# On Approximation of Classifications, Rough Equalities and Rough Equivalences

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**Summary.** In this chapter we mainly focus on the study of some topological aspects of rough sets and approximations of classifications. The topological classification of rough sets deals with their types. We find out types of intersection and union of rough sets, New concepts of rough equivalence (top, bottom and total) are defined, which capture approximate equality of sets at a higher level than rough equality (top, bottom and total) of sets introduced and studied by Novotny and Pawlak [23,24,25] and is also more realistic. Properties are established when top and bottom rough equalities are interchanged. Also, parallel properties for rough equivalences are established. We study approximation of classifications (introduced and studied by Busse [12]) and find the different types of classifications of an universe completely. We find out properties of rules generated from information systems and observations on the structure of such rules. The algebraic properties which hold for crisp sets and deal with equalities loose their meaning when crisp sets are replaced with rough sets. We analyze the validity of such properties with respect to rough equivalences.

## 1 Introduction

The notion of rough sets was introduced by Pawlak [26,27,28] as an extension of the concept of crisp sets to capture impreciseness. Imprecision in this approach is expressed by the boundary region of a set. In fact, the idea of rough set is based upon approximation of a set by a pair of sets, called the *lower* and *upper approximations* of the set.

In real life situations, fruitful and accurate applications of rough sets require two aspects, called accuracy measure and topological characterization. We shall mainly concentrate on topological characterization of rough sets in this chapter. The other related aspect to be dealt with is approximations of classifications, which are in a sense extensions of the concept of approximation of sets but their characteristics are not exactly same. We shall study the types of union and intersection of rough sets which are also used in dealing with types of classifications. New notions of approximate equalities, called rough equivalences are introduced and their properties are studied. Using this notion, some basic algebraic properties for crisp sets are extended to rough sets. We also touch the topic of rule generation from information systems.

A. Abraham, R. Falcón, and R. Bello (Eds.): Rough Set Theory, SCI 174, pp. 85–133.
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Now, we present the detailed structure of the chapter here. In sect. 2, we establish two theorems on rough approximations which provide necessary and sufficient conditions for equality to hold in two of the properties, where in general inclusions hold true. There are several applications of these results as we shall see in sections 4.3, 5.5 and 9.4.

As mentioned by Pawlak [30], one important difference between the concept of rough set and the classical notion of set is the equality of sets. In classical set theory, two sets are equal if they have exactly the same elements. But a more practically applicable form of equality (approximate equality) called rough equality was introduced in [23,24,25]. Here, two sets may not be equal in the classical sense but they have enough of close features (that is they differ slightly from each other) to be assumed to be approximately equal. These types of equalities of sets refer to the topological structure of compared sets but not to the elements they consist of. In fact two sets can be exactly equal in one knowledge base but approximately equal or not equal in another. The practicality of this notion depends upon the common observation that things are equal or not equal from the point of view of our knowledge about them. Certain properties of these equalities were established by Novotny and Pawlak [23,24,25]. But they have remarked that these properties cease to be true when top and bottom rough equalities are replaced one by the other. In sect. 3, we see that some of these properties are true under replacement and others hold true if some additional conditions are imposed.

A topological characterization of imprecision defined through the lower and upper approximation of sets is the notion of type of rough sets. There are four such types [30]. This method of characterization of imprecision complements the other method of characterization of imprecision through accuracy measures, which expresses degree of completeness of our knowledge about a set. As observed by Pawlak ([30], p. 22), in practical applications of rough sets we combine both kinds of information. As far as the information available, no further study is found in rough set literature on this topic after its introduction. In sect. 4, we study the types of rough sets obtained by union and intersection of rough sets. We shall deal with applications of these results in sects. 5, 7 and 9.

As mentioned above, rough equalities deal with topological structures of the compared sets. In sect. 5, we introduce and study another type of approximate equality, called rough equivalence of sets, which captures topological structures of the compared sets at a higher level than rough equality. By this, we mean that any two sets comparable with the notions of rough equalities (bottom, top and total) are also comparable with the corresponding notion of rough equivalence (bottom, top and total) and the converse is not necessarily true. In fact, there are many practical situations, where we can talk of approximate equality of the compared sets with new notion but can not do so with the old one. More importantly, this new comparison very much matches with our perception of equality depending upon our knowledge about the universe. We illustrate this with some examples. Also, properties rough equivalences, which are in parallel with those for rough equalities along with the corresponding replacement properties are analyzed and established.

To deal with knowledge acquisition under uncertainty, Busse [12] considered the approximations of classifications as a new approach. Some earlier approaches to the acquisition of knowledge and reasoning under uncertainty by expert systems research community are in [1,11,19,44]. Uncertainty may be caused by ambiguous meanings of the terms used, corrupted data or uncertainty in the knowledge itself [12]. One of the popular ways to acquire the knowledge is based upon learning from examples [12]. The information system (a data base-like system) represents what is called an 'instant space' in learning from examples. In the approach of Busse, inconsistencies are not corrected. Instead, produced rules are categorized into certain rules and possible rules. Some other authors who have dealt without correcting inconsistencies in information systems are Mamdani et. al.[19] and Quinlan [35]. Four results were established by Busse on approximation of classifications. In sect. 6, we generalize these results to necessary and sufficient type ones from which, along with the results of Busse many other results can be obtained as corollaries. The types of classifications are thoroughly analyzed and their properties are studied in sect. 7. We find that the eleven numbers of possible types reduce either directly or transitively to the five types considered by Busse. In sect. 8, we present some of the properties of rules generated from information systems and obtain many observations on the structure of such rules.

There are many fundamental algebraic properties of crisp sets with respect to the operations of union, intersection and complementation. All these properties involve equality of two such expressions. When the involved sets are taken to rough sets the equalities bear very little meaning (particularly, after the introduction of the concepts of rough equalities and rough equivalences). To make them more and more meaningful, one has to consider rough equality or rough equivalence in general. In sect. 9, we consider the validity of many of these basic properties with crisp equality being replaced by rough equivalence. Rough equalities being special cases of rough equivalences, we can derive the corresponding validities easily. We shall end the chapter with some concluding remarks and finally provide a bibliography of papers and other related materials, which are referred during the compilation of the materials of the chapter.

## 2 Rough Sets and Properties of Approximations

In this sect. we shall first introduce the definitions of rough set and related concepts in sect. 2.1. In sect. 2.2 we introduce some properties of lower and upper approximations and establish two theorems related to these properties, which are to be used in later sections.

### 2.1 Rough Sets

Let $U$ be a universe of discourse and $R$ be an equivalence relation over $U$. By $U/R$ we denote the family of all equivalence classes of $R$, referred to as *categories* or *concepts* of $R$ and the equivalence class of an element $x \in U$ is denoted by $[x]_R$. The basic philosophy of rough set is that knowledge is deep-seated in the classificatory abilities of human beings and other species. Knowledge is connected with the variety of classification patterns related to specific parts of real or abstract world, called the universe. Knowledge consists of a family of various classification patterns of a domain of interest, which provide explicit facts about reality-together with the reasoning capacity able to deliver implicit facts derivable from explicit knowledge ([30], p. 2).

There is, however a variety of opinions and approaches in this area, as to how to understand, represent and manipulate knowledge [3,4,6,7,9,15,20,21].

Usually, we do not deal with a single classification, but with families of classifications over U. A family of classifications over U is called a knowledge base over U. This provides us with a variety of classification patterns which constitute the fundamental equipment to define its relation to the environment. More precisely, by a knowledge base we mean a relational system $\mathbf{K}=(U,\boldsymbol{R})$, where $U$ is as above and $\boldsymbol{R}$ is a non-empty family of equivalence relations over $U$.

For any subset $P(\neq \phi) \subseteq \boldsymbol{R}$, the intersection of all equivalence relations in $P$ is denoted by *IND(P)* and is called the *indiscernilibity relation* over $P$. By $IND(K)$ we denote the family of all equivalence relations defined in $K$, that is $IND(K) = \{IND(P) : P \subseteq \ \boldsymbol{R}, P \neq \phi\}$.

Given any $X \subseteq U$ and $R \in$ *IND(K)*, we associate two subsets, $\underline{R}X = \bigcup\{Y \in U/R : Y \subseteq X\}$ and $\bar{R}X = \bigcup\{Y \in U/R : Y \cap X \neq \phi\}$, called the R-*lower* and R-*upper approximations* of $X$ respectively. The R-*boundary* of $X$ is denoted by $BN_R(X)$ and is given by $BN_R(X) = \bar{R}X - \underline{R}X$. The elements of $\underline{R}X$ are those elements of $U$ which can certainly be classified as elements of $X$ and elements of $\bar{R}X$ are those elements of $U$ which can possibly be classified as elements of $X$, employing the knowledge of $R$. We say that $X$ is rough with respect to $R$ if and only if $\underline{R}X \neq \bar{R}X$, equivalently $BN_R(X) \neq \phi$. $X$ is said to be R-*definable* if and only if $\underline{R}X = \bar{R}X$, or $BN_R(X) = \phi$.

### 2.2 Properties of Approximations

The lower and upper approximations of rough sets have several properties [30]. We shall be using the following four properties in our discussions:

$$\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y) \tag{1}$$

$$\bar{R}(X \cap Y) \subseteq \bar{R}X \cap \bar{R}Y \tag{2}$$

$$\bar{R}(X \cup Y) = \bar{R}(X) \cup \bar{R}(Y) \tag{3}$$

$$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y) \tag{4}$$

The inclusions in (1) and (2) can be proper [30] and also can be extended to a finite number of sets. These results confirm to the obvious observation that, in general knowledge included in a distributed knowledge base is less than the integrated one. That is, in general, dividing the knowledge base into smaller fragments causes loss of information [30]. This leads to the interesting problem of determining the exact circumstances under which there will be no loss of information even if one distributes the knowledge base or equivalently under what circumstances there will definitely be loss of information. The following two theorems [37] establish necessary and sufficient conditions for the inclusions (1) and (2) to be proper. The corollaries derived from these results provide necessary and sufficient conditions for equalities to hold in (1) and (2). Thus answers to the questions raised above have been obtained. We shall find many applications of these results in this chapter.

**Theorem 1.** Let $\{E_1, E_2, E_3, ..., E_n\}$ be the partition of any universe $U$ with respect to an equivalence relation $R$. Then for any finite number of subsets $X_1$, $X_2$, $X_3$, . . ., $X_m$, of $U$,

$$\bigcup_{i=1}^{m} \underline{R}(X_i) \subset \underline{R}(\bigcup_{i=1}^{m} X_i) \tag{5}$$

if and only if there exists at least one $E_j$ such that

$$X_i \cap E_j \subset E_j, \text{ for } i = 1, 2, ..., m \textit{ and } \bigcup_{i=1}^{m} X_i \supseteq E_j \tag{6}$$

**Proof.** The sufficiency follows from the fact that $E_j \not\subset \underline{R}(X_i)$, for $i = 1, 2, ..., m$, but $E_j \subset \underline{R}(\bigcup_{i=1}^{m} X_i)$. Conversely, suppose $\bigcup_{i=1}^{m} \underline{R}(X_i) \subset \underline{R}(\bigcup_{i=1}^{m} X_i)$. As $\underline{R}X$ for any $X$ is the union of $E_j$'s only, there is at least one $E_j$ such that $E_j \subseteq \underline{R}(\bigcup_{i=1}^{m} X_i)$ and $E_j \not\subset \underline{R}(X_i)$ for any $i = 1, 2, ..., m$. So, $E_j \subseteq \bigcup_{i=1}^{m} X_i$, but $E_j \not\subset X_j$, for any $i = 1, 2, ..., m$. Thus $X_i \cap E_j \subset E_j$ and $E_j \subseteq \bigcup_{i=1}^{m} X_i$.

**Corollary 1.** A necessary and sufficient condition for

$$\bigcup_{i=1}^{m} \underline{R}(X_i) = \underline{R}(\bigcup_{i=1}^{m} X_i) \tag{7}$$

is that there exist no $E_j$ such that

$$X_i \cap E_j \subset E_j, i = 1, 2, ..., m \textit{ and } \bigcup_{i=1}^{m} X_i \supseteq E_j. \tag{8}$$

We shall be using the following example to illustrate the results of this sect.

**Example 1.** Let us consider an organization having four different sites. For simplicity in computation we assume that there are 20 employees only in the organization who are distributed over four sites.

Further, suppose that these employees are working on different projects $p_i, i = 1, 2, 3, 4$; irrespective of their branch. Some of the employees are involved in more than one project whereas some are not involved in any of the projects. Let the sets $E_1$, $E_2$, $E_3$, $E_4$ denote employees working at the four sites and $X_1$, $X_2$, $X_3$, $X_4$ be the set of employees working for the projects $p_1, p_2, p_3$ and $p_4$ respectively. Let

$E_1 = \{e_1, e_2, e_3, e_4, e_5\}$
$E_2 = \{e_6, e_7, e_8, e_9, e_{10}\}$
$E_3 = \{e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}$
$E_4 = \{e_{16}, e_{17}, e_{18}, e_{19}, e_{20}\}$

$X_1 = \{e_1, e_2, e_4, e_7, e_{11}, e_{13}, e_{19}\}$
$X_2 = \{e_4, e_7, e_{11}, e_{12}, e_{15}, e_{19}\}$
$X_3 = \{e_4, e_7, e_{11}, e_{16}, e_{18}, e_{19}\}$ and
$X_4 = \{e_4, e_7, e_{11}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}\}$

Let us define a relation R over the set of employees U in the organization as $e_i R e_j$ if and only if both $e_i$ and $e_j$ work in the same branch.

The lower approximation of a set $X_i, i = 1, 2, 3, 4$ here provides the fact whether all the employees in a particular site work in a given project or not. Similarly, the upper approximation of these sets provide the fact whether any employee in a particular site works in a project or not. For example, $\underline{R}X_4 = E_4$, says that all the employees in site 4 work in project 4. Similarly, $\bar{R}X_1 = U$ means that some employees of every site work in project 1.

**Illustration for Corollary 1.** Here, $(\bigcup_{i=1}^{4} X_i) \not\supseteq E_j \; for \; j = 1, 2, 3$ and for $E_4$,

$$(\bigcup_{i=1}^{4} X_i) \supseteq E_4, \text{ but } X_4 \cap E_4 = E_4.$$

So, the conditions of Corollary 1 are satisfied. Hence we must have the equality true.

In fact, we see that

$$\underline{R}(\bigcup_{i=1}^{4} X_i) = E_4 \text{ and } (\bigcup_{i=1}^{4} \underline{R}X_i) = E_4 \; as \; \underline{R}X_1 = \underline{R}X_2 = \underline{R}X_3 = \phi$$

and $\underline{R}X_4 = E_4$.

