Context Algebras, Context Frames, and Their Discrete Duality*

Ewa Orłowska[1] and Ingrid Rewitzky[2]

[1] National Institute of Telecommunications, Warsaw, Poland
E.Orlowska@itl.waw.pl
[2] Department of Mathematical Sciences, University of Stellenbosch, South Africa
rewitzky@sun.ac.za

Abstract. The data structures dealt with in formal concept analysis are referred to as contexts. In this paper we study contexts within the framework of discrete duality. We show that contexts can be adequately represented by a class of sufficiency algebras called context algebras. On the logical side we define a class of context frames which are the semantic structures for context logic, a lattice-based logic associated with the class of context algebras. We prove a discrete duality between context algebras and context frames, and we develop a Hilbert style axiomatization of context logic and prove its completeness with respect to context frames. Then we prove a duality via truth theorem showing that both context algebras and context frames provide the adequate semantic structures for context logic. We discuss applications of context algebras and context logic to the specification and verification of various problems concerning contexts such as implications (attribute dependencies) in contexts, and derivation of implications from finite sets of implications.

Keywords: Duality, duality via truth, representation theorem, formal concept analysis, context, concept, attribute dependency, implication.

1 Introduction

A fundamental structure arising in formal concept analysis (FCA) [7,21] is that of a 'context'. In this paper we will consider this notion within the framework of what we refer to as discrete duality. While a classical duality, such as that of, for example, Stone [19] and Priestley [18], includes a representation of a class of algebras in terms of a topological structure, a discrete duality includes a representation for a class of algebras in terms of the relational structures that provide the frame semantics (or equivalently, Kripke-style semantics) of the lattice-based logic associated with the class of algebras. The frame semantics is given in terms of a relational structure without a (non-discrete) topology which

* Ewa Orłowska acknowledges partial support from the Polish Ministry of Science and Higher Education grant N N206 399134, and Ingrid Rewitzky acknowledges partial support from National Research Foundation of South Africa. Both authors acknowledge support from the Poland/SA Bilateral Collaboration Programme.

J.F. Peters et al. (Eds.): Transactions on Rough Sets IX, LNCS 5390, pp. 212–229, 2008.

explains the name of this type of duality. General principles of establishing a discrete duality are presented in [14]. In this paper we elaborate in detail a discrete duality for the structures arising in connection with problems considered in formal concept analysis.

First, we show that contexts can be adequately represented by an axiomatic and signature extension of the class of sufficiency algebras [6], referred to as context algebras. The lattice-based logic associated with the class of context algebras we call context logic. Context algebras are Boolean algebras with a pair of sufficiency operators forming a Galois connection. The sufficiency operators of context algebras are the abstract counterparts of the maps of extent and intent determined by a context. On the logical side we define a class of context frames which are the semantic structures for context logic. We prove a discrete duality between context algebras and context frames. This duality is established in two steps: we show a representation theorem for context algebras such that a representation algebra is constructed from a context frame and is shown to contain (up to isomorphism) a given algebra as a subalgebra; next we show a representation theorem for context frames such that a representation frame is constructed from a context algebra and is shown to contain (up to isomorphism) a given context frame as a substructure. A discrete duality for sufficiency algebras is presented in [6], see also [12]. It extends Jónsson-Tarski duality [10] for Boolean algebras with normal and additive operators to Boolean algebras with a sufficiency operator. The discrete duality proved in the present paper extends this result further to context algebras. Discrete dualities for distributive lattices with operators forming a Galois connection and also some other similar connections are developed in [15]. Next, we develop a Hilbert-style axiomatization of context logic and prove its completeness with respect to context frames.

Second, we discuss applications of context algebras and context logic to the specification and verification of various problems concerning contexts and concepts. We consider three groups of problems: first, problems related to the verification of whether a pair 'set of objects, set of attributes' is a formal concept having some properties; second, attribute dependencies in information systems which are closely related to contexts; third, implications in contexts and derivation of implications from finite sets of implications. We indicate that the tasks from all these three groups can be specified within the framework of context algebras and context logic presented in this paper. The deduction tools of context logic and the theories of context algebras can then be used for solving these problems.

2 Context Algebras and Frames

Central in formal concept analysis is the notion of a *Galois connection* between two types of entities. Algebraically this may be captured by two maps e and i over a Boolean algebra, and relationally by a relation between the two types of entities. In this section we formalise this in the notions of context algebra and context frame.

