$, *then for any natural* $n \geq 2$ *the recursive formula that enables one to compute the crisp system output is the same as (4.57).*

Proof. For $n = 1$ the system is defined by the rules (4.12). Thus, the system output is as follows

$$S_1(z_1 \mid q_0, q_1, q_2) = \frac{N_1(z_1)\, q_0}{N_1(z_1) + Z_1(z_1) + P_1(z_1)} + \frac{Z_1(z_1)\, q_1}{N_1(z_1) + Z_1(z_1) + P_1(z_1)} + \frac{P_1(z_1)\, q_2}{N_1(z_1) + Z_1(z_1) + P_1(z_1)}. \tag{4.69}$$

It is the same as in (4.68) since the normalization condition (4.7) is satisfied. Let us use a simplified notation: $N_k(z_k) = N_k$, $Z_k(z_k) = Z_k$ and $P_k(z_k) = P_k$. For $n = 2$, due to the rule-base (4.32) we have

$$\begin{aligned} S_2(z_1, z_2 \mid q_{00}, \ldots, q_{22}) = {} & N_1 N_2 q_{00}/D_2 + Z_1 N_2 q_{10}/D_2 + P_1 N_2 q_{20}/D_2 \\ & + N_1 Z_2 q_{01}/D_2 + Z_1 Z_2 q_{11}/D_2 + P_1 Z_2 q_{21}/D_2 \\ & + N_1 P_2 q_{02}/D_2 + Z_1 P_2 q_{12}/D_2 + P_1 P_2 q_{22}/D_2. \end{aligned}$$

But $D_2 = \prod_{k=1}^{2} (N_k(z_k) + Z_k(z_k) + P_k(z_k)) = 1$. Thus,

$$\begin{aligned} S_2(z_1, z_2 \mid q_{00}, \ldots, q_{22}) = {} & N_2 (N_1 q_{00} + Z_1 q_{10} + P_1 q_{20}) \\ & + Z_2 (N_1 q_{01} + Z_1 q_{11} + P_1 q_{21}) \\ & + P_2 (N_1 q_{02} + Z_1 q_{12} + P_1 q_{22}) \end{aligned}$$

and S_2 is the same as in (4.67), i.e. for $n = 2$ the Theorem 4.15 is true.

According to the rule-base decomposition (4.55) for $n = k+1 \geq 3$ we obtain

$$\begin{aligned} S_{k+1} &= N_{k+1}\left(N_1 N_2 \ldots N_k q_{0,0,\ldots,0,0} + \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,0}\right)/D_{k+1} \\ &+ Z_{k+1}\left(N_1 N_2 \ldots N_k q_{0,0,\ldots,0,1} + \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,1}\right)/D_{k+1} \\ &+ P_{k+1}\left(N_1 N_2 \ldots N_k q_{0,0,\ldots,0,2} + \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,2}\right)/D_{k+1}, \end{aligned}$$

where the denominator $D_{k+1} = \prod_{i=1}^{k+1}\left(N_i(z_i) + Z_i(z_i) + P_i(z_i)\right) = 1$. Knowing that $D_k = 1$ for $k = 1, 2, \ldots$ we have

$$\begin{aligned} S_{k+1}\left(\mathbf{z}_{k+1} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}\right) &= N_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}\right) \\ &+ Z_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}\right) \\ &+ P_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}\right), \end{aligned}$$

where

$$\begin{aligned} S_k\left(\mathbf{z}_k \mid q_{0,0,\ldots,0,0}, \ldots, q_{2,2,\ldots,2,2}\right) &= N_1 N_2 \ldots N_k q_{0,0,\ldots,0,0} \\ &+ Z_1 N_2 \ldots N_k q_{1,0,\ldots,0,0} \\ &+ P_1 N_2 \ldots N_k q_{2,0,\ldots,0,0} \\ &+ \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,2}\ , \end{aligned}$$

$$\begin{aligned} S_k\left(\mathbf{z}_k \mid q_{0,0,\ldots,0,1}, \ldots, q_{2,2,\ldots,2,1}\right) &= N_1 N_2 \ldots N_k q_{0,0,\ldots,0,1} \\ &+ Z_1 N_2 \ldots N_k q_{1,0,\ldots,0,1} \\ &+ P_1 N_2 \ldots N_k q_{2,0,\ldots,0,1} \\ &+ \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,1}\ , \end{aligned}$$

$$\begin{aligned} S_k\left(\mathbf{z}_k \mid q_{0,0,\ldots,0,2}, \ldots, q_{2,2,\ldots,2,2}\right) &= N_1 N_2 \ldots N_k q_{0,0,\ldots,0,2} \\ &+ Z_1 N_2 \ldots N_k q_{1,0,\ldots,0,2} \\ &+ P_1 N_2 \ldots N_k q_{2,0,\ldots,0,2} \\ &+ \ldots + P_1 P_2 \ldots P_k q_{2,2,\ldots,2,2}\ . \end{aligned}$$

Thus,

$$\begin{aligned} S_{k+1}\left(\mathbf{z}_{k+1} \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,2}\right) &= N_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,0}, \ldots, q_{2,\ldots,2,0}\right) \\ &+ Z_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,1}, \ldots, q_{2,\ldots,2,1}\right) \\ &+ P_{k+1} S_k\left(\mathbf{z}_k \mid q_{0,\ldots,0,2}, \ldots, q_{2,\ldots,2,2}\right). \end{aligned}$$

This finishes the proof of Theorem 4.15. □

The above Theorem can be used for rather large rule-bases. For $n = 3$ inputs, taking into account (4.66), it can be graphically interpreted as shown in Fig. 4.7. In the case of the TS system with n inputs, the architecture can be viewed as n-layer neural network with linear activation functions f for all neurons, where $f\,(input) = input$. In the layer number k, $(k = 1, \ldots, n)$, the