**Theorem 2.** Let $\{E_1, E_2, ..., E_n\}$ be a partition of any universe $U$ with respect to an equivalence relation $R$. Then for a finite number of subsets $X_1, X_2, ..., X_m$ of $U$, the necessary and sufficient condition for

$$\bar{R}(\bigcap_{i=1}^{m} X_i) \subset \bigcap_{i=1}^{m} \bar{R}(X_i) \tag{9}$$

is that there exists at least one $E_j$ such that

$$X_i \cap E_j \neq \phi \text{ for } i = 1, 2, ..., m \; and \; (\bigcap_{i=1}^{m} X_i) \cap E_j = \phi \tag{10}$$

**Proof.** The sufficiency follows from the fact that

$$E_j \not\subset \bar{R}(\bigcap_{i=1}^{m} X_i) \text{ and } \bar{R}(X_i) \supseteq E_j \text{ for } i = 1, 2, ..., m.$$

Conversely, suppose the conclusion is true. Then for some $E_j$,

$$E_j \subseteq \bigcap_{i=1}^{m} \bar{R}(X_i) \text{ but } E_j \not\subset \bar{R}(\bigcap_{i=1}^{m} X_i).$$

So, $$E_j \subseteq \bar{R}(X_i) \text{ for } i = 1, 2, ..., m \text{ and } E_j \bigcap (\bigcap_{i=1}^{m} X_i) = \phi.$$

That is $$E_j \cap X_i \neq \phi, i = 1, 2, ..., m \text{ and } (\bigcap_{i=1}^{m} X_i) \bigcap E_j = \phi.$$

This completes the proof.

**Corollary 2.** Let $\{E_1, E_2, ..., E_n\}$ be a partition of $U$ with respect to an equivalence relation $R$. Then for any finite number of subsets $X_1, X_2, ..., X_m$ of $U$,

$$\bar{R}(\bigcap_{i=1}^{m} X_i) = \bigcap_{i=1}^{m} \bar{R}(X_i) \tag{11}$$

if and only if there is no $E_j$ such that

$$X_i \cap E_j \neq \phi \ \textit{for} \ i = 1, 2, ..., m \ \textit{and} \ (\bigcap_{i=1}^{m} X_i) \cap E_j = \phi. \tag{12}$$

**Illustration for Corollary 2**

$$\textit{Here} \ \bigcap_{i=1}^{4} X_i = \{e_4, e_7, e_{11}, e_{19}\}. \ \text{ So, } E_j \cap \bigcap_{i=1}^{4} X_i \neq \phi \text{ for } j = 1, 2, 3, 4.$$

Also, $X_i \cap E_j \neq \phi$, for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$. Hence conditions of Corollary 2 are satisfied. Also, we see that

$$\bar{R}(\bigcap_{i=1}^{4} X_i) = U = \bigcap_{i=1}^{4} \bar{R}(X_i).$$

## 3 Rough Equality of Sets

Comparison of sets plays a major role in classical set theory. When we move to the representation of approximate knowledge through rough sets the usual comparisons loose their meaning and in a sense are of no use. To bring about more meaning into such comparisons of rough sets which translate into approximate comparison of knowledge bases, Novotny and Pawlak [23,24,25] introduced three notions of rough equalities (bottom, top and total) and established several of their properties. However, it is mentioned [30] that these properties fail to hold when notions of bottom and top rough equalities are replaced one by the other. We show in this sect. that some of these properties hold under such interchanges and establish suitable conditions under which these interchanges are valid. Some other papers which have dealt with rough equalities are [2,5,8].

Two sets are said to be equal in crisp set theory if and only if they have the same elements. The concept has been extended to define rough equalities of sets by Novotny and Pawlak [23,24,25]. In the next sect. we state these equalities.

### 3.1 Definitions

**Definition 1.** Let $K = (U, \boldsymbol{R})$ be a knowledge base, $X, Y \subseteq U$ and $R \in IND(K)$. We say that

**(i)** Two sets $X$ and $Y$ are *bottom R-equal* ($X{=_B}Y$) if $\underline{R}X = \underline{R}Y$;
**(ii)** Two sets $X$ and $Y$ are *top R-equal* ($X{=_T}Y$) if $\bar{R}X = \bar{R}Y$;
**(iii)** Two sets $X$ and $Y$ are *R-equal* ($X{=}Y$) if ($X{=_B}Y$) and ($X{=_T}Y$); equivalently, $\underline{R}X = \underline{R}Y$ and $\bar{R}X = \bar{R}Y$.

We have dropped the suffix $R$ in the notations to make them look simpler and easy to use. Also the notations used are different from the original ones. This has been done due to non-availability of the original notations in the symbol set. It can be easily verified that the relations bottom R-equal, top R-equal and R-equal are equivalence relations on *P(U)*, the power set of *U*.

The concept of approximate equality of sets refers to the topological structure of the compared sets but not the elements they consist of. Thus, sets having significantly different elements may be rough equal. In fact, if $X =_B Y$ then $\underline{R}X = \underline{R}Y$ and as $X \supseteq \underline{R}X, Y \supseteq \underline{R}Y$, $X$ and $Y$ can differ only in elements of $X - \underline{R}X$ and $Y - \underline{R}Y$. However, taking the example; $U = \{x_1, x_2, ..., x_8\}$ and $R = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}\}$, we see that the two sets $X = \{x_1, x_3, x_5\}$ and $Y = \{x_2, x_4, x_6\}$ are top R-equal, even though $X \cap Y = \phi$.

As noted by Pawlak [30, p.26], this concept of rough equality of sets is of relative character, that is things are equal or not equal from our point of view depending on what we know about them. So, in a sense the definition of rough equality refers to our knowledge about the universe.

### 3.2 Properties of Rough Equalities

The following properties of rough equalities are well known [30].

$$X{=_B}Y \textit{ if and only if } X \cap Y{=_B}X \textit{ and } X \cap Y{=_B}Y. \tag{13}$$

$$X{=_T}Y \textit{ if and only if } X \cap Y{=_T}X \textit{ and } X \cap Y{=_T}Y. \tag{14}$$

$$If\ X =_T X'\ and\ Y =_T Y'\ then\ X \cup Y =_T X' \cup Y'. \tag{15}$$

$$If\ X =_B X'\ and\ Y =_B Y'\ then\ X \cap Y =_B X' \cap Y'. \tag{16}$$

$$If\ X =_T Y\ then\ X \cup -Y =_T U. \tag{17}$$

$$If\ X =_B Y\ then\ X \cap -Y =_B \phi. \tag{18}$$

$$If\ X \subseteq Y\ and\ Y =_T \phi\ then\ X =_T \phi. \tag{19}$$

$$If\ X \subseteq Y\ and\ X =_T U\ then\ Y =_T U. \tag{20}$$

$$X =_T Y\ if\ and\ only\ if\ -X =_B -Y. \tag{21}$$

$$If\ X =_B \phi\ or\ Y =_B U\ then\ X \cap Y =_B \phi. \tag{22}$$

$$If\ X =_T U\ or\ Y =_T U\ then\ X \cup Y =_T U. \tag{23}$$

It has been pointed out that (see for instance [30]) the above properties fail to hold if $=_T$ is replaced by $=_B$ or conversely. However, we have the following observations in connection with this interchange.

**I.** The properties (19) to (23) hold true under the interchange.
That is we have

$$X \subseteq Y\ and\ Y =_B \phi \Rightarrow X =_B \phi. \tag{19'}$$

$$If\ X \subseteq Y\ and\ X =_B U \Rightarrow Y =_B U. \tag{20'}$$

$$X =_B Y\ if\ and\ only\ if\ -X =_T -Y. \tag{21'}$$

$$If\ X =_T \phi\ or\ Y =_T \phi\ then\ X \cap Y =_T \phi\ ,\ and \tag{22'}$$

$$If\ X =_B U\ or\ Y =_B U\ then\ X \cup Y =_B U. \tag{23'}$$

**II.** The properties (17) and (18) holds true under the interchange in the following form:

$$If\ X =_B Y\ then\ X \cup -Y =_B U\ if\ BN_R(Y) = \phi. \tag{17'}$$

$$If\ X =_T Y\ then\ X \cap -Y =_T \phi\ if\ BN_R(Y) = \phi. \tag{18'}$$

**Proof of (17').** $\underline{R}(X \cup -Y) \supseteq \underline{R}(X) \cup \underline{R}(-Y)$
$= \underline{R}(Y) \cup (-\bar{R}(Y))$
$= \underline{R}(Y) \cup (-\underline{R}(-Y)) \cup BN_R(Y)$
$= \underline{R}(Y) \cup (-\underline{R}(Y)) \cap (-BN_R(Y))$
$= \underline{R}(Y) \cup (-\underline{R}(Y)) \cap (\underline{R}Y \cup (-BN_R(Y)))$
$= U \cap (\underline{R}Y \cup (-BN_R(Y)))$
$= \underline{R}Y \cup (-BN_R(Y)))$
$= \underline{R}Y \cup (\underline{R}Y \cup (-\bar{R}Y))$
$= \underline{R}Y \cup (-\bar{R}Y)$
$= (-BN_R(Y))$
$= U.$
So, $X \cup (-Y) {=_B} U$.

**Proof of (18').** $\bar{R}(X \cap -Y) \subseteq \bar{R}(X) \cap \bar{R}(-Y)$
$= \bar{R}(Y) \cap \bar{R}(-Y))$
$= \bar{R}(Y) \cap (-\underline{R}(Y))$
$= \bar{R}(Y) \cap ((-\bar{R}(Y)) \cup (-BN_R(Y)))$
$= (\bar{R}(Y) \cap ((-\bar{R}(Y)))) \cup (\bar{R}Y \cap BN_R(Y))$
$= \phi \cup (\bar{R}Y \cap BN_R(Y))$
$= BN_R(Y)$
$= \phi.$

**III.** (i) The properties (13) and (16) hold under the interchange, if conditions of Corollary 2 hold with m = 2.
(ii) The properties (14) and (15) hold under the interchange, if conditions of Corollary 1 hold with m = 2.
So, we get

$$X {=_T} Y \textit{ if and only if } X \cap Y {=_T} X \textit{ and } X \cap Y {=_T} Y, \tag{13'}$$

$$X {=_B} Y \textit{ if and only if } X \cup Y {=_B} X \textit{ and } X \cup Y {=_B} Y, \tag{14'}$$

$$X {=_B} X' \textit{ and } Y {=_B} Y' \Rightarrow X \cup Y {=_B} X' \cup Y', \tag{15'}$$

$$X {=_T} X' \textit{ and } Y {=_T} Y' \Rightarrow X \cap Y {=_B} X' \cap Y'. \tag{16'}$$

**Proof of (13').** $X {=_T} Y \Rightarrow \bar{R}X = \bar{R}Y \Rightarrow \bar{R}(X \cap Y) = \bar{R}X \cap \bar{R}Y = \bar{R}X = \bar{R}Y$.
So, $X \cap Y {=_B} X$ and $X \cap Y {=_T} Y$. The converse is trivial.

**Proof of (14').** $X {=_B} Y \Rightarrow \underline{R}X = \underline{R}Y \Rightarrow \underline{R}(X \cap Y) = \underline{R}X \cup \underline{R}Y = \underline{R}X = \underline{R}Y$.
So, $X \cap Y {=_T} X$ and $X \cap Y {=_T} Y$. The converse is trivial.

The proofs of (15') and (16') are similar.

## 4 Types of Rough Sets

We have mentioned in the introduction there are four important and different topological characterizations of rough sets called their types. In this sect., we shall start with the introduction of these types. The physical interpretation and intuitive meanings of these types can be found in [30].

Type 1: If $\underline{R}X \neq \phi$ and $\bar{R}X \neq U$, then we say that $X$ is *roughly R-definable.*
Type 2: If $\underline{R}X = \phi$ and $\bar{R}X \neq U$, then we say that $X$ is *internally R-undefinable.*
Type 3: If $\underline{R}X \neq \phi$ and $\bar{R}X = U$, then we say that $X$ is *externally R-undefinable.*
Type 4: If $\underline{R}X = \phi$ and $\bar{R}X = U$, then we say that $X$ is *totally R-undefinable.*

The union and intersection of rough sets have importance from the point of distribution of knowledge base and common knowledge respectively. In this context the study of types of union and intersection of different types of rough sets have significance. For example, if two rough sets are roughly R-definable (Type 1), then there are some objects in the universe which can be positively classified, based on the available information to belong to each these sets. Now, one would like to get information about elements in the universe which can be positively classified to be in both. If the intersection is of Type 1/Type 3, then one can obviously conclude this. On the contrary if the intersection is of Type 2/Type 4, then no such element exists. From the table in sect. 4.1 we see that the intersection is Type 1/Type 2. So, it can not be said definitely that the element is in both. In fact this matches with our normal observation. Similarly, for such sets there are some other elements which can be negatively classified without any ambiguity as being outside the sets. Now, what can one say about the union of two such sets ? That is, are there are still some elements which can be negatively classified without any ambiguity being outside the union of their elements ? If the type of the union is Type 1/Type 2, then we are sure of such elements. On the other hand if it is of Type 3/Type 4 no such elements exist. From the table in sect. 4.2 we see that the union is of Type 1/Type 3. So, one can not be sure about some elements being negatively classified as outside the union. This again matches with our observation. In this sect. we shall produce general results on the types of union and intersection of rough sets of different types. We shall also try to reduce the ambiguities in the possible cases under suitable conditions through establishment of theorems.

So far nothing has been said in the literature regarding the type of a rough set which is obtained as union or intersection of different types of rough sets. In the next two sub-sections we obtain the results of union and intersection of any two types of rough sets. This study was initiated in [37].

### 4.1 Intersection

In the next sub-section we establish and present in a table, the results of intersection of two rough sets of different types. It is interesting to note that out of sixteen cases, as many as nine are unambiguous. The other ambiguous cases are

**Table 1.** Intersection of different types of rough sets

| $\bigcap$ | **Type1** | **Type2** | **Type3** | **Type4** |
|---|---|---|---|---|
| **Type1** | Type1/Type2 | Type2 | Type1/Type2 | Type2 |
| **Type2** | Type2 | Type2 | Type2 | Type2 |
| **Type3** | Type1 / Type2 | Type2 | Type1 to Type4 | Type2/Type4 |
| **Type4** | Type2 | Type2 | Type2 / Type4 | Type2 / Type4 |

mainly due to the inclusion (2). Applying Theorem 2 above, some of the ambiguities of the table can be reduced or removed under suitable conditions which are provided by the theorem. These conditions being of necessary and sufficient type, cannot be improved further.

**Proofs**

We shall denote the entry in $i^{th}$ row and $j^{th}$ column of the table by $(i, j)$. In the proofs, we shall be using (2) and the property that for any two rough sets $X$ and $Y$

$$\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y \tag{24}$$

We shall provide the proofs for the cases (1,2) and (3,3). The rest of the proofs can be worked out similarly.

**Proof of** (1,2)
Here, $X$ is of Type 1 and $Y$ is of Type 2. So $\underline{R}X \neq \phi, \bar{R}X \neq U$ and $\underline{R}Y = \phi, \bar{R}Y \neq U$. Hence by (24) $\underline{R}(X \cap Y) = \phi$ , and by (2) $\bar{R}(X \cap Y) \subseteq \bar{R}X \cap \bar{R}Y \neq U$. So, $X \cap Y$ is of Type 2.

**Proof of** (3,3)
Let both $X$ and $Y$ be of Type 3.

Then $\underline{R}X \neq \phi$, $\bar{R}X = U$ and $\underline{R}Y \neq \phi$, $\bar{R}Y = U$. Now, by (24) $\underline{R}(X \cap Y)$ may or may not be $\phi$ and by (2) $\bar{R}(X \cap Y)$ may or may not be U. $X \cap Y$ can be of any of the four Types.

**Examples**

In this sect. we provide examples to show that the ambiguous cases in the table can actually arise for (3). The other cases can be justified similarly. We continue with the same example of sect. 3.