Definition 1. *A context algebra* $(W, \vee, \wedge, \neg, 0, 1, e, i)$ *is a Boolean algebra* $(W, \vee, \wedge, \neg, 0, 1)$ *endowed with unary operators* e, i *satisfying, for any* $a, b \in W$*,*

(AC1) $g(a \vee b) = g(a) \wedge g(b)$ *for* $g = e, i$
(AC2) $g(0) = 1$ *for* $g = e, i$
(AC3) $a \leq e(i(a))$
(AC4) $a \leq i(e(a))$.

An operator $g : W \to W$ satisfying (AC1) and (AC2) is called a *sufficiency operator*, see [6]. It follows that the sufficiency operators e and i are antitone and form a Galois connection, that is,

$$a \leq i(b) \quad \text{iff} \quad b \leq e(a), \qquad \text{for any } a, b \in W.$$

From this Galois connection we can separate the two types of entities and the relation between them thereby deriving a formal context arising in formal concept analysis. From a context algebra $\mathcal{A} = (W, \vee, \wedge, \neg, 0, 1, e, i)$ we may define the formal context $\mathcal{C}_{\mathcal{A}} = (G_{\mathcal{A}}, M_{\mathcal{A}}, I_{\mathcal{A}})$, where $G_{\mathcal{A}} = \{o \mid \exists a, o \leq e(a)\}$, $M_{\mathcal{A}} = \{a \mid \exists o, a \leq i(o)\}$ and $I_{\mathcal{A}} = \{(o, a) \mid o \leq e(a)\} = \{(o, a) \mid a \leq i(o)\}$. Then $G_{\mathcal{A}} = \mathrm{dom}(I_{\mathcal{A}})$ and $M_{\mathcal{A}} = \mathrm{ran}(I_{\mathcal{A}})$.

Lemma 1. *Let* $\mathcal{A} = (W, \vee, \wedge, \neg, 0, 1, e, i)$ *be a context algebra. The sufficiency operators* e *and* i *of* $\mathcal{A}$ *are the mappings of extent and intent determined by a formal context* $\mathcal{C}_{\mathcal{A}} = (G_{\mathcal{A}}, M_{\mathcal{A}}, I_{\mathcal{A}})$*, in the sense that, for any* $a, o \in W$*,*

$$o \leq e(a) \text{ iff } o \in e_{\mathcal{A}}(\{a\}) = \{o \mid oI_{\mathcal{A}}a\}$$

and

$$a \leq i(o) \text{ iff } a \in i_{\mathcal{A}}(\{o\}) = \{a \mid oI_{\mathcal{A}}a\}.$$

On the other hand, given a formal context $\mathcal{C} = (G, M, I)$, we may define a context algebra $(W_{\mathcal{C}}, e_{\mathcal{C}}, i_{\mathcal{C}})$ where $W_{\mathcal{C}} = 2^{G \cup M}$, $e_{\mathcal{C}} = [\![I]\!]$ and $i_{\mathcal{C}} = [\![I^{-1}]\!]$, where for any $A \in W_{\mathcal{C}}$ and $T = I, I^{-1}$,

$$[\![T]\!](A) = \{x \in G \cup M \mid \forall y,\ y \in A \ \Rightarrow\ xTy\}.$$

Lemma 2. *Let* $\mathcal{C} = (G, M, I)$ *be a formal context. The mappings of extent and intent determined by* $\mathcal{C}$ *are the sufficiency operators of the context algebra* $\mathcal{A}_{\mathcal{C}} = (W_{\mathcal{C}}, e_{\mathcal{C}}, i_{\mathcal{C}})$*, that is,* $e = e_{\mathcal{C}}$ *and* $i = i_{\mathcal{C}}$*.*

Theorem 1

(a) *If a formal context* $\mathcal{C} = (G, M, I)$ *satisfies* $G = \mathrm{dom}(I)$ *and* $M = \mathrm{ran}(I)$*, then* $\mathcal{C} = \mathcal{C}_{\mathcal{A}_{\mathcal{C}}}$*.*
(b) *If a context algebra* $\mathcal{A} = (W, \vee, \wedge, \neg, 0, 1, e, i)$ *is complete and atomic and such that* $W = 2^X$ *where* $X = \{o \mid o \in e(\{a\})\} \cup \{a \mid a \in i(\{o\})\}$*, then* $\mathcal{A} = \mathcal{A}_{\mathcal{C}_{\mathcal{A}}}$*.*

Definition 2. *A context frame* $\mathcal{F} = (X, R, S)$ *is a non-empty set* X *endowed with binary relations* R *and* S *such that* $S = R^{-1}$*.*

Although relation S is definable from R, the setting with two relations enables us to avoid any relational-algebraic structure (in this case the operation of converse of a relation) in the language of context logic developed in Section 3. In this way the intended object language is singular, the required constraint is formulated only in the definition of its semantics, that is, in the metalanguage.