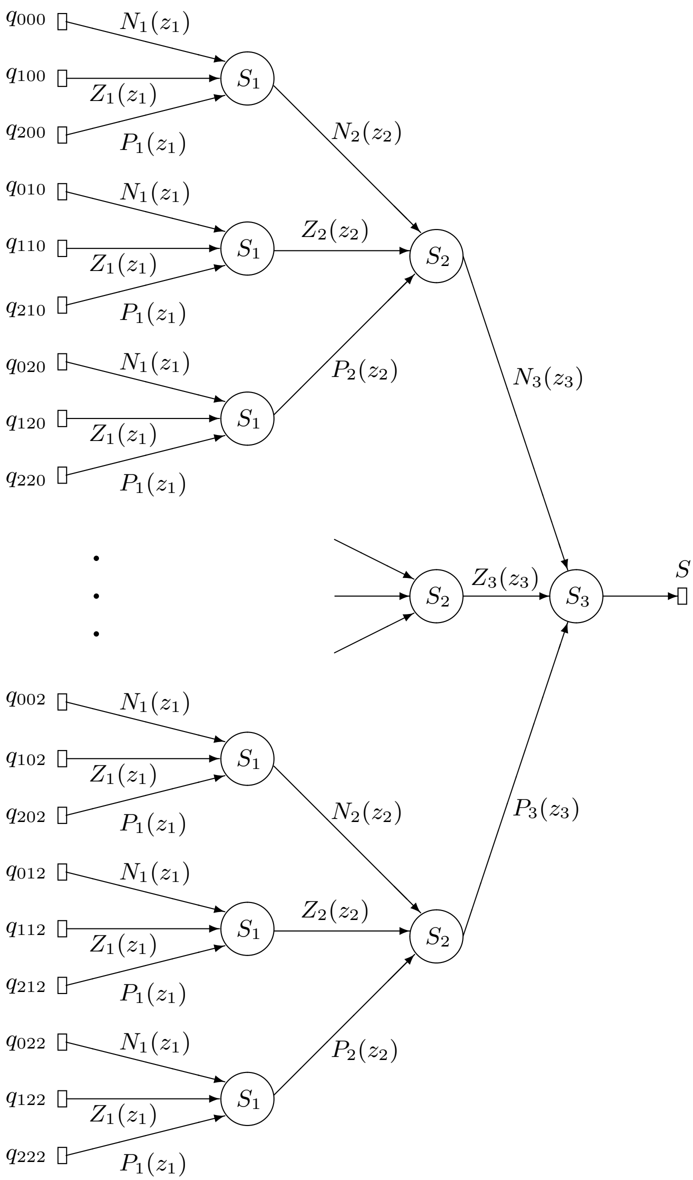


Fig. 4.7 Graphic interpretation of Theorem 4.15 for a TS system with $n = 3$ inputs and the output $S = S_3\left(z_1, z_2, z_3 | q_{000}, \ldots, q_{222}\right)$

network contains exactly the same neurons S_k and every neuron has three inputs and the same weights, namely $N_k\left(z_k\right)$, $Z_k\left(z_k\right)$ and $P_k\left(z_k\right)$.

A generalization of the Theorem 4.15 for MIMO systems is straightforward and will be omitted. A computational architecture of the recursion for MIMO P2-TS systems as a generalization of (4.57) can be easily drawn, similarly to the one of Fig. 4.7, as well.

4.8 Summary

We considered the TS systems which use the second degree polynomials as the membership functions of fuzzy sets for the inputs. It was shown that it is not possible to obtain any second degree polynomial function, to which a TS rule-based system is equivalent, on the assumption that only two complementary membership functions as the second degree polynomials are defined for the input variables. However, three quadratic membership functions suffice to model every second degree polynomial function.

For the considered zero-order TS system, we defined for every input variable the set of three highly interpretable normalized membership functions as the second degree polynomials (N, Z and P). They contain one free design parameter. The TS systems that use such fuzzy sets were called P2-TS systems and they were thoroughly investigated. One of theorems says that the crisp output of the MISO P2-TS system in the vertex of the hypercuboid D^n is exactly the same as the appropriate conclusion of the fuzzy rule contained in the rule-base.

For the P2-TS systems both the generator and the fundamental matrix were defined. The fundamental matrix and its inverse are important, since they enable one to establish an exact relationship between the consequents of the "If-then" rules and the parameters that define the crisp function, to which the rule-based system is equivalent. Therefore, the procedure of how to compute the fundamental matrix and its inverse was given.

Examples 4.12-4.14 show that P2-TS systems have highly interpretable rule-bases when we use individual fuzzy rules or the metarules.

The P2-TS system with n-inputs, which normally contains a complete and noncontradictory set of fuzzy rules, consists of 3^n individual fuzzy rules. Thus, the curse of dimensionality problem is much more serious for the P2-TS systems than the one for the P1-TS systems. Therefore, we developed the recursive procedures for the computation of both the inverse of the fundamental matrix and the crisp output of the P2-TS systems. Theorem 4.10 and its generalization say that we do not need to inverse large matrices to obtain the crisp output of the P2-TS systems. As a result of these theorems, the curse of dimensionality in P2-TS systems was substantially weakened. Although we considered the MISO systems, all the results can be easily generalized for the MIMO case.

After this chapter we are able to thoroughly generalize the results for the TS systems with the membership functions that are polynomials of the degree $d \geq 3$. However, we should realize that the number of complete and noncontradictory rules will rapidly grow and the analysis will become more and more complicated. Both P1- and P2-TS systems are able to model a large class of real nonlinear processes. Therefore, if it is not necessary, we should not complicate our models in the engineering practice.