**Examples for (3,3)**
Let $X = \{e_1, e_2, ...., e_{10}, e_{14}, e_{19}\}$ and $Y = \{e_4, e_9, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{17}\}$. Then X and Y are of Type 3 as $\underline{R}X = E_1 \cup E_2$, $\bar{R}X = E_1 \cup E_2 \cup E_3 \cup E_4$,

**Table 2.** Union of different types of rough sets

| ⋃ | **Type1** | **Type2** | **Type3** | **Type4** |
|---|---|---|---|---|
| **Type1** | Type1/Type3 | Type1/Type3 | Type3 | Type3 |
| **Type2** | Type1/Type3 | Type1 to Type4 | Type3 | Type3/Type4 |
| **Type3** | Type3 | Type3 | Type3 | Type3 |
| **Type4** | Type3 | Type3/Type4 | Type3 | Type3/Type4 |

$\underline{R}Y = E_3$ and $\bar{R}Y = E_1 \cup E_2 \cup E_3 \cup E_4$. But $X \cap Y = \{e_4, e_9, e_{14}\}$. So that $\underline{R}(X \cap Y) = \phi$, $\bar{R}(X \cap Y) = E_1 \cup E_2 \cup E_3$ and hence, $X \cap Y$ is of Type 2.

Again, considering $X = \{e_1, e_2, ...., e_{10}, e_{14}, e_{19}\}$ and $Y = \{e_1, e_2, e_7, e_{14}, e_{20}\}$, both X and Y are of Type 3 as $\underline{R}X = E_1 \cup E_2$, $\bar{R}X = E_1 \cup E_2 \cup E_3 \cup E_4$, $\underline{R}Y = E_1$ and $\bar{R}Y = E_1 \cup E_2 \cup E_3 \cup E_4$. But $X \cap Y = \{e_1, e_2, , e_7, e_{14}\}$. So that $\underline{R}(X \cap Y) = E_1$ and $\bar{R}(X \cap Y) = E_1 \cup E_2 \cup E_3$. Hence, $X \cap Y$ is of Type 1.

Also, taking $X = \{e_1, e_2, ...., e_{10}, e_{14}, e_{19}\}$ and $Y = \{e_4, e_9, e_{14}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}\}$, both X and Y are of Type 3 as $\underline{R}X = E_1 \cup E_2$, $\bar{R}X = E_1 \cup E_2 \cup E_3 \cup E_4$, $\underline{R}Y = E_4$ and $\bar{R}Y = E_1 \cup E_2 \cup E_3 \cup E_4$. But $X \cap Y = \{e_4, e_9, e_{14}, e_{19}\}$. So that $\underline{R}(X \cap Y) = \phi$ and $\bar{R}(X \cap Y) = E_1 \cup E_2 \cup E_3 \cup E_4$. Hence, $X \cap Y$ is of Type 4.

Finally, taking $X = \{e_1, e_2, ..., e_{10}, e_{14}, e_{19}\}$ and $Y = \{e_1, e_6, ..., e_{10}, e_{11}, e_{16}, ..., e_{20}\}$, both X and Y are of Type 3 as $\underline{R}X = E_1 \cup E_2$, $\bar{R}X = E_1 \cup E_2 \cup E_3 \cup E_4$, $\underline{R}Y = E_2 \cup E_4$ and $\bar{R}X = E_1 \cup E_2 \cup E_3 \cup E_4$. But $X \cap Y = \{e_1, e_6, e_7, e_8, e_9, e_{10}, e_{19}\}$. So that $\underline{R}(X \cap Y) = E_2$ and $\bar{R}(X \cap Y) = E_1 \cup E_2 \cup E_3 \cup E_4$. Hence, $X \cap Y$ is of Type 3.

### 4.2 Union

In this sub-sect. we establish and present in a table, the results of union of two rough sets of different types. Like the cases of intersection, here also nine cases are unambiguous. The other ambiguous cases are mostly due to the inclusion (1). Applying Theorem 1 above, some of the ambiguities in the table can be reduced or removed under suitable conditions which are provided by the theorem. These conditions being of necessary and sufficient type, cannot be improved further.

#### Proofs

We shall denote the entry in $i^{th}$ row and $j^{th}$ column of the table by $(i, j)$ to represent the different possible cases. In the proof, we shall be using (1) and the property that for any two rough sets $X$ and $Y$

$$\bar{R}(X \cup Y) = \bar{R}X \cup \bar{R}Y \tag{25}$$

We shall provide the proof for the cases (1,2) and (2,2). The rest of the proofs can be worked out similarly.

**Proof of** (1,2)
Let $X$ be of Type 1 and $Y$ be of Type 2. Then $\underline{R}X \neq \phi$,$\bar{R}X \neq U$ and $\underline{R}Y = \phi$, $\bar{R}Y \neq U$. So, by (1) $\underline{R}(X \cup Y)$ is not $\phi$. But, by (25), $\bar{R}(X \cup Y)$ may or may not be U. So, $X \cup Y$ can be of Type 1 or of Type 3.

**Proof of** (2,2)
Let $X$ and $Y$ be of Type 2. Then $\underline{R}X = \phi, \bar{R}X \neq U$ and $\underline{R}Y = \phi, \bar{R}Y \neq U$. By (1) $\underline{R}(X \cup Y)$ may or may not be $\phi$ and by (25) $\bar{R}(X \cup Y)$ may or may not be U. So, $X \cup Y$ can be of any of the four Types.

**Examples**

Below, we provide examples to show that all the possibilities in the ambiguous cases can actually arise for (2,2). The other cases can be justified similarly. We continue with the same Example of sect. 3.

**Examples for (2,2)**
Let $X = \{e_4, e_9, e_{14}\}$ and $Y = \{e_9, e_{14}, e_{19}\}$. Then both X and Y are of Type 2 as $\underline{R}X = \phi$, $\bar{R}X = E_1 \cup E_2 \cup E_3$, $\underline{R}Y = \phi$ and $\bar{R}Y = E_2 \cup E_3 \cup E_4$. But $X \cup Y = \{e_4, e_9, e_{14}, e_{19}\}$. So that $\underline{R}(X \cup Y) = \phi$, $\bar{R}(X \cup Y) = E_1 \cup E_2 \cup E_3 \cup E_4 = U$ and hence, $X \cup Y$ is of Type 4.

Again considering $X = \{e_4, e_9, e_{14}\}$ and $Y = \{e_4, e_9\}$, both X and Y are of Type 2 as $\underline{R}X = \phi$, $\bar{R}X = E_1 \cup E_2 \cup E_3$, $\underline{R}Y = \phi$ and $\bar{R}Y = E_1 \cup E_2$. But $X \cup Y = \{e_4, e_9, e_{14}\}$. So that $\underline{R}(X \cup Y) = \phi$ and $\bar{R}(X \cup Y) = E_1 \cup E_2 \cup E_3$. Hence, $X \cup Y$ is of Type 2.

Also, taking $X = \{e_4, e_9, e_{14}\}$ and $Y = \{e_6, e_7, e_8, e_{10}, e_{14}\}$, both X and Y are of Type 2 as $\underline{R}X = \phi$, $\bar{R}X = E_1 \cup E_2 \cup E_3$, $\underline{R}Y = \phi$ and $\bar{R}Y = E_2 \cup E_3$. But $X \cup Y = \{e_4, e_6, e_7, e_8, e_9, e_{10}, e_{14}\}$. So that $\underline{R}(X \cup Y) = E_2$ and $\bar{R}(X \cup Y) = E_1 \cup E_2 \cup E_3$. Hence, $X \cup Y$ is of Type 1.

Finally, taking $X = \{e_4, e_9, e_{14}\}$ and $Y = \{e_4, e_6, e_7, e_8, e_{10}, e_{19}\}$, both X and Y are of Type 2 as $\underline{R}X = \phi$, $\bar{R}X = E_1 \cup E_2 \cup E_3$, $\underline{R}Y = \phi$ and $\bar{R}X = E_1 \cup E_2 \cup E_4$. But $X \cup Y = \{e_4, e_6, e_7, e_8, e_9, e_{10}, e_{14}, e_{19}\}$. So that $\underline{R}(X \cup Y) = E_2$ and $\bar{R}(X \cup Y) = E_1 \cup E_2 \cup E_3 \cup E_4$. Hence, $X \cup Y$ is of Type 3.

### 4.3 Application of Theorems 1 and 2

As we have seen in sect. 3, there are a number of ambiguous entries in the union and intersection tables. However, if the conditions of corollaries 1 and 2 are satisfied, equalities hold in (1) and (2) and as a result the number of ambiguities decreases. This provides a much more convenient and improved situation. The conditions being of necessary and sufficient types cannot be improved further, under the circumstances.

**Table for Intersection**

As observed above, there were seven ambiguous cases in the table for intersection. Now, if hypotheses of Corollary 2 are satisfied with m = 2, then the number

**Table 3.** Intersection of different types of rough sets after applying Corollary 2

| $\bigcap$ | **Type 1** | **Type 2** | **Type 3** | **Type 4** |
|---|---|---|---|---|
| **Type 1** | Type 1/Type 2 | Type 2 | Type 1/Type 2 | Type 2 |
| **Type 2** | Type 2 | Type 2 | Type 2 | Type 2 |
| **Type 3** | Type 1/Type 2 | Type 2 | Type 3/Type 4 | Type 4 |
| **Type 4** | Type 2 | Type 2 | Type 4 | Type 4 |

reduces to four. In the new table presented below, we find that there is no ambiguous entry having all four Types.

**Table for Union**

As in case of intersection, there were seven ambiguous cases in the union table also. Now, if the hypotheses of Corollary 1 are satisfied with m = 2, then the number reduces to four. As in case of intersection, there are no ambiguous entries in the improved table, which we present below.

**Table 4.** Union of different types of rough sets after applying Corollary 1 with $m = 2$

| $\bigcup$ | **Type 1** | **Type 2** | **Type 3** | **Type 4** |
|---|---|---|---|---|
| **Type 1** | Type 1/Type 3 | Type 1/Type 3 | Type 3 | Type 3 |
| **Type 2** | Type 1/Type 3 | Type 2/Type 4 | Type 3 | Type 4 |
| **Type 3** | Type 3 | Type 3 | Type 3 | Type 3 |
| **Type 4** | Type 3 | Type 4 | Type 3 | Type 4 |

## 5 Rough Equivalence of Sets

A new concept of rough equivalence is to be introduced in this sect. As mentioned in the introduction, this concept captures approximate equality of sets at a higher level than rough equality. In parallel to rough equalities (bottom, top and total) we shall deal with three corresponding types of rough equivalences. Obviously, these concepts deal with topological structures of the lower and upper approximations of the sets. The rough equalities depend upon the elements of the approximation sets but on the contrary rough equivalences depend upon only the structure of the approximation sets. As shall be evident from the definitions, rough equalities (bottom, top and total) imply the corresponding rough equivalences (bottom, top and total) but the converse is not true. However, we shall see through a real life example that the new concepts are very much used by us to infer imprecise information.

### 5.1 Definitions

**I.** We say that two sets $X$ and $Y$ are *bottom R-equivalent* if and only if both $\underline{R}X$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together (we write, $X$ is b_eqv. to $Y$). We put the restriction here that for bottom R-equivalence of $X$ and $Y$ either both $\underline{R}X$ and $\underline{R}Y$ are equal to $U$ or none of them is equal to $U$.

**II.** We say that two sets $X$ and $Y$ are *top R-equivalent* if and only if both $\bar{R}X$ and $\bar{R}Y$ are $U$ or not $U$ together (we write, $X$ is t_eqv. to $Y$). We put the restriction here that for top R-equivalence of $X$ and $Y$ either both $\bar{R}X$ and $\bar{R}Y$ are equal to $\phi$ or none of them is equal to $\phi$.

**III.** We say that two sets $X$ and $Y$ are *R-equivalent* if and only if $X$ and $Y$ are bottom $R$-equivalent and top $R$-equivalent (we write, $X$ is eqv. to $Y$). We would like to note here that when two sets $X$ and $Y$ are R-equivalent, the restrictions in **I** and **II** become redundant.

For example, in case **I**, if one of the $\underline{R}X$ and $\underline{R}Y$ are equal to $U$ then the corresponding upper approximation must be $U$ and for rough equivalence it is necessary that the other upper approximation must also be $U$. Similarly, the other case.

### 5.2 Elementary Properties

**I.** It is clear from the definition above that in all cases (bottom,top,total) $R$-equality implies $R$-equivalence.

**II.** Obviously, the converses are not true.

**III.** Bottom $R$-equivalence, top $R$-equivalence and $R$-equivalence are equivalence relations on P(U).

**IV.** The concept of approximate equality of sets refers to the topological structure of compared sets but not to the elements they consist of.

If two sets are roughly equivalent then by using our present knowledge, we may not be able to say whether two sets are approximately equal as described above, but, we can say that they are approximately equivalent. That is both the sets have or not have positive elements with respect to $R$ and both the sets have or not have negative elements with respect to $R$.

### 5.3 Example 2

Let us consider all the cattle in a locality as our universal set $U$. We define a relation $R$ over $U$ by $xRy$ if and only if $x$ and $y$ are cattle of the same kind. Suppose for example, this equivalence relation decomposes the universe into disjoint equivalence classes as given below.

$C = \{\text{Cow}, \text{Buffalo}, \text{Goat}, \text{Sheep}, \text{Bullock}\}$.

Let $P_1$ and $P_2$ be two persons in the locality having their set of cattle represented by $X$ and $Y$.

We cannot talk about the equality of $X$ and $Y$ in the usual sense as the cattle can not be owned by two different people.

Similarly we can not talk about the rough equality of $X$ and $Y$ except the trivial case when both the persons do not own any cattle.

We find that rough equivalence is a better concept which can be used to decide the equality of the sets $X$ and $Y$ in a very approximate and real sense.

There are four different cases in which we can talk about equivalence of $P_1$ and $P_2$.

**Case I.** $\bar{R}X, \bar{R}Y$ are not $U$ and $\underline{R}X, \underline{R}Y$ are $\phi$. That is $P_1$ and $P_2$ both have some kind of cattle but do not have all cattle of any kind in the locality. So, they are equivalent.

**Case II.** $\bar{R}X, \bar{R}Y$ are not $U$ and $\underline{R}X, \underline{R}Y$ are not $\phi$. That is $P_1$ and $P_2$ both have some kind of cattle and also have all cattle of some kind in the locality. So, they are equivalent.

**Case III.** $\bar{R}X, \bar{R}Y$ are $U$ and $\underline{R}X, \underline{R}Y$ are $\phi$. That is $P_1$ and $P_2$ both have all kinds of cattle but do not have all cattle of any kind in the locality. So, they are equivalent.

**Case IV.** $\bar{R}X, \bar{R}Y$ are $U$ and $\underline{R}X, \underline{R}Y$ are not $\phi$. That is $P_1$ and $P_2$ both have all kinds of cattle and also have all cattle of some kind in the locality. So, they are equivalent.

There are two different cases under which we can talk about the non - equivalence of $P_1$ and $P_2$.

**Case V.** One of $\bar{R}X$ and $\bar{R}Y$ is $U$ and the other one is not. Then, out of $P_1$ and $P_2$ one has cattle of all kinds and other one dose not have so. So, they are not equivalent. Here the structures of $\underline{R}X$ and $\underline{R}Y$ are unimportant.

**Case VI.** Out of $\underline{R}X$ and $\underline{R}Y$ one is $\phi$ and other one is not. Then, one of $P_1$ and $P_2$ does not have all cattle of any kind, whereas the other one has all cattle of some kind. So, they are not equivalent. Here the structures of $\bar{R}X$ and $\bar{R}Y$ are unimportant.