From a context frame $\mathcal{F} = (X, R, S)$ we may define the formal context $\mathcal{C}_{\mathcal{F}} = (G_{\mathcal{F}}, M_{\mathcal{F}}, I_{\mathcal{F}})$, where $G_{\mathcal{F}} = dom(R)$, $M_{\mathcal{F}} = ran(R)$, and $I_{\mathcal{F}} = R$.

Lemma 3. *Let $\mathcal{F} = (X, R, S)$ be a context frame. The sufficiency operators determined by $\mathcal{F}$ are the mappings of extent and intent determined by a formal context $\mathcal{C}_{\mathcal{F}} = (G_{\mathcal{F}}, M_{\mathcal{F}}, I_{\mathcal{F}})$, that is, $[\![R]\!] = e_{\mathcal{F}}$ and $[\![S]\!] = i_{\mathcal{F}}$.*

On the other hand, given a formal context $\mathcal{C} = (G, M, I)$, we may define a context frame $\mathcal{F}_{\mathcal{C}} = (X_{\mathcal{C}}, R_{\mathcal{C}}, S_{\mathcal{C}})$, where $X_{\mathcal{C}} = G \cup M$, $R_{\mathcal{C}} = I$ and $S_{\mathcal{C}} = I^{-1}$.

Lemma 4. *Let $\mathcal{C} = (G, M, I)$ be a formal context. The mappings of extent and intent determined by $\mathcal{C}$ are the sufficiency operators determined by the context frame $\mathcal{F}_{\mathcal{C}} = (X_{\mathcal{C}}, R_{\mathcal{C}}, S_{\mathcal{C}})$, that is, $e = [\![R_{\mathcal{C}}]\!]$ and $i = [\![S_{\mathcal{C}}]\!]$.*

Theorem 2

(a) *If a formal context $\mathcal{C} = (G, M, I)$ satisfies $G = \mathrm{dom}(I)$ and $M = \mathrm{ran}(I)$, then $\mathcal{C} = \mathcal{C}_{\mathcal{F}_{\mathcal{C}}}$.*
(b) *If a context frame $\mathcal{F} = (X, R, S)$ satisfies $X = \mathrm{dom}(R) \cup \mathrm{ran}(R)$, then $\mathcal{F} = \mathcal{F}_{\mathcal{C}_{\mathcal{F}}}$.*

We now establish a discrete duality between context algebras and context frames. First, we show that from any context frame we can define a context algebra. Let (X, R, S) be a context frame. Then the binary relations R and S over X induce sufficiency operators over 2^X, namely, $e^c : 2^X \to 2^X$ defined, for any $A \in 2^X$, by

$$e^c(A) = [\![R]\!](A) = \{x \in X \mid \forall y \in X,\ y \in A \ \Rightarrow\ xRy\},$$

and $i^c : 2^X \to 2^X$ defined for any $A \in 2^X$, by

$$i^c(A) = [\![S]\!](A) = \{x \in X \mid \forall y \in X,\ y \in A \ \Rightarrow\ xSy\}.$$

Thus,

Definition 3. *Let (X, R, S) be a context frame. Then its complex algebra is the powerset Boolean algebra with sufficiency operators $(2^X, \cup, \cap, -, \emptyset, X, e^c, i^c)$.*

Theorem 3. *The complex algebra of a context frame is a context algebra.*

Proof: The operators e^c and i^c are sufficiency operators as shown in [12]. By way of example we show that (AC3) is satisfied. Let $A \subseteq X$, we show that $A \subseteq e^c(i^c(A))$. Let $x \in X$ and suppose that $x \in A$ but $x \notin e^c(i^c(A))$. It follows that there is $y_0 \in i^c(A)$ such that $(x, y_0) \notin R$. By definition of i^c, for every $z \in A$, $y_0 S z$. In particular, taking z to be x, we have $y_0 S x$. Since $S = R^{-1}$, we have xRy_0, a contradiction. The proof of (AC4) is similar. □