It may be noted that we have put the restriction for top rough equivalence that in the case when $\bar{R}X$ and $\bar{R}Y$ are not equal to $U$, it should be the case that both are $\phi$ or not $\phi$ together. It will remove the cases when one set is $\phi$ and the other has elements from all but one of the equivalence classes but does not have all the elements of any class completely being rough equivalent. Taking the example into consideration it removes cases like when a person has no cattle being rough equivalent to a person, who has some cattle of every kind except one.

Similarly, for bottom rough equivalence we have put the restriction that when $\underline{R}X$ and $\underline{R}Y$ are not equal to $\phi$, it should be the case that both are $U$ or not $U$ together.

## 5.4 General Properties

In this sect. we establish some properties of rough equivalences of sets, which are parallel to those stated in sect. 4.2. Some of these properties hold, some are partially true and some do not hold at all. For those properties, which do hold partially or do not hold at all, we shall provide some sufficient conditions for the conclusion to be true. Also, we shall verify the necessity of such conditions. The sufficient conditions depend upon the concepts of different rough inclusions (Pawlak [30], p.27) and rough comparabilities which we introduce below.

**Definition 2.** Let $K = (U, \boldsymbol{R})$ be a knowledge base, $X, Y \subseteq U$ and $R \in IND(K)$. Then

**(i)**We say that $X$ is *bottom R-included* in $Y(X \sqsubseteq_{BR} Y)$ if and only if $\underline{R}X \subseteq \underline{R}Y$.

**(ii)**We say that $X$ is *top R-included* in $Y(X \sqsubseteq_{TR} Y)$ if and only if $\bar{R}X \subseteq \bar{R}Y$.

**(iii)**We say that $X$ is *R-included* in $Y(X \sqsubseteq_R Y)$ if and only if $X \sqsubseteq_{BR} Y$ and $X \sqsubseteq_{TR} Y$.

We shall drop the suffixes $R$ from the notations above in their use of make them simpler.

**Definition 3**

**(i)** We say $X$, $Y \subseteq U$ are *bottom rough comparable* if and only if $X \sqsubseteq_B Y$ or $Y \sqsubseteq_B X$ holds.

**(ii)** We say $X$, $Y \subseteq U$ are *top rough comparable* if and only if $X \sqsubseteq_T Y$ or $Y \sqsubseteq_T X$ holds.

**(iii)** We say $X$, $Y \subseteq U$ are *rough comparable* if and only if $X$ and $Y$ are both top rough comparable and bottom rough comparable.

**Property 1**

**(i)**If $X \cap Y$ is *b_eqv* to $X$ and $X \cap Y$ is *b_eqv* to $Y$ then $X$ is *b_eqv* to $Y$.

**(ii)** The converse of (i) is not necessarily true.

**(iii)** The converse is true if in addition $X$ and $Y$ are bottom rough comparable.

**(iv)** The condition in (iii) is not necessary.

**Proof**

**(i)** Since $\underline{R}(X \cap Y)$ and $\underline{R}X$ are $\phi$ or not $\phi$ together and $\underline{R}(X \cap Y)$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together, $\underline{R}(X \cap Y)$ being common we get that $\underline{R}X$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together. Hence $X$ is bottom equivalent to $Y$.

**(ii)** The cases when $\underline{R}X$ and $\underline{R}Y$ are both not $\phi$ but $\underline{R}(X \cap Y) = \phi$ the converse is not true.

**(iii)** We have $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y = \underline{R}X$ or $\underline{R}Y$, as the case may be. Since $X$ and $Y$ are bottom rough comparable.

So,$X \cap Y$ is *b_eqv* to $X$ and $X \cap Y$ is *b_eq* to $Y$.

**(iv)**We provide an example to show that this condition is not necessary. Let us take $U = \{x_1, x_2, .., x_8\}$ and the partition induced by an equivalence relation $R$ be $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}\}$.

Now, for $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{x_3, x_4, x_5, x_6\}$, we have $\underline{R}X = X \neq \phi, \underline{R}Y = Y \neq \phi, X \cap Y = \{x_3, x_4\}$ and $\underline{R}(X \cap Y) = \{x_3, x_4\} \neq \phi$. So, $X \cap Y$ is *b_eqv* to both $X$ and $Y$. But $X$ and $Y$ are not bottom rough comparable.

**Property 2**

**(i)** If $X \cup Y$ is *t_eqv* to $X$ and $X \cup Y$ is *t_eqv* to $Y$ then $X$ is *t_eqv* to $Y$.

**(ii)** The converse of (i) may not be true.

**(iii)**A sufficient condition for the converse of (i) to be true is that $X$ and $Y$ are top rough comparable.
**(iv)** The condition in (iii) is not necessary.

**Proof**

**(i)** Similar to part(i) of property 1.

**(ii)** The cases when $\bar{R}(X) \neq U$ and $\bar{R}(Y) \neq U$ but $\bar{R}(X \cup Y) = U$, the converse is not true.

**(iii)** Similar to part(iii) of property 1.

**(iv)** We take the same example as above to show that this condition is not necessary. Here, we have $\bar{R}X = X \neq U, \bar{R}Y = Y \neq U, \bar{R}(X \cup Y) = \{x_1, x_2, x_3, x_4, x_5, x_6\} \neq U$. So, $X$ is *t_eqv* to $Y$.Also, $X \cup Y$ is *t_eqv* to both $X$ and $Y$. But $X$ and $Y$ are not top rough comparable.

**Property 3**

**(i)** If $X$ is *t_eqv* to $X'$ and $Y$ is *t_eqv* to $Y'$ then it may or may not be true that $X \cup Y$ is *t_eqv* to $X' \cup Y'$.

**(ii)** A sufficient condition for the result in (i) to be true is that $X$ and $Y$ are top rough comparable and $X'$ and $Y'$ are top rough comparable.

**(iii)** The condition in (ii) is not necessary for result in (i) to be true.

**Proof**

**(i)** The result fails to be true when all of $\bar{R}(X)$, $\bar{R}(X')$, $\bar{R}(Y)$ and $\bar{R}(Y')$ are not $U$ and exactly one of $X \cup Y$ and $X' \cup Y'$ is $U$.

**(ii)** We have $\bar{R}(X) \neq U$, $\bar{R}(X') \neq U$, $\bar{R}(Y) \neq U$ and $\bar{R}(Y') \neq U$. So, under the hypothesis,$\bar{R}(X \cup Y) = \bar{R}X \cup \bar{R}Y = \bar{R}(X)$ or $\bar{R}(Y)$, which is not equal to $U$. Similarly, $\bar{R}(X' \cup Y') \neq U$. Hence, $X \cup Y$ is *t_eqv* to $X' \cup Y'$.

**(iii)** Continuing with the same example, taking $X = \{x_1, x_2, x_3\}$, $X' = \{x_1, x_2, x_4\}$, $Y = \{x_4, x_5, x_6\}$ and $Y' = \{x_3, x_5, x_6\}$, we find that $\bar{R}X = \{x_1, x_2, x_3, x_4\} = \bar{R}X' \neq U$ and $\bar{R}Y = \{x_3, x_4, x_5, x_6\} = \bar{R}Y' \neq U$. So, $X$ and $Y$ are not top rough comparable. $X'$ and $Y'$ are not top rough comparable. But, $\bar{R}(X \cup Y) = \{x_1, x_2, x_3, x_4, x_5, x_6\} = \bar{R}(X' \cup Y')$. So, $X \cup Y$ is top equivalent to $X' \cup Y'$.

**Property 4**

**(i)** $X$ is *b_eqv* to $X'$ and $Y$ is *b_eqv* to $Y'$ may or may not imply that $X \cap Y$ is *b_eqv* to $X' \cap Y'$.

**(ii)** A sufficient condition for the result in (i) to be true is that $X$ and $Y$ are bottom rough comparable and $X'$ and $Y'$ are bottom rough comparable.

**(iii)** The condition in (ii) is not necessary for result in (i) to be true.

**Proof**

**(i)** When all of $\underline{R}(X)$, $\underline{R}(X')$, $\underline{R}(Y)$ and $\underline{R}(Y')$ are not $\phi$ and exactly one of the $X \cap Y$ and $X' \cap Y'$ is $\phi$, the result fails.

**(ii)** Now, under the hypothesis, we have $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y) = \underline{R}(X)$ or $\underline{R}(Y) \neq \phi$, Similarly, $\underline{R}(X' \cap Y') \neq \phi$. So, $X \cap Y$ is *b_eq* to $X' \cap Y'$.

**(iii)** Continuing with the same example and taking $X = \{x_1, x_2, x_3\}$, $X' = \{x_3, x_4, x_5\}$, $Y = \{x_3, x_7, x_8\}$ and $Y' = \{x_5, x_7, x_8\}$, we find that $\underline{R}X \neq \phi$, $\underline{R}X' \neq \phi$, $\underline{R}Y \neq \phi$ and $\underline{R}Y' \neq \phi$. So, $X$ is *b_eqv* to $X'$ and $Y$ is *b_eqv* to $Y'$. Also, $\underline{R}(X \cap Y) = \phi$ and $\underline{R}(X' \cap Y') = \phi$. So, $X \cap Y$ is *b_eqv* to $X' \cap Y'$. However, $X$ and $Y$ are not bottom rough comparable and so are $X'$ and $Y'$.

**Property 5**
**(i)** $X$ is *t_eqv* to $Y$ may or may not imply that $X \cup (-Y)$ is *t_eqv* to $U$.
**(ii)** A sufficient condition for result in (i) to hold is that $X {=}_B Y$.
**(iii)** The condition in (ii) is not necessary for the result in (i) to hold.

**Proof**
**(i)** The result fails to hold true when $\bar{R}(X) \neq U$, $\bar{R}(Y) \neq U$ and still $\bar{R}(X \cup (-Y)) = U$.
**(ii)** As $X {=}_B Y$, we have $\underline{R}X = \underline{R}Y$. So, $-\underline{R}X = -\underline{R}Y$. Equivalently, $\bar{R}(-X) = \bar{R}(-Y)$. Now, $\bar{R}(X \cup -Y) = \bar{R}(X) \cup \bar{R}(-Y) = \bar{R}(X) \cup \bar{R}(-X) = \bar{R}(X \cup -X) = \bar{R}(U) = U$. So, $X \cup -Y$ is *t_eqv* to $U$.
**(iii)** Continuing with the same example and taking $X = \{x_1, x_2, x_3\}$, $Y = \{x_2, x_3, x_4\}$ we get $-Y = \{x_1, x_5, x_6, x_7, x_8\}$. So that $\underline{R}X = \{x_1, x_2\}$ and $\underline{R}Y = \{x_3, x_4\}$. Hence, it is not true that $X {=}_B Y$. But, $X \cup -Y = \{x_1, x_2, x_3, x_5, x_6,$
$x_7, x_8\}$. So, $\bar{R}(X \cup -Y) = U$. That is, $X \cup -Y t$*_eqv* to $U$.

**Property 6**
**(i)** $X$ is *b_eqv* to $Y$ may or may not imply that $X \cap (-Y)$ is *b_eqv* to $\phi$.
**(ii)** A sufficient condition for the result in (i) to hold true is that $X {=}_T Y$.
**(iii)** The condition in (ii) is not necessary for the result in (i) to hold true.

**Proof**
**(i)** The result fails to hold true when $\underline{R}(X) \neq \phi$, $\underline{R}(Y) \neq \phi$ and $\underline{R}(X) \cap \underline{R}(-Y) = \phi$.
**(ii)** As $X {=}_T Y$, we have $\bar{R}X = \bar{R}Y$. So, $-\bar{R}X = -\bar{R}Y$. Equivalently, $\underline{R}(-X) = \underline{R}(-Y)$. Now, $\underline{R}(X \cap -Y) = \underline{R}(X) \cap \underline{R}(-Y) = \underline{R}(X) \cap \underline{R}(-X) = \underline{R}(X \cap -X) = \underline{R}(\phi) = \phi$. Hence, $X \cap -Y$ is *b_eqv* to $\phi$.
**(iii)** Continuing with the same example by taking $X = \{x_1, x_2, x_3\}$, $Y = \{x_1, x_2, x_5\}$ we have $-Y = \{x_3, x_4, x_6, x_7, x_8\}$. So, $X$ is *b_eqv* to $Y$. But $X$ is not top equal to $Y$. However, $X \cap -Y = \{x_3\}$ and so, $\underline{R}(X \cap -Y) = \phi$. Hence, $X \cap -Y$ is *b_eqv* to $\phi$.

**Property 7.** If $X \subseteq Y$ and $Y$ is *b_eqv* to $\phi$ then $X$ is *b_eqv* to $\phi$.

**Proof.** As $Y$ is *b_eqv* to $\phi$, we have $\underline{R}(Y) = \phi$. So, if $X \subseteq Y$, $\underline{R}(X) \subseteq \underline{R}(Y) = \phi$.

**Property 8.** If $X \subseteq Y$ and $X$ is *t_eqv* to $U$ then $Y$ is *t_eqv* to $U$.

**Proof.** The proof is similar to that of Property 7.

**Property 9.** $X$ is *t_eqv* to $Y$ if and only if $-X$ is *b_eqv* to $-Y$.

**Proof.** The proof follows from the property, $\underline{R}(-X) = -\bar{R}(X)$.

**Property 10.** $X$ is *b_eqv* to $\phi$, $Y$ is *b_eqv* to $\phi \Rightarrow X \cap Y$ is *b_eqv* to $\phi$.

**Proof.** The proof follows directly from the fact that under the hypothesis the only possibility is $\underline{R}(X) = \underline{R}(Y) = \phi$.

**Property 11.** If $X$ is *t_eqv* to $U$ or $Y$ is *t_eqv* to $U$ then $X \cup Y$ is *t_eqv* to $U$.

**Proof.** The proof follows directly from the fact that under the hypothesis the only possibility is $\bar{R}(X) = \bar{R}(Y) = U$.

### 5.5 Replacement Properties

In this sect. we shall consider properties obtained from the properties of sect. 5.4 by interchanging top and bottom rough equivalences. We shall provide proofs whenever these properties hold true. Otherwise, sufficient conditions are to be established under which these properties are valid. In addition, we shall test if such conditions are also necessary for the validity of the properties. Invariably, it has been found that such conditions are not necessary. We shall show it by providing suitable examples.

**Property 12**

**(i)** If $X \cap Y$ is *t_eqv* to $X$ and $X \cap Y$ is *t_eqv* to $Y$ then $X$ is *t_eqv* $Y$.

**(ii)** The converse of (i) is not necessarily true.

**(iii)** A sufficient condition for the converse of (i) to hold true is that conditions of Corollary 2 hold with m = 2.

**(iv)** The condition in (iii) is not necessary.

**Proof**

**(i)** Here $\bar{R}X$ and $\bar{R}(X \cap Y)$ are $U$ or not $U$ together and $\bar{R}Y$ and $\bar{R}(X \cap Y)$ are $U$ or not $U$ together being common, we get $\bar{R}X$ and $\bar{R}(Y)$ are $U$ or not $U$ together. So, $X$ is *t_eqv*$Y$.

**(ii)** The result fails when $\bar{R}X$ and $\bar{R}(X) = U\bar{R}(Y)$ and $\bar{R}(X \cap Y) \neq U$.

**(iii)** Under the hypothesis, we have $\bar{R}(X \cap Y) = \bar{R}(X) \cap \bar{R}(Y)$. If $X$ is *t_eqv* to $Y$ then both $\bar{R}X$ and $\bar{R}Y$ are equal to $U$ or not equal to $U$ together. So, accordingly we get $\bar{R}(X \cap Y)$ equal to $U$ or not equal to $U$. Hence the conclusion follows.

**(iv)** We see that the sufficient condition for the equality to hold when m = 2 in Corollary 2 is that there is no $E_j$ such that $X \cap E_j \neq \phi$, $Y \cap E_j \neq \phi$ and $X \cap Y \cap E_j = \phi$.

Let us take $U$ and the relation as above. Now, taking $X = \{x_1, x_3, x_6\}$, $Y = \{x_3, x_5, x_6\}$. The above sufficiency conditions are not satisfied as $\{x_5, x_6\} \cap$

$X \neq \phi$, $\{x_5, x_6\} \cap Y \neq \phi$ and $\{x_5, x_6\} \cap X \cap Y = \phi$. However, $\bar{R}X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \neq U$.

**Property 13**

**(i)** $X \cup Y$ is *b_eqv* to $X$ and $X \cup Y$ is *b_eqv* to $Y$ then $X$ is *b_eqv* to $Y$.

**(ii)** The converse of (i) is not necessarily true.

**(iii)** A sufficient condition for the converse of (i) to hold true is that the condition of Corollary 1 holds for m = 2.

**(iv)** The condition in (iii) is not necessary.

**Proof**

**(i)** $\underline{R}X$ and $\underline{R}(X \cup Y)$ are $\phi$ or not $\phi$ together and $\underline{R}Y$ and $\underline{R}(X \cup Y)$ are $\phi$ or not $\phi$ together. Since $\underline{R}(X \cup Y)$ is common, $\underline{R}X$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together. So, $X$ is *b_eqv* to $Y$.

**(ii)** Suppose $X$ and $Y$ are such that $\underline{R}X$ and $\underline{R}Y$ are both $\phi$ but $\underline{R}(X \cup Y) \neq \phi$. Then $X$ is *b_eqv* to $Y$ but $X \cup Y$ is not *b_eqv* to any one of $X$ and $Y$.

**(iii)** Suppose $X$ is *b_eqv* to $Y$. Then $\underline{R}X$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together. If the conditions are satisfied then $\underline{R}(X \cup Y) = \underline{R}X \cup \underline{R}Y$. So, if both $\underline{R}X$ and $\underline{R}Y$ are $\phi$ or not $\phi$ together then $\underline{R}(X \cup Y)$ is $\phi$ or not $\phi$ accordingly and the conclusion holds.

**(iv)** Let us take $U$ as above. The classification corresponding to the equivalence relation be given by $\{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \{x_7, x_8\}\}$.

Let $X = \{x_1, x_3, x_6\}$, $Y = \{x_2, x_5, x_6\}$. Then $\underline{R}(X) \neq \phi$, $\underline{R}(Y) \neq \phi$ and $\underline{R}(X \cup Y) \neq \phi$. The condition in (iii) is not satisfied as taking $E = \{x_1, x_2\}$ we see that $X \cap E \subset E$, $Y \cap E \subset E$ and $X \cup Y \supseteq E$.

**Property 14**

**(i)** $X$ is *b_eqv* to $X'$ and $Y$ is *b_eqv* to $Y'$ may not imply $X \cup Y$ is *b_eqv* to $X' \cup Y'$.

**(ii)** A sufficient condition for the conclusion of (i) to hold is that the conditions of corollary 2 are satisfied for both $X$, $Y$ and $X'$, $Y'$ separately with m = 2.

**(iii)** The condition in (ii) is not necessary for the conclusion in (i) to be true

**Proof**

**(i)** When $\underline{R}X$, $\underline{R}Y$,$\underline{R}X'$, $\underline{R}Y'$ are all $\phi$ and out of $X \cup Y$ and $X' \cup Y'$ one is $\phi$ but the other one is not $\phi$, the result fails to be true.

**(ii)** Under the additional hypothesis, we have $\underline{R}(X \cup Y) = \underline{R}X \cup \underline{R}Y$ and $\underline{R}(X' \cup Y') = \underline{R}X' \cup \underline{R}Y'$. Here both $\underline{R}X$ and $\underline{R}X'$ are $\phi$ or not $\phi$ together and both $\underline{R}Y$ and $\underline{R}Y'$ are $\phi$ or not $\phi$ together. If all are $\phi$ then both $\underline{R}(X \cup Y)$ and $\underline{R}(X' \cup Y')$ are $\phi$. So, they are *b_eqv*. On the other hand, if at least one pair is not $\phi$ then we get both $\underline{R}(X \cup Y)$ and $\underline{R}(X' \cup Y')$ are not $\phi$ and so they are *b_eqv*.

**(iii)** The condition is not satisfied means there is $E_i$ with $X \cap E_i \subset E_i$, $Y \cap E_i \subset E_i$ and $X \cup Y \supseteq E_i$; there exists $E_j$ ( not necessarily different from $E_i$) such that $X' \cap E_j \subset E_j$, $Y' \cap E_j \subset E_j$ and $X' \cup Y' \supseteq E_j$.

Let us consider the example, $U = x_1, x_2, ..., x_8$ and the partition induced by an equivalence relation R be $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\{x_7, x_8\}\}$. $X = \{x_1, x_5\}$,

$Y = \{x_3, x_6\}$, $X' = \{x_1, x_4\}$ and $Y' = \{x_3, x_7\}$. Then $\underline{R}X = \underline{R}X' = \underline{R}Y = \underline{R}Y' = \phi$. Also, $\underline{R}(X \cup Y) \neq \phi$, $\underline{R}(X' \cup Y') \neq \phi$. So, $X$ is *b_eqv* to $X'$, $Y$ is *b_eqv* to $Y'$ and $X \cup Y$ is *b_eqv* to $X' \cup Y'$. However, $X' \cap \{x_3, x_4\} \subset \{x_3, x_4\}$, $Y' \cap \{x_3, x_4\} \subset \{x_3, x_4\}$ and $X' \cup Y' \supseteq \{x_3, x_4\}$. So, the condition are not satisfied.

**Property 15**

**(i)** $X$ is *t_eqv* to $X'$ and $Y$ is *t_eqv* to $Y'$ may not necessarily imply that $X \cap Y$ is *t_eqv* to $X' \cap Y'$.

**(ii)** A sufficient condition for the conclusion in (i) to hold is the conditions of corollary 1 are satisfied for both X, Y and X', Y' separately with m = 2.

**(iii)** The condition in (ii) is not necessary for the conclusion in (i) to hold.

**Proof**

**(i)**When $\bar{R}X = \bar{R}X' = \bar{R}Y = \bar{R}Y' = U$ and out of $\bar{R}(X \cap Y)$, $\bar{R}(X' \cap Y')$ one is $U$ whereas the other one is not $U$ the result fails to be true.

**(ii)**If the conditions of corollary 1 are satisfied for X, Y and X', Y' separately then the case when $\bar{R}X = \bar{R}X' = \bar{R}Y = \bar{R}Y' = U$, we have $\bar{R}(X' \cap Y') = \bar{R}X' \cap \bar{R}Y' = U$ and $\bar{R}(X \cap Y) = \bar{R}X \cap \bar{R}Y = U$. In other cases, if $\bar{R}X$ and $\bar{R}X'$ not $U$ or $\bar{R}Y$ and $\bar{R}Y'$ not U then as $\bar{R}(X' \cap Y') \neq U$ and $\bar{R}(X \cap Y) \neq U$. So, in any case $X \cap Y$ and $X' \cap Y'$ are *t_eqv* to each other.

**(iii)** We continue with the same example. The conditions are not satisfied means there is no $E_j$ such that $X \cap E_j \neq \phi$, $Y \cap E_j \neq \phi$ and $X \cap Y \cap E_j = \phi$ or $X' \cap E_j \neq \phi$, $Y' \cap E_j \neq \phi$ and $X' \cap Y' \cap E_j = \phi$. Taking $X = \{x_1, x_5\}$, $Y = \{x_3, x_5\}, X' = \{x_1, x_4\}$ and $Y' = \{x_2, x_4\}$ we have $X \cap \{x_5, x_6\} \neq \phi$, $Y' \cap \{x_5, x_6\} \neq \phi$ and $X \cap Y \cap \{x_5, x_6\} = \phi$. $X' \cap \{x_3, x_4\}, Y' \cap \{x_3, x_4\} \neq \phi$ and $X' \cap Y' \cap \{x_3, x_4\}$. So, the conditions are violated. But $\bar{R}X \neq U$, $\bar{R}X' \neq U$, $\bar{R}Y \neq U$, $\bar{R}Y' \neq U$. So, X is *t_eqv* and Y is *t_eqv* Y'. Also, $\bar{R}(X \cap Y) \neq U$ and $\bar{R}(X' \cap Y') \neq U$. Hence, $X \cap Y$ is *t_eqv* to $X' \cap Y'$.

**Remark**

We would like to make the following comments in connection with the properties 16 to 19, 21 and 22:

**(i)** We know that $\underline{R}U = U$. So, bottom R-equivalent to $U$ can be considered under the case that $\underline{R}U \neq \phi$.

**(ii)** We know that $\bar{R}\phi = \phi$. So, top R-equivalent to $\phi$ can be considered under the case that $\bar{R}\phi \neq U$.

The proofs of the properties 16, 17, 18 and 19 are trivial and we omit them.

**Property 16.** $X$ is *b_eqv* to $Y$ may or may not imply that $X \cup -Y$ is *b_eqv* to $U$.

**Property 17.** $X$ is *t_eqv* to $Y$ may or may not imply that $X \cap -Y$ is *t_eqv* to $\phi$.

**Property 18.** If $X \subseteq Y$ and $Y$ is *t_eqv* to $\phi$ then $X$ is *t_eqv* to $\phi$.

**Property 19.** If $X \subseteq Y$ and $X$ is *b_eqv* to $U$ then $Y$ is *b_eqv* to $U$.

**Property 20.** $X$ is *b_eqv* to $Y$ if and only if $-X$ is *t_eqv* to $-Y$.

**Proof.** Follows from the identity $\bar{R}(-X) = -\bar{R}(X)$.
The proofs of the following two properties are also trivial.

**Property 21.** $X$ is *t_eqv* to $\phi$ and $Y$ is *t_eqv* to $\phi \Rightarrow X \cap Y$ is *t_eqv* to $\phi$.

**Property 22.** $X$ is *b_eqv* to $U$ and $Y$ is *b_eqv* to $U \Rightarrow X \cup Y$ is *b_eqv* to $U$.

## 6 Approximation of Classifications

Approximation of classifications is a simple extension of the definition of approximation of sets. Let $F = \{X_1, X_2, ..., X_n\}$ be a family of non empty sets, which is a classification of $U$ in the sense that $X_i \cap X_j = \phi$ for $i \neq j$ and

$$\bigcup_{i=1}^{n} X_i = U.$$

Then $\underline{R}F = \{\underline{R}X_1, \underline{R}X_2, ..., \underline{R}X_n\}$ and $\bar{R}F = \{\bar{R}X_1, \bar{R}X_2, ..., \bar{R}X_n\}$ are called the R-*lower* and R-*upper approximations* of the family $F$, respectively.

Grzymala-Busse [12] has established some properties of approximation of classifications. But, these results are irreversible in nature. Pawlak [30, p.24] has remarked that these results of Busse establish that the two concepts, approximation of sets and approximation of families of sets (or classifications) are two different issues and the equivalence classes of approximate classifications cannot be arbitrary sets. He has further stated that if we have positive example of each category in the *approximate classification* then we must have also negative examples of each category. In this sect., we further analyze these aspects of theorems of Busse and provide physical interpretation of each one of them by taking a standard example.

One primary objective is to extend the results of Busse by obtaining necessary and sufficient type theorems and show how the results of Busse can be derived from them. The results of Busse we discuss here are in their slightly modified form as presented by Pawlak [30]. Some more work in dealing with incomplete data are due to Busse [13,14].

### 6.1 Theorems on Approximation of Classifications

In this sect., we shall establish two theorems which have many corollaries generalizing the four theorems established by Busse [12] in their modified forms [30]. We shall also provide interpretations for most of these results including those of Busse and illustrate them through a simple example of toys [30].

**Example 3.** Suppose we have a set of toys of different colours red, blue, yellow and different shapes square, circular, triangular. We define the first description as a classification of the set of toys and represent the second description as an equivalence relation $R$. We say for two toys $x$ and $y$, $xRy$ if $x$ and $y$ are of the same shape.

We shall use the following notations for representational convenience :

$N_n = \{1, 2, ..., n\}$ and for any $I \subset N_n$, by $I^c$ we mean the complement of $I$ in $N_n$.

**Theorem 3.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. Then for any $I \subset N_n$,

$$\bar{R}(\bigcup_{i \in I}(X_i)) = U \ if \ and \ only \ if \ \underline{R}(\bigcup_{j \in I^c}(X_j)) = \phi.$$

**Proof.** We have

$$\underline{R}(\bigcup_{j \in I^c}(X_j)) = \phi \Leftrightarrow \underline{R}(U - \bigcup_{i \in I^c}(X_i)) = \phi \Leftrightarrow -\bar{R}\bigcup_{i \in I}(X_i) = \phi \Leftrightarrow \bar{R}(\bigcup_{i \in I}(X_i)) = U.$$

This completes the proof.

**Corollary 3.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. Then for $I \subset N_n$,

$$if \ \bar{R}(\bigcup_{i \in I}(X_i)) = U \ then \ \underline{R}X_j = \phi \ for \ each \ j \in I^c.$$

**Proof.** By the above theorem, using the hypothesis we get

$$\underline{R}(\bigcup_{j \in I^c}(X_j)) = \phi.$$

As

$$\underline{R}X_j \subseteq \underline{R}(\bigcup_{i \in I^c}(X_j))$$

for each $j \in I^c$, the conclusion follows.

**Interpretation**

Suppose, in a classification of a universe, there is no negative element for the union of some elements of the classification taken together with respect to an equivalence relation. Then for all other elements of the classification there is no positive element with respect to the equivalence relation. Referring to the example, if we have circular or triangular toys of all different colours then all the toys of no particular colour are rectangular in shape.

**Corollary 4.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and $R$ be an equivalence relation on it. Then for each $i \in N_n$, $\bar{R}X_i = U$ if and only if

$$\underline{R}(\bigcup_{j \neq i}(X_j) = \phi.$$

**Proof.** Taking $I = \{i\}$, in Theorem 3 we get this.

**Corollary 5.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. Then for each $i \in N_n$, $\underline{R}X_i = \phi$ if and only if

$$\bar{R}(\bigcup_{j \neq i} X_j) = U.$$

**Proof.** Taking $I = \{i\}^c$, in Theorem 3 we get this.

**Corollary 6.** [30, Proposition 2.6 ] Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. If there exists $i \in N_n$ such that $\bar{R}X_i = U$ then for each $j$ other than $i$ in $N_n$, then $\underline{R}X_j = \phi$.

**Proof.** From Corollary 4., $\bar{R}X_i = U$

$$\Rightarrow \underline{R}(\bigcup_{j \neq i} X_j) = \phi.$$

$$\Rightarrow \underline{R}X_j = \phi \text{ for each } j \neq i.$$

**Interpretation**
Suppose in a classification of a universe, there are positive elements of one member of the classification with respect to a equivalence relation. Then there are negative elements of all other members of the classification with respect to the equivalence relation.