Next we show that any context algebra in turn gives rise to a context frame. In the case of a sufficiency operator g over a powerset Boolean algebra 2^X, a relation r_g over X may be defined, as in [6], by

$$xr_g y \quad \text{iff} \quad x \in g(\{y\}), \qquad \text{for any } x, y \in X.$$

In general, as in [12], we invoke Stone's representation theorem and then define, from each sufficiency operator $g : W \to W$, a binary relation R_g over the family $\mathcal{X}(W)$ of prime filters of W by

$$FR_g G \quad \text{iff} \quad g(G) \cap F \neq \emptyset, \quad \text{for any } F, G \in \mathcal{X}(W)$$

where for $A \subseteq W$, $g(A) = \{g(a) \in W \mid a \in A\}$. It is an easy exercise to show that R_g is an extension of r_g. Thus,

Definition 4. *The* canonical frame *of a context algebra* $(W, \vee, \wedge, \neg, 0, 1, e, i)$ *is the relational structure* $(\mathcal{X}(W), R^c, S^c)$*, where* $\mathcal{X}(W)$ *is the family of prime filters of* W*,* $R^c = R_e$ *and* $S^c = R_i$*.*

Theorem 4. *The canonical frame of a context algebra is a context frame.*

Proof: We show that $(R^c)^{-1} \subseteq S^c$. Let $(F, G) \in (R^c)^{-1}$. Then $(G, F) \in R^c$, that is $e(F) \cap G \neq \emptyset$. Hence, there is some a_0 such that $a_0 \in G$ and $a_0 \in e(F)$. Take b_0 to be $i(a_0)$. Then $b_0 \in i(G)$ and $b_0 \in i(e(F))$. Now $i(e(F)) \subseteq F$ since if $a \in i(e(F))$ then $a = i(e(x))$ for some $x \in F$, so, by (AC4) and since F is up-closed, $a = i(e(x)) \in F$. Thus $b_0 \in i(G) \cap F$, that is, $i(G) \cap F \neq \emptyset$. The proof of the other inclusion is similar. □

Let $(W, \vee, \wedge, \neg, 0, 1, e, i)$ be a context algebra. Then

$$(2^{\mathcal{X}(W)}, \cup, \cap, -, \emptyset, \mathcal{X}(W), e^c, i^c)$$

is the complex algebra of the canonical frame $(\mathcal{X}(W), R^c, S^c)$ of the original context algebra. The relationship between these algebras is captured by the Stone mapping $h : W \to 2^{\mathcal{X}(W)}$ defined, for any $a \in W$, by

$$h(a) = \{F \in \mathcal{X}(W) \mid a \in F\}.$$

This mapping is an embedding and preserves operators i and e over W. That is,

Theorem 5. *For any context algebra* $(W, \vee, \wedge, \neg, 0, 1, e, i)$ *and any* $a \in W$*,*

$$h(e(a)) = e^c(h(a)) \qquad \text{and} \qquad h(i(a)) = i^c(h(a)).$$

Proof: We give the proof for e; that for i is similar. We need to show, for any $a \in W$ and any $F \in \mathcal{X}(W)$, that

$$e(a) \in F \quad \text{iff} \quad \forall G \in \mathcal{X}(W),\ a \in G \ \Rightarrow\ e(G) \cap F \neq \emptyset.$$

Assume $a \in G$ and $e(G) \cap F = \emptyset$. Then $e(a) \in e(G)$ and hence $e(a) \notin F$. On the other hand, assume $e(a) \notin F$. Let a dual of e, denoted e^d, be defined, for any $b \in W$, by $e^d(b) = -e(-b)$. Consider the set $Z_e = \{b \in W \mid e^d(b) \notin F\}$. Let F' be the filter generated by $Z_e \cup \{a\}$, that is, $F' = \{b \in W \mid \exists a_1, \ldots, a_n \in Z_e, \quad a_1 \wedge \ldots \wedge a_n \wedge a \leq b\}$. Then F' is proper. Suppose otherwise. Then for some $a_1, \ldots, a_n \in Z_e, \quad a_1 \wedge \ldots \wedge a_n \wedge a = 0$, that is, $a \leq -(a_1 \wedge \ldots \wedge a_n) = -a_1 \vee \ldots \vee -a_n$. Since e is antitone, $e(-a_1 \vee \ldots \vee -a_n) \leq e(a)$. Thus $e(-a_1) \wedge \ldots \wedge e(-a_n) \leq e(a)$, that is, $-e^d(a_1) \wedge \ldots \wedge -e^d(a_n) \leq e(a)$. By definition of Z_e we have $e^d(a_1), \ldots, e^d(a_n) \notin F$ so $-e^d(a_1), \ldots, -e^d(a_n) \in F$. Since F is a filter, $-e^d(a_1) \wedge \ldots \wedge -e^d(a_n) \in F$ and hence $e(a) \in F$ which contradicts the original assumption. So, by ([4], p188), there is a prime filter G containing F'. Since $a \in F'$, $a \in G$ and hence $G \in h(a)$. Also $e(G) \cap F = \emptyset$ since if there is some $b \in W$ with $b \in e(G)$ and $b \in F$, then $b = e(c)$ for some $c \in G$ and thus $e(c) \in F$, so $e^d(-c) \notin F$ hence $-c \in Z_e \subseteq F' \subseteq G$ and thus $c \notin G$, which is a contradiction. □