Taking the above example into consideration if all red toys are of triangular shape (say) then for toys of circular and rectangular shape at least one colour is absent.

**Corollary 7.** [30, proposition 2.8] Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on it. If for all $i \in N_n$, $\bar{R}X_i = U$ holds then $\underline{R}X_i = \phi$ for all $i \in N_n$.

**Proof.** If for some $i$, $1 \leq i \leq n$, $\underline{R}X_i \neq \phi$, then by Corollary 6 $\bar{R}X_j \neq U$ for some $j(\neq i)$ in $N_n$ ; which is a contradiction.

This completes the proof.

**Interpretation**
Suppose in a classification of a universe, there is no negative element of one member of the classification with respect to an equivalence relation. Then for all other members of the classifications there is no positive element with respect to the equivalence relation.

Referring to the example, if there are triangular toys of all different colours then for any other shape (circular or rectangular) all the toys of no particular colour are of that shape.

**Theorem 4.** Let $F = \{X_1, X_2, ..., X_n\}$ be a partition of $U$ and $R$ be an equivalence relation on $U$. Then for any $I \subset N_n$,

$$\underline{R}(\bigcup_{i \in I} X_i) \neq \phi \;\; if \; and \; only \; if \bigcup_{j \in I^C} \bar{R}(X_j) \neq U.$$

**Proof.** *(Sufficiency)* By property of lower approximation,

$$\bar{R}(\bigcup_{j \in I^c} X_j) = (\bigcup_{j \in I^c} \bar{R}X_j) \neq U.$$

So, there exists $[x]_R$ for some $x \in U$ such that

$$[x]_R \cap (\bigcup_{j \in I^c} X_j) = \phi.$$

Hence,

$$\underline{R}(\bigcup_{i \in I} X_i) \neq \phi.$$

(Necessity) Suppose,

$$\underline{R}(\bigcup_{i \in I} X_i) \neq \phi.$$

Then there exists $x \in U$ such that

$$[x]_R \subseteq (\bigcup_{i \in I} X_i).$$

Thus, $[x]_R \cap X_j = \phi$ for $j \notin I$. So, $x \notin \bar{R}X_j$, for $j \notin I$. Hence

$$(\bigcup_{j \in I^c} \bar{R}X_j) \neq U.$$

**Corollary 8.** Let $F = \{X_1, X_2, ..., X_n$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. Then for $I \subset N_n$,

$$if \; \underline{R}(\bigcup_{i \in I} X_i) \neq \phi \;\; then \; \bar{R}X_j \neq U \; for \; each \; j \in I^c.$$

**Proof.** By Theorem 4,

$$\underline{R}(\bigcup_{i \in I} X_i) \neq \phi$$

$$\Rightarrow (\bigcup_{j \in I^c} \bar{R}X_j) \neq U$$

$$\Rightarrow \bar{R}X_j \neq U \text{ for each } j \in I^c.$$

This completes the proof.

**Interpretation**
Suppose in a classification of a universe, there are positive elements for the union of some elements of the classification taken together with respect to an equivalence relation. Then for all other elements of the classification there are negative elements with respect to the equivalence relation. Referring to the same example, if all toys of red colour are rectangular or triangular in shape then circular toys of at least one colour is absent.

**Corollary 9.** Let $F = \{X_1, X_2, ..., X_n\}$ be a partition of $U$ and $R$ be an equivalence relation on $U$. Then for each $i \in N_n$,

$$\underline{R}X_i \neq \phi \ if \ and \ only \ if \ (\bigcup_{j \neq i} \bar{R}X_j) \neq U.$$

**Proof.** Taking $I = \{i\}$ in Theorem 4 we get this.

**Corollary 10.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and $R$ be an equivalence relation on $U$. Then for all $i$, $1 \leq i \leq n$, $\bar{R}X_i \neq U$ if and only if

$$\underline{R}(\bigcup_{j \neq i} X_j) \neq \phi.$$

**Proof.** Taking $I = \{i\}^C$ in Theorem 4. we get this. Also, this result can be obtained as a contrapositive of Corollary 9.

**Corollary 11.** [30, proposition 2.5] Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. If there exist $i \in N_n$ such that $\underline{R}X_i \neq \phi$ then for each $j(\neq i) \in N_n$, $\underline{R}X_j \neq U$.

**Proof.** By Corollary 9,

$$\underline{R}X_i \neq \phi \Rightarrow (\bigcup_{j \neq i} \bar{R}X_j) \neq U \Rightarrow \bar{R}X_j \neq U,$$

for each $j \neq i$, $1 \leq i \leq n$.

**Interpretation**
Suppose in a classification of a universe, there are positive elements of one member of classification with respect to an equivalence relation. Then there are negative elements of all other numbers of the classification with respect to equivalence relation. Taking the example into consideration if all red toys are of triangular shape (say) then for toys for circular or rectangular shape at least one colour is absent.

**Corollary 12.** [30, proposition 2.7] Let $F = \{X_1, X_2, ..., X_n\}, n > 1$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. If for all $i \in N_n$, $\underline{R}X_i \neq \phi$ holds then $\bar{R}X_i \neq U$ for all $i \in N_n$.

**Proof.** As $\underline{R}X_i \neq \phi$ for all $i \in N_n$, we have

$$\underline{R}(\bigcup_{j \neq i} X_j) \neq \phi \textit{ for all } i \in N_n. \textit{ So, by Corollary } 10\ \bar{R}X_i \neq U \textit{ for all } i \in N_n.$$

**Interpretation**
Suppose in a classification, there is a positive element in each member of the classification with respect to an equivalence relation. Then there is a negative element in each member of the classification with respect to the equivalence relation.

Referring to the example, if all toys of red colour are triangular, all the toys of green colour are circular and all the toys of blue colour are rectangular in shape then there is no green colour toy of triangular shape and so on.

## 7 Some Properties of Classifications

In this sect. we shall establish some properties of measures of uncertainty [12] and discuss in detail on properties of classifications with two elements and three elements.

### 7.1 Measures of Uncertainty

The following definitions are taken from Grzymala-Busse [12].

**Definition 4.** Let $F = \{X_1, X_2, ..., X_n\}$ be a classification of $U$ and $R$ be an equivalence relation on $U$. Then the *accuracy of approximation* of $F$ by $R$, denoted by $\beta_R(F)$ and is defined as

$$\beta_R(F) = (\sum_{i=1}^{n} \mid \underline{R}X_i \mid)/(\sum_{i=1}^{n} \mid \bar{R}X_i \mid). \tag{26}$$

**Definition 5.** Let $F$ and $R$ be as above. Then the *quality of approximation of* $F$ by $R$ is denoted by $\gamma_R(F)$ and is defined as

$$\gamma_R(F) = (\sum_{i=1}^{n} \mid \underline{R}X_i \mid)/ \mid U \mid . \tag{27}$$

The accuracy of classification expresses the percentage of possible correct decision when classifying objects employing the knowledge $R$. The quality of classification expresses the percentage of objects which can be correctly classified to classes of $F$ employing knowledge $R$.

Let $R_1$ and $R_2$ be any two equivalence relations on $U$. $F_1$ and $F_2$ be the classification of $U$ generated by $R_1$ and $R_2$ respectively.

**Definition 6**

**(i)** We say that $R_2$ *depends in degree* $k$ on $R_1$ in $U$ and denote it by

$$R_1 \longrightarrow^k R_2 \ \ \textit{if and only if} \ \ \gamma_{R_1}(F_2) = k. \tag{28}$$

**(ii)**We say that $R_2$ *totally depends* on $R_1$ in $U$ if and only if $k = 1$.
**(iii)** We say that $R_2$ *roughly depends* on $R_1$ in $U$ if and only if $0 < k < 1$.
**(iv)** We say that $R_2$ *totally independent* on $R_1$ in $U$ if and only if $k = 0$.
**(v)** We say $F_2$ *depends in degree* $k$ on $F_1$ in $U$, written as

$$F_1 \longrightarrow^k F_2 \;\; if\ and\ \ only\ if \;\; R_1 \longrightarrow^k R_2.$$

**Property 23.** For any R-definable classification $F$ in $U$, $\beta_R(F) = \gamma_R(F) = 1$ So, if a classification $F$ is R-definable then it is totally independent on $R$.

**Proof.** For all R-definable classifications $F$,$\underline{R}F = \bar{R}F$. So, by definition $\beta_R(F) = 1$. Again, by property of upper approximation and as $F$ is a classification of $U$, we have

$$\sum_{i=1}^{n} | \bar{R}X_i | \geq \sum_{i=1}^{n} | X_i | = | \bigcup_{i=1}^{n} X_i | = | U | .$$

Also,

$$\sum_{i=1}^{n} | \underline{R}X_i | \leq \sum_{i=1}^{n} | X_i | = | \bigcup_{i=1}^{n} X_i | = | U | .$$

But, for R-definable classifications

$$\sum_{i=1}^{n} | \underline{R}X_i | = \sum_{i=1}^{n} | \bar{R}X_i | .$$

Hence,

$$\sum_{i=1}^{n} | \underline{R}X_i | = | U | .$$

So, we get $\gamma_R(F) = 1$.

**Property 24.** For any classification $F$ in $U$ and an equivalence relation $R$ on $U$, $\beta_R(F) \leq \gamma_R(F) \leq 1$.

**Proof.** Since $| \underline{R}X_i | \leq | X_i |$, we have

$$\sum_{i=1}^{n} | \underline{R}X_i | \leq \sum_{i=1}^{n} | X_i | = | U | .$$

So, $\gamma_R(F) \leq 1$. Again, as shown above,

$$\sum_{i=1}^{n} | \bar{R}X_i | \geq | U | .$$

Hence,

$$\beta_R(F) = (\sum_{i=1}^{n} | \underline{R}X_i |)/(\sum_{i=1}^{n} | \bar{R}X_i |) \leq (\sum_{i=1}^{n} | \underline{R}X_i |)/ | U | = \gamma_R(F).$$

### 7.2 Classification Types

In this sect. we present Types of classifications and their rough definability as stated by Busse [12]. As mentioned, classifications are of great interest in the process of learning from examples, rules are derived from classifications generated by single decisions.

**Definition 7.** Let $R$ be an equivalence relation on $U$. A classification $F = \{X_1, X_2, ..., X_n\}$ of $U$ will be called *roughly R-definable, weak* in $U$ if and only if there exists a number $i \in N_n$ such that $\underline{R}X_i \neq \phi$.

It can be noted from Corollary 9 that for a R-definable weak classification $F$ of $U$, there exists $j(\neq i) \in N_n$ such that $\bar{R}X_j \neq U$.

**Definition 8.** Let $R$ and $F$ be as above. Then $F$ will be called *roughly R-definable strong* in $U$ (Type 1) if $\underline{R}X_i \neq \phi$ and only if $i \in N_n$ for each. By Corollary, in roughly R-definable strong classification in $U$, $\bar{R}X_i \neq U$ for each $i \in N_n$.

**Definition 9.** Let $R$ and $F$ be as above. Then $F$ will be called *internally R-undefinable weak* in $U$ if and only if $\underline{R}X_i = \phi$ for each $i \in N_n$ and there exists $j \in N_n$ such that $\bar{R}X_j \neq U$.

**Definition 10.** Let $R$ and $F$ be as above. Then $F$ will be called *internally R-undefinable strong* in $U$ (Type 2) if and only if $\underline{R}X_i = \phi$ and $\bar{R}X_i \neq U$ for each $i \in N_n$.

It has been observed by Busse [12] that due to Corollary 10 no *externally P-undefinable* set $X$ exists in $U$. So, extension of Types of rough sets, classified on Type 3 is not possible to the case of classifications.

**Definition 11.** Let $R$ and $F$ be as above. Then $F$ will be called *totally R-undefinable* in $U$ (Type 4) if and only if $\underline{R}X_i = \phi$ and $\bar{R}X_i = U$ for each $i \in N_n$.

### 7.3 Classifications with Two or Three Elements

As remarked by Pawlak [30], approximation of sets and approximation of families of sets are two different issues and equivalence classes of approximate classification cannot be arbitrary sets, although they are strongly related. The concepts of compliments in case of sets and in case of classifications are different, which may lead to new concepts of negation in the case of binary and multivalued logics.

In this sect. we shall analyze the structure and properties of classifications having 2 elements and classifications having 3 elements. This may shed some light on the above statement of Pawlak. We shall use $T - i$ to represent $Type - i$, i=1,2,3,4 from this sect. onwards.

**Classifications with Two Elements**

Let $\chi = \{X_1, X_2\}$. Then $X_2 = X_1^C$. Since complements of T-1/T-4 rough sets are T-1/T-4 respectively and T-2/T-3 rough sets have complements of T-3/T-2 respectively, ([30], proposition 2.4), out of 16 possibilities for $\chi$ with respect to Typing only four alternates are possible. Namely, {T-1,T-1}, {T-2, T-3}, {T-3, T-2} and {T-4, T-4}. Again, the second and third possibilities are similar. So, there are only three distinct alternates. Hence, we have.

**Property 25.** A classification with two elements is roughly R-definable weak or of T-1 or of T-4 only.

**Classifications with Three Elements**

In a classification with 3 elements, say $\{X_1, X_2, X_3\}$ there are supposed to be 64 possibilities. But we shall show that only 8 of these possibilities can actually occur and other possibilities are not there.

**Property 26.** In a classification $F = \{X_1, X_2, X_3\}$ of $U$ there are 8 possibilities for $F$ with respect to Types of $X_1, X_2, X_3$. These are {T-1,T-1, T-1}, {T-1,T-1,T-2}, {T-1,T-2,T-2}, {T-2,T-2,T-2}, {T-2,T-2,T-4}, {T-2,T-4,T-4}, {T-3,T-2,T-2} and { T-4,T-4,T-4}.

**Proof.** We shall consider four cases depending upon the Type of $X_1$.
**Case 1.** $X_1$ is of T-1. Then $X_2 \cup X_3$ being compliment of $X_1$, must be of T-1. So, from the table of sect. 5.2, three cases arise for $X_2$ and $X_3$, that is {T-1,T-1}, {T-1,T-2}, and {T-2,T-2}.

**Case 2.** $X_1$ is of T-2. Complement of T-2 being Type -3, $X_2 \cup X_3$ is of T-3. Now, from the table of sect. 5.2, there are nine cases for $X_2$ and $X_3$. Out of these {T-1,T-1} and {T-1,T-2} have been covered in Case 1. {T-1,T-3} cannot occur as {T-2 ∪ T-3} = T-3 and $(\text{T-1})^C$ = T-1. Similarly, {T-1,T-4} cannot occur as {T-2 ∪ T-4} = T-3 and $(\text{T-4})^C$ = T-4. So, four cases remains {T-2,T-2}, {T-2,T-4} and {T-4,T-4}.

**Case 3.** $X_1$ is of T-3, $X_2 \cup X_3$ must be of T-2. Referring to the table of sect. 5.2. There is only one possibility for $X_2$, $X_3$ that is {T-2,T-2} which has been covered in Case 2.

**Case 4.** $X_1$ is of T-4. Then $X_2 \cup X_3$ must be of T-4. Referring to the table of sect. 5.2, there are three possibilities for $X_2$ and $X_3$. Out of these cases {T-2,T-2} and {T-2,T-4} have been considered in Case 2. So, only one case remains {T-4,T-4}.

This completes the proof of the property.

## 7.4 Further Types of Classifications

First we present two theorems which shows that the hypothesis in the theorems of Busse, as presented in Corollary 7 and Corollary 12 can be further relaxed to get the conclusions. However, even these hypothesis are to be shown as not necessary for the conclusions to be true.

**Theorem 5.** Let $F = \{X_1, X_2, ..., X_n\}$, where $n > 1$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. If there exists $p$ and $q$, $1 \leq p$, $q \leq n$ and $p \neq q$ such that $\underline{R}X_p \neq \phi$, $\underline{R}X_q \neq \phi$ then for each $i \in N_n$, $\bar{R}X_i \neq U$.

**Proof.** Since $\underline{R}X_p \neq \phi$, by Corollary 11, $\bar{R}X_i \neq U$ for $i \neq p$ and since $\underline{R}X_q \neq \phi$, by the same Corollary, $\bar{R}X_i \neq U$ for $i \neq q$. So, from these two we get $\bar{R}X_i \neq U$ for all $i$ as $p \neq q$.

**Note 1.** The above condition is not necessary. Let us consider $U = \{x_1, x_2, ..., x_8\}$ and $R$ be an equivalence relation on $U$ with equivalence classes $X_1 = \{x_1, x_3, x_5\}$, $X_2 = \{x_2, x_4\}$ and $X_3 = \{x_6, x_7, x_8\}$. Then taking the classification $\{Z_1, Z_2, Z_3\}$ defined by $Z_1 = \{x_2, x_4\}$, $Z_2 = \{x_1, x_3, x_6\}$ and $Z_3 = \{x_5, x_7, x_8\}$, we find that $\bar{R}Z_1 \neq U$, $\bar{R}Z_2 \neq U$, $\bar{R}Z_3 \neq U$. But $\bar{R}Z_1 \neq \phi$, $\bar{R}Z_2 \neq \phi$, $\bar{R}Z_3 \neq \phi$.

**Theorem 6.** $F = \{X_1, X_2, ..., X_n\}$, where $n > 1$ be a classification of $U$ and let $R$ be an equivalence relation on $U$. If there exists $p$ and $q$ $1 \leq p, q \leq n$ and $p \neq q$ such that $\bar{R}X_p = \bar{R}X_q = U$ then for each $i \in N_n$, $\underline{R}X_i = \phi$.

**Proof.** Since $\bar{R}X_p = U$, by Corollary 6, $\underline{R}X_i = \phi$ for $i \neq p$ and since $\underline{R}X_q = U$, by the same Corollary, $\underline{R}X_i = \phi$ for $i \neq q$. So, from these two we get $\underline{R}X_i = \phi$ for all $i$ as $p \neq q$.

**Note 2.** The above condition is not necessary. Let us consider $U$, $R$ and $X_1$, $X_2$ and $X_3$ as in the above note. We take the classification defined by $Z_1, Z_2, Z_3$ defined by $Z_1 = \{x_2, x_6\}$, $Z_2 = \{x_1, x_3, x_4\}$ and $Z_3 = \{x_5, x_7, x_8\}$. We find that $\underline{R}Z_1 = \phi$, $\underline{R}Z_2 = \phi$ and $\underline{R}Z_3 = \phi$ whereas $\bar{R}Z_1 \neq U$, $\bar{R}Z_2 \neq U$ and $\bar{R}Z_3 \neq U$.

**Observation 1.** In Corollary 11, we have $\bar{R}X_j \neq U$ for all $j \neq i$ if $\exists X_i$ such that $\underline{R}X_i \neq \phi$. It is easy to observe that $\bar{R}X_i$ may or may not be $U$ under the circumstances.

**Observation 2.** In Corollary 6, we have $\underline{R}X_j = \phi$ for all $j \neq i$ if $\exists X_i$ such that $\bar{R}X_i = U$. It is easy to observe that $\underline{R}X_i$ may or may not be $\phi$ under the circumstances.

For any classification $F = \{X_1, X_2, ..., X_n\}$ of $U$ we have the following possibilities with respect to lower and upper approximations.

Basing upon the above table, possible combinations for classifications are $(i, j)$; $i = 1, 2, 3, 4$ and $j = 5, 6, 7, 8$.

Out of these, several cases have been considered by Busse [12]. We shall examine all the possibilities closely. In fact we have the following table of combinations.

We shall be using the following abbreviations in Table 6:

Roughly R-definable weak 2 = RRdW2
Internally R-undefinable weak 2 = IRudW2
Internally R-undefinable weak 3 = IRudW3
Roughly R-definable weak 1 = RRdW1
Internally R-definable weak 1 = IRdW1
Totally R-undefinable weak 3 = TRudW3
Externally R-undefinable = ERud
Totally R-undefinable weak 1 = TRudW1

**Table 5.** Possibilities w.r.t. lower and upper approximations

| F | **Lower** $\neq \phi$ | **Lower** $= \phi$ | **Upper** $\neq U$ | **Upper** $= U$ |
|---|---|---|---|---|
| $\forall$ | 1 | 2 | 5 | 6 |
| $\exists$ | 3 | 4 | 7 | 8 |

**Table 6.** Possible combinations

| | **5** | **6** | **7** | **8** |
|---|---|---|---|---|
| **1** | T-1 | Not Possible | T-1 | Not Possible |
| **2** | T-2 | T-4 | IRudW3 | TRudW3 |
| **3** | RRdW2 | Not Possible | RRdW1 | ERud |
| **4** | IRudW2 | T-4 | IRdW1 | TRudW1 |

The cases (1,6) and (1,8) are not possible by Corollary 12. The case (3,6) is not possible by Corollary 7. The case (1,7) reduces to (1,5) by Corollary 12. The case (1,5) has been termed as roughly R-definable strong classification by Busse and we call it T-1 as all the elements of the classifications are of T-1. So far as row-1 of the table is concerned, the only possible classification is roughly R-definable strong or T-1.

The case (2,5) has been termed by Busse as internally R-undefinable strong. We call it T-2 as all the elements of the classifications are of T-2. The case (2,6) has been termed as totally R-undefinable by Busse and we call it T-4 as all the elements of the classification are of T-4. The case (2,7) has been termed as internally R-undefinable weak by Busse.

**The Characterisation.** We have the following conventions in connection with types of classifications:

**(I)** *Internal definability:* Lower approximation $\neq \phi$
**(II)** *Internal undefinibility:* Lower approximation $= \phi$

**(III)** *External definibility:* Upper approximation $\neq U$
**(IV)** *External undefinability:* Upper approximation $= U$

Also, from the set of elements in a classification, if we have $\exists$ some element satisfying a typing property, it leads to a weak type. On the other hand, if a typing property is true $\forall$ element, it leads to a strong type.

So, we have the following general types of classifications.

**(I)** *Roughly R-definable* $\Leftrightarrow$ Internally R-definable and Externally R-definable
**(II)** *Internally R-undefinable* $\Leftrightarrow$ Internally R-undefinable and Externally R-definable
**(III)** *Externally R-undefinable* $\Leftrightarrow$ Internally R-definable and Externally R-undefinable
**(IV)** *Totally R-undefinable* $\Leftrightarrow$ Internally R-undefinable and Externally R-undefinable

In case (I) we have one strong type, we call it T-1. This is the case when $\forall i$, $\underline{R}X_i \neq \phi$ and $\forall j$, $\underline{R}X_j \neq U$.

Also there are two weak types. We set them as:

**(i)** *Roughly R-definable* (weak -1) if and only if $\exists i$, $\underline{R}X_i \neq \phi$ and $\exists j$, $\underline{R}X_j \neq U$ and
**(ii)** *Roughly R-definable* (weak -2) if and only if and $\exists i$, $\underline{R}X_i \neq \phi$ and $\forall j$, $\underline{R}X_j \neq U$.

In case (II) we have one strong type, we call it T-2. This is the case when $\forall i$, $\underline{R}X_i \neq \phi$ and $\forall j$, $\bar{R}X_j \neq U$.

Also there are three weak types. We set them as:

**(i)** *Internally R-definable* (weak -1) if and only if $\exists i$, $\underline{R}X_i = \phi$ and $\exists j$, $\bar{R}X_j \neq U$.
**(ii)** *Internally R-definable* (weak -2) if and only if $\exists i$, $\underline{R}X_i = \phi$ and $\forall j$, $\bar{R}X_j \neq U$.
**(iii)** *Internally R-definable* (weak -3) if and only if $\forall i$, $\underline{R}X_i = \phi$ and $\exists j$, $\bar{R}X_j \neq U$.

In case (III) we have one strong type, we call it Externally R-undefinable only as there is no weak type possible in this case. This is the case when $\exists i$, $\underline{R}X_i \neq \phi$ and $\exists j$, $\bar{R}X_j = U$.

In case (IV) we have one strong type, we call it T-4. This is the case when $\forall i$, $\underline{R}X_i = \phi$ and $\exists j$, $\bar{R}X_j = U$.

Also there are two weak types. We set them as

**(i)** *Totally r-undefinable* (weak -1) if and only if $\exists i$, $\underline{R}X_i = \phi$ and $\exists j$, $\bar{R}X_j = U$.
**(ii)** *Totally R-undefinable* (weak -2) if and only if $\forall i$, $\underline{R}X_i = \phi$ and $\exists j$, $\bar{R}X_j = U$.

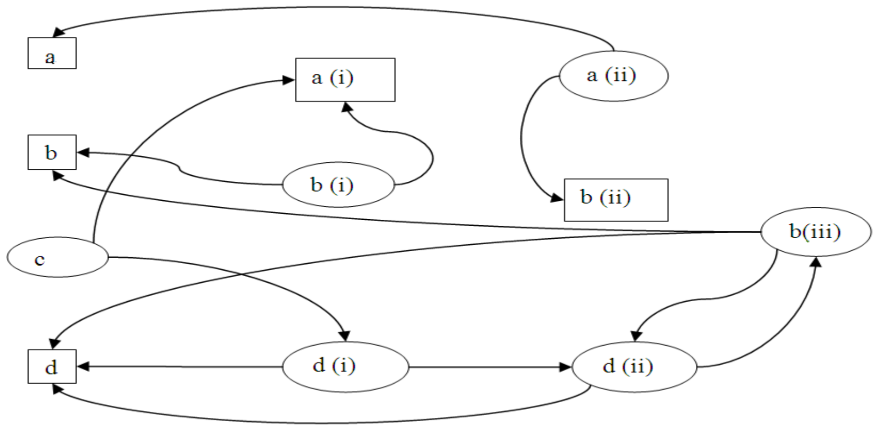


**Fig. 1.** A schematic representation of Busse's cases

Out of these eleven possibilities only five have been considered by Busse [12]. In Fig. 1, we represent the cases considered by Busse inside rectangles and those not considered by him inside ellipses. The arrows show the reduction of the six cases not considered by Busse to those considered by him either directly or transitively.

## 7.5 Application

A new approach to knowledge acquisition under uncertainty based on rough set theory was presented by Busse [12]. The real world phenomena are represented

**Table 7.** An example of inconsistent information system

| Q | $c_1$ | $c_2$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | $v_1$ | $w_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_2$ | $v_1$ | $w_2$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $x_3$ | $v_1$ | $w_1$ | 0 | 0 | 0 | 1 | 2 | 1 |
| $x_4$ | $v_1$ | $w_2$ | 1 | 0 | 0 | 1 | 2 | 1 |
| $x_5$ | $v_2$ | $w_2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $x_6$ | $v_2$ | $w_2$ | 0 | 1 | 1 | 1 | 1 | 1 |
| $x_7$ | $v_3$ | $w_1$ | 1 | 1 | 0 | 0 | 1 | 0 |
| $x_8$ | $v_3$ | $w_1$ | 1 | 1 | 1 | 1 | 2 | 1 |
| $x_9$ | $v_3$ | $w_1$ | 1 | 1 | 1 | 1 | 2 | 2 |

by information system, where inconsistencies are included. For this he considered the example of opinion of six doctors $d_1$, $d_2$, $d_3$, $d_4$, $d_5$ and $d_6$ on nine patients $x_1$, $x_2$,..., $x_9$ based upon the result of two tests $c_1$ and $c_2$. On the basis of values of tests, experts classify patients as being on some level of disease. The information system is represented in a tabular form, which is clearly inconsistent.

The classification, generated by the set $C$ of conditions is equal to $\{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}\}$.

If we denote the classification $X_i$ generated by the opinion of doctor $d_i$, $i = 1, 2, 3, 4, 5, 6$ then

$X_1 = \{\{x_1, x_3, x_5, x_6\}, \{x_2, x_4, x_7, x_8, x_9\}\}$,
$X_2 = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9\}\}$,
$X_3 = \{\{x_1, x_2, x_3, x_4, x_5, x_7\}, \{x_6, x_8, x_9\}\}$,
$X_4 = \{\{x_1, x_2, x_5, x_7\}, \{x_3, x_4, x_6, x_8, x_9\}\}$,
$X_5 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_8, x_9\}, \{x_6, x_7\}\}$
and
$X_6 = \{\{x_1, x_2, x_5, x_7\}, \{x_3, x_4, x_6, x_8\ \{x_9\}\}$

It is easy to see that the above classifications are of type C-definable, roughly C-definable strong, roughly C-definable weak, totally C-undefinable, internally C-undefinable and internally C-undefinable weak respectively.

## 8 Rule Generation

By rules on information systems we mean conditional statements that specify actions under conditions. The following notations are used :

Constants: 0,1
Atomic expression: $a := v \equiv \{\rho(x, a) = v : x \in U\}$
Boolean Operations: $\neg, \vee, \wedge$
$0 \equiv$ Empty set.
$1 \equiv U$.

### 8.1 Definitions

**(i)** Rules computed from lower approximations are *certain rules.*
**(ii)** Rules computed from upper approximations are *possible rules.*
The following properties hold for rule generation.

**I.** Necessary and sufficient condition for a classification $\chi$ to induce certain rules is $\underline{C}\chi \neq \phi$.
**II.** The number of certain rules is equal to the number of non-empty lower approximations in the classification.
**III.** The number of possible rules is equal to the number of non-empty boundaries in the classification.

### 8.2 Observations

**I.** For C-definable classifications, all the rules are certain rules.
**II.** For roughly C-definable strong and roughly C-definable weak classifications both certain and possible rules exist.
**III.** For totally C-undefinable, internally C-undefinable strong and internally C-undefinable weak classifications there are no certain rules.
**IV.** For roughly C-definable strong sets the number of certain rules is equal to the number of elements in the classification.
**V.** All types of classifications other than C-definable classifications have the property that there is at least one possible rule.
**VI.** For roughly C-definable weak classifications there is at least one certain rule.
**VII.** For totally C-undefinable classifications, there is no certain rule. The number of possible rules is equal to the number of elements in the classification.
**VIII.** For intrenally C-undefinable strong classifications, there is no certain rule. The number of possible rules is at most equal to the number of elements in the classification.
**IX.** For internally C-undefinable weak classifications, there is no certain rule. There is no guarantee about the existence of possible rules.

### 8.3 Examples

Let us see how some certain and possible rules can be generated from the example 7.5.