On the other hand, let (X, R, S) be a context frame. The $(\mathcal{X}(2^X), R^c, S^c)$ is the canonical frame of the complex algebra $(2^X, \cup, \cap, -, \emptyset, X, e^c, i^c)$ of the original context frame. The relationship between these frames is captured by the mapping $k : X \to \mathcal{X}(2^X)$ defined, for any $x \in X$, by $k(x) = \{A \in 2^X \mid x \in A\}$. It is easy to show that k is well-defined and an embedding. All that remains is to show that k preserves structure, that is,

Theorem 6. *For any context frame (X, R, S) and any $x, y \in X$,*

$$xRy \text{ iff } k(x)R^c k(y) \qquad \text{and} \qquad xSy \text{ iff } k(x)S^c k(y).$$

Proof: We give the proof for R; that for S is similar. Note, for any $x, y \in X$,

$$\begin{aligned} k(x)R^c k(y) \quad &\text{iff} \quad [\![R]\!](k(y)) \cap k(x) \neq \emptyset \\ &\text{iff} \quad \exists A \in 2^X,\ y \in A \ \wedge\ \forall z \in X, z \in A \Rightarrow xRz. \end{aligned}$$

Suppose $k(x)R^c k(y)$ does not hold. Let $A = \{y\}$. Then $y \in A$ and hence, for some $z \in A$, xRz does not hold. Therefore, $z = y$ and xRy does not hold. Suppose $k(x)R^c k(y)$. Let $A = \{y\}$. Then $y \in A$ and hence xRy. □

Therefore, we have a discrete duality between context algebras and context frames.

Theorem 7

(a) *Every context algebra can be embedded into the complex algebra of its canonical frame.*
(b) *Every context frame can be embedded into the canonical frame of its complex algebra.*

3 Context Logic

In order to extend the duality established in Theorem 7 to a Duality via Truth as considered in [13], we need a logical language presented in [8].

Definition 5. *Let* LC *be a modal language extending the language of classical propositional calculus, that is, its formulas are built from propositional variables taken from an infinite denumerable set* V *and the constants true* (1) *and false* (0)*, with the classical propositional operations of negation* ($\neg$)*, disjunction* ($\vee$)*, conjunction* ($\wedge$)*, and with two unary operators* $[\![]\!]_1$ *and* $[\![]\!]_2$*. As usual,* $\rightarrow$ *and* $\leftrightarrow$ *are definable:*

$$\phi \rightarrow \psi := \neg\phi \vee \psi \quad \text{and} \quad \phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi).$$

Also the constants are definable: $0 := \phi \wedge \neg\phi$ and $1 := \phi \vee \neg\phi$*. We define* $\langle\!\langle\rangle\!\rangle_i$*, by* $\langle\!\langle\rangle\!\rangle_i\phi := \neg[\![]\!]_i\neg\phi$ *(for* $i = 1, 2$*). Let* For *denote the set of all formulae of* LC*.*

Within LC the conditions on a context algebra can be captured, using a Hilbert-style axiomatisation, as follows:

Axioms:

(LC0) Axioms of the classical propositional calculus (see eg [16])
(LC1) $[\![]\!]_i(\phi \vee \psi) \leftrightarrow [\![]\!]_i\phi \wedge [\![]\!]_i\psi$ (for $i = 1, 2$)
(LC2) $[\![]\!]_i 0 = 1$ (for $i = 1, 2$)
(LC3) $\phi \rightarrow [\![]\!]_1[\![]\!]_2\phi$
(LC4) $\phi \rightarrow [\![]\!]_2[\![]\!]_1\phi$.