**(I)**$X_1$ is C-definable and hence all the rules corresponding to it are certain rules. In fact, the rules are
**(i)**$((c_1 = v_1) \wedge (c_2 = w_1)) \vee ((c_1 = v_2) \wedge (c_2 = w_2)) \Rightarrow (d_1 = 0)$ and
**(ii)** $((c_1 = v_1) \wedge (c_2 = w_2)) \vee ((c_1 = v_3) \wedge (c_2 = w_1)) \Rightarrow (d_1 = 1)$

**(II)**$X_2$ is roughly C-definable strong. So, it has both type of rules,
**(i)**$((c_1 = v_1) \Rightarrow (d_2 = 0)$ (Certain rule) and
**(ii)** $((c_1 = v_2) \wedge (c_2 = w_2)) \Rightarrow (d_2 = 0)$ (Possible rule).

**(III)**$X_5$ is internally C-undefinable strong. So, it has no certain rules. As it has three elements, by Observation VIII it can have at most three possible rules. In fact the rules are
**(i)**$(c_1 = v_1) \vee (c_1 = v_2) \vee (c_2 = w_2) \Rightarrow (d_5 = 0)$
**(ii)** $((c_1 = v_2) \vee (c_1 = v_3) \Rightarrow (d_5 = 1)$ and
**(iii)** $(c_1 = v_1) \vee (c_1 = v_3) \vee (c_2 = w_1) \Rightarrow (d_5 = 2)$.

## 9 Rough Equivalence of Algebraic Rules

We have several algebraic properties with respect to the set theoretic operations of union, intersection and complementation. Ordinary equality when the sets involved are taken to be rough sets bears little meaning and does not comply with common sense reasoning. So, rough equality or rough equivalence seems to be a possible solution. In this sect. we continue with rough equivalence and verify the validity of rough equivalence of left and right hand sides of these properties. This study was initiated in [39].

### 9.1 Associative Rule

The two Associative laws for crisp sets are:
For any three sets $A$, $B$ and $C$,

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{29}$$

and

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{30}$$

Now, it is interesting to verify whether the left and right hand side of (29) and (30) match with their Types. For this, we consider four different cases depending upon Types of A with B and C being of any of the four Types. We take it as case $i$, when A is of T-$i$, $i = 1, 2, 3, 4$. It is observed that in all these cases the left hand side and right hand side of the above equalities match with each other as is evident from the corresponding tables. First we consider the four cases for union and than for intersection. Tables 1 and 2 are used to derive the tables below.

**Union**

**Table 8.** Union: case 1

| $\bigcup$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-3 | T-1/T-3 | T-3 | T-3 |
| **T-2** | T-1/T-3 | T-1/T-3 | T-3 | T-3 |
| **T-3** | T-3 | T-3 | T-3 | T-3 |
| **T-4** | T-3 | T-3 | T-3 | T-3 |

**Table 9.** Union: case 2

| $\bigcup$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-3 | T-1/T-3 | T-3 | T-3 |
| **T-2** | T-1/T-3 | T-1/T-2/T-3/T-4 | T-3 | T-3/T-4 |
| **T-3** | T-3 | T-3 | T-3 | T-3 |
| **T-4** | T-3 | T-4 | T-3 | T-3/T-4 |

**Table 10.** Union: case 3

| $\bigcup$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-3 | T-3 | T-3 | T-3 |
| **T-2** | T-3 | T-3 | T-3 | T-3 |
| **T-3** | T-3 | T-3 | T-3 | T-3 |
| **T-4** | T-3 | T-3 | T-3 | T-3 |

**Table 11.** Union: case 4

| $\bigcup$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-3 | T-3 | T-3 | T-3 |
| **T-2** | T-3 | T-3/T-4 | T-3 | T-3/T-4 |
| **T-3** | T-3 | T-3 | T-3 | T-3 |
| **T-4** | T-3 | T-3/T-4 | T-3 | T-3/T-4 |

**Table 12.** Intersection: case 1

| $\bigcap$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-2 | T-2 | T-1/T-2 | T-2 |
| **T-2** | T-2 | T-2 | T-2 | T-2 |
| **T-3** | T-1/T-2 | T-2 | T-1/T-2 | T-2 |
| **T-4** | T-2 | T-2 | T-2 | T-2 |

**Table 13.** Intersection: case 2

| $\bigcap$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-2 | T-2 | T-2 | T-2 |
| **T-2** | T-2 | T-2 | T-2 | T-2 |
| **T-3** | T-2 | T-2 | T-2 | T-2 |
| **T-4** | T-2 | T-2 | T-2 | T-2 |

**Table 14.** Intersection: case 3

| $\bigcap$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-2 | T-2 | T-1/T-2 | T-2 |
| **T-2** | T-2 | T-2 | T-2 | T-2 |
| **T-3** | T-1/T-2 | T-2 | T-1/T-2/T-3/T-4 | T-2/T-4 |
| **Type 4** | T-2 | T-2 | T-2/T-4 | T-2/T-4 |

**Table 15.** Intersection: case 4

| $\bigcap$ | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-2 | T-2 | T-2 | T-2 |
| **T-2** | T-2 | T-2 | T-2 | T-2 |
| **T-3** | T-2 | T-2 | T-2/T-4 | T-2/T-4 |
| **T-4** | T-2 | T-2 | T-2/T-4 | T-2/T-4 |

**Table 16.** Double negations for different types of rough sets

| A | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| $(A)^C$ | T-1 | T-3 | T-2 | T-4 |
| $((A)^C)^C$ | T-1 | T-2 | T-3 | T-4 |

**Intersection**

### 9.2 Complement and Double Negation

The Types of complement of rough sets of different Types have been obtained by (Pawlak [30], Theorem 2.4). Using this, it is easy to compute the double negations of different Types of rough sets as provided in the following table and see that the complementation law holds for rough equivalence.

### 9.3 De Morgan's Theorems

De Morgan's Theorems for crisp sets state that for any two sets $X$ and $Y$,

$$(X \cup Y)^C = (X)^C \cap (Y)^C \tag{31}$$

and

$$(X \cap Y)^C = (X)^C \cup (Y)^C \tag{32}$$

Also, when both X and Y are rough sets of different types we observe that both sides of (31) and (32) are rough equivalent as is evident from the following tables.

Table for both the sides of (31) is:

**Table 17**

**Table 17.** De Morgan's union for different types of rough sets

| | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-2 | T-1/T-2 | T-2 | T-2 |
| **T-2** | T-1/T-2 | T-1/T-2/T-3/T-4 | T-2 | T-2/T-4 |
| **T-3** | T-2 | T-2 | T-2 | T-2 |
| **T-4** | T-2 | T-2/T-4 | T-2 | T-2/T-4 |

Table for both the sides of (32) is:

**Table 18.** De Morgan's intersection for different types of rough sets

| | **T-1** | **T-2** | **T-3** | **T-4** |
|---|---|---|---|---|
| **T-1** | T-1/T-3 | T-3 | T-1/T-3 | T-3 |
| **T-2** | T-3 | T-3 | T-3 | T-3 |
| **T-3** | T-2 | T-2 | T-2 | T-2 |
| **T-4** | T-3 | T-3 | T-3/T-4 | T-3/T-4 |

### 9.4 Distributive Property

The two distributive properties for crisp sets state that, for any three sets $A$, $B$ and $C$,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{33}$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{34}$$

We have the following observations with respect to the rough equivalence of Left hand side and Right hand side of (33) and (34):

When A is T-2, the left hand side and right hand side of (33) are rough equivalent and the case is similar for (34) when A is of T-3.

In other cases, we have following observations:

**(i)** When A is of T-1, left hand side of (33) is of T-1 or T-3, whereas right hand side can be any of the four types. The result remains same even by using our Corollaries 1 and 2.

So left hand side and right hand side are not rough equivalent of any kind.

When A is of T-1, left hand side of (34) is of T-1 or T-2, whereas right hand side can be any of the four types.

The result remains unchanged even by using our Corollaries. So left hand side and right hand side are not rough equivalent of any kind.

**(ii)** When A is of T-2, both left hand sides and right hand sides of (33) can be any of the four types.

So, they are not rough equivalent.

When A is of T-2, left hand side of (34) is of T-2, whereas right hand side can be any of the four types.

So, left hand sides and right hand sides of (34) are not rough equivalent.

However, left hand side is Bottom Rough equivalent to its right hand side when condition of Corollaries 1 and 2 are satisfied.

**(iii)** When A is of T-3, left hand side of (33) is of T-3, whereas right hand side can be any of the four types.

So, left hand sides and right hand sides are not rough equivalent.

However, left hand side is Top Rough equivalent to its right hand side when condition of Corollaries 1 and 2 are satisfied.

When A is of T-3, left hand side and right hand side of (34) can be of any of the four types.

So, left hand sides and right hand sides are not rough equivalent.

**(iv)** When A is of T-4, left hand side of (33) is of T-3 or T-4, whereas right hand side can be any of the four types.

So, again left hand sides and right hand sides of (34) are not rough equivalent.

However, left hand side is Top Rough equivalent to its right hand side when Corollaries 1 and 2 are used.

When A is of T-4, left hand side of (34) is of T-2 or T-4, whereas right hand side can be any of the four types.

However, left hand side is Top Rough equivalent to its right hand side when Corollaries 1 and 2 are used. So, in general the left hand side and right hand side of distributive properties are not rough equivalent.

## 9.5 Idempotent and Absorption Property

### Idempotent Property

The two idempotent properties for crisp sets state that for any set $A$,

$$A \cap A = A \tag{35}$$

and

$$A \cup A = A \tag{36}$$

When A is a Rough set, it is clear from the diagonal entries of the union and intersection table in Sect. 3 that (35) holds good with Type matching only when A is of T-2. For rest of types, the left hand side is not Rough equivalent to its right hand side. However we observe that in (35), for A of T-1, the left hand side is Top Rough equivalent to its right hand side. For A of T-3, the left hand side is Top Rough equivalent to its right hand side when conditions of Corollary 2 are satisfied. For A of T-4, the left hand side is Bottom Rough equivalent to its right hand side.

When A is a Rough set, the left hand side and right hand side of (36) are rough equivalent only when A is of T-3 and for rest of the types, the left hand side is not Rough equivalent to its right hand side. However we observe that in (36), for A of T-1, the left hand side is Bottom Rough equivalent to its Right Hand Side. For A of T-2, the left hand side is Bottom Rough equivalent to its right hand side, when conditions of Corollary 1 are satisfied. For A of T-4, the left hand side is Top Rough equivalent to its right hand side.

### Absorption Property

The two absorption properties for crisp sets state that for any two sets $A$ and $B$

$$A \cup (A \cap B) = A \tag{37}$$

and

$$A \cap (A \cup B) = A \tag{38}$$

Taking A and B as Rough sets, we find that when A is of T-3, both the sides of (37) are of T-3 and when A is of T-2, both the sides of (38) are of T-2. Hence, the left hand side and rough hand side are rough equivalent. In the rest of the cases left hand side and right hand sides of (37) and (38) are not rough equivalent. In fact the following properties hold good:

**(i)** When A is of T-1, left hand side of (37) is of T-1 or T-3.

So, left hand side is Bottom Rough equivalent to its right hand side.

**(ii)** When A is of T-2, left hand side of (37) is any of the four types. However, left hand side is Bottom Rough equivalent to its right hand side when condition of Corollaries 1 and 2 are satisfied.

**(iii)** When A is of T-4, left hand side of (37) is of T-3 or T-4.

So, left hand side is Top Rough equivalent to its right hand side.

**(iv)** When A is of T-1, left hand side of (38) is of T-1 or T-2.

So, left hand side is Top Rough equivalent to its right hand side.

**(v)** When A is of T-3, left hand side of (38) is any of the four types.

So, left hand side is Top Rough equivalent to its right hand side when conditions of Corollaries 1 and 2 are satisfied.

**(vi)** When A is of T-4, left hand side of (38) is of T-2 or T-4.

However, left hand side is Bottom Rough equivalent to its right hand side.

So, Left hand side and Right hand sides of absorption rules are not rough equivalent in general.

### 9.6 Kleene's Property

The Kleene's property states that for any two sets $A$ and $B$ ,

$$(A \cup A^C) \cup (B \cap B^C) = A \cup A^C \tag{39}$$

and

$$(A \cup A^C) \cap (B \cap B^C) = B \cap B^C \tag{40}$$

We show below that for Rough sets A and B, both sides of (39) and (40) match with each other with respect to types. Due to symmetry of the operations of union and intersection, it is enough to consider the ten cases; case ( i, j ) being A of Type i and B of Type j; i, j =1,2,3,4 and $j \geq i$.

**Proof of (39)**

In cases (1,1), (1,2), (1,3) and (1,4) both the left hand side and right hand side of (39) are of Type 1 or Type 3.

In cases (2,2), (2,3), (2,4), (3,3) and (3,4) both the left hand side and right hand side of (39) are of Type 3.

Finally, in case of (4, 4) both the left hand side and right hand side of (39) are of Type 3 or Type 4.

**Proof of (40)**

In case of (1,1) both the left hand side and right hand side of (40) are of Type 1 or Type 2.

In cases (1,2), (1,3), (2,2), (2,3) and (3,3) both the left hand side and right hand side of (40) are of Type 2.

In cases (1,4), (2,4), (3,4) and (4,4) both the left hand side and right hand side of (40) are of Type 2 or Type 4.

Hence, from the above observations, it is clear that the left hand side and right hand sides of Kleene's property are rough equivalent.

### 9.7 Maximum and Minimum Elements' Properties

It is obvious that both $\phi$ and $U$ are crisp sets with respect to any equivalence relation defined over the universe.

**Maximum Element**

The Maximum element property for crisp sets state that for any set $A$

$$A \cup U = U \tag{41}$$

and

$$A \cap U = A \tag{42}$$

For any Rough set A, as $A$ is a subset of $U$ both (41) and (42) hold true. So, the rough equivalence of both side are obvious.

**Minimum Element**

The Minimum element property for crisp sets state that for any set $A$,

$$A \cup \phi = A \tag{43}$$

and

$$A \cap \phi = \phi \tag{44}$$

For any Rough set A, as $\phi$ is a subset of $A$ both (43) and (44) hold true. So, the rough equivalence of both side is automatically satisfied.

### 9.8 Complementary Laws

The complementary laws for crisp sets state that for any set A,

$$A \cup A^C = U \tag{45}$$

and

$$A \cap A^C = \phi \tag{46}$$

For any Rough set A, A is a subset of $U$ and also $A^C$ is a subset of $U$. So, both (45) and (46) hold true. Hence, the rough equivalence of both sides is automatically satisfied.

## 10 Conclusions

Study of topological classification of sets under consideration provides some insight into how the boundary regions are structured. This has a major role in practical application of rough sets. In this chapter we studied some properties of topological classification of sets starting with types of rough sets, then we moved to find properties of types of union and intersection of rough sets. The concept of rough equivalences of sets introduced by Tripathy and Mitra [38], which captures approximate equality of rough sets at a higher level than rough equalities of Novotny and Pawlak [23,24,25] was discussed in detail. Some real life examples were considered in support of the above claim. Properties of rough equalities which were noted to be not true when bottom and top rough equalities are interchanged, were dealt with and established along with parallel properties for rough equivalences. Approximation of classifications of universes was introduced and studied by Busse [12]. The types of classifications were studied completely by us in this chapter. Also, theorems of Busse establishing properties of approximation of classifications wee completely generalized to their necessary and sufficient type form. From these results new results could be obtained as corollaries. All these results were interpreted with the help of simple examples. Complete characterizations of classifications having 2 or 3 elements are done. A characterization of a general classification having n elements is still awaited. Such a solution would shed light on negation in case of multivalued logic. Continuing with the study of rough equivalences, the algebraic properties involving rough sets were analyzed and established.

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