Rules of inference: modus ponens and

$$\frac{\phi \rightarrow \psi}{[\![]\!]_i\psi \rightarrow [\![]\!]_i\phi} \qquad \text{(for } i = 1, 2\text{)}.$$

If ϕ is obtained from the axioms by repeated applications of the inference rules, then ϕ is called a *theorem* of LC, written $\vdash \phi$. Axioms (LC3) and (LC4) reflect the fact that the two sufficiency operators form a Galois connection. Some logics arising from a Galois connection are also considered in [20].

The semantics of context logic LC is based on context frames (X, R, S) where X is a non-empty set endowed with binary relations R and S such that $S = R^{-1}$. A LC-model based on a context frame (X, R, S) is a system $M = (X, R, S, m)$, where $m : V \cup \{0, 1\} \rightarrow 2^X$ is a meaning function such that $m(p) \subseteq X$ for $p \in V$, $m(0) = \emptyset$, $m(1) = X$. The satisfaction relation $\models$ is defined as follows, where $M, x \models \phi$ means that the state x satisfies formula ϕ in model M:

$$\begin{aligned}
M, x \models p &\ \text{ iff } \ x \in m(p), \text{ for every } p \in V \\
M, x \models \phi \vee \psi &\ \text{ iff } \ M, x \models \phi \text{ or } M, x \models \psi \\
M, x \models \phi \wedge \psi &\ \text{ iff } \ M, x \models \phi \text{ and } M, x \models \psi \\
M, x \models \neg\phi &\ \text{ iff } \ \text{not } M, x \models \phi \\
M, x \models [\![]\!]_1\phi &\ \text{ iff } \ \forall y \in X,\ M, y \models \phi \text{ implies } xRy \\
M, x \models [\![]\!]_2\phi &\ \text{ iff } \ \forall y \in X,\ M, y \models \phi \text{ implies } xSy.
\end{aligned}$$

From now on we shall write $[\![R]\!]$ and $[\![S]\!]$ instead of $[\![]\!]_1$ and $[\![]\!]_2$, respectively. A notion of truth of formulae based on the LC semantics is defined as usual. A formula $\phi \in \mathsf{LC}$ is true in a context model M, written $M \models \phi$, whenever for every $x \in X$ we have $M, x \models \phi$. A formula $\phi \in \mathsf{LC}$ is true in a context frame (X, R, S) iff ϕ is true in every model based on this frame. And finally a formula $\phi \in \mathsf{LC}$ is valid in the logic LC, called LC-valid and written $\models \phi$, iff it is true in every context frame.

Theorem 8. (Soundness) *For any* LC*-formula* $\phi \in \mathsf{For}$*, if* ϕ *is a theorem of* LC *then* ϕ *is* LC*-valid.*

Proof: Proving soundness is an easy task — it involves showing that the axioms of LC are LC-valid and the rules preserve LC-validity. □

Following a technique due to Rasiowa [16] to prove completeness, we need some constructions and lemmas. The relation $\approx$ defined on the set For of formulae of LC by:

$$\phi \approx \psi \quad \text{iff} \quad \vdash (\phi \leftrightarrow \psi)$$

is an equivalence relation compatible with the operations $\vee, \wedge, \neg, [\![R]\!], [\![S]\!], 0, 1$. This induces a quotient algebra

$$\mathcal{A}_{\approx} = (\mathsf{For}|_{\approx}, \cup, \cap, -, 0_{\approx}, 1_{\approx}, [\![R]\!]_{\approx}, [\![S]\!]_{\approx})$$

where $\mathsf{For}|_{\approx}$ is the family of equivalence classes of $\approx$ and, for any $\phi, \psi \in \mathsf{For}$,

$$|\phi| \cup |\psi| = |\phi \vee \psi| \quad |\phi| \cap |\psi| = |\phi \wedge \psi| \quad -|\phi| = |\neg\phi| \quad 0_{\approx} = |\phi \wedge \neg\phi| \quad 1_{\approx} = |\phi \vee \neg\phi|$$

$$[\![T]\!]_{\approx}|\phi| = |[\![T]\!]\phi| \ \ (\text{for } T = R, S).$$

Then for the definable connectives, $\rightarrow$ and $\leftrightarrow$, we postulate:

$$|\phi| \rightarrow |\psi| = |\phi \rightarrow \psi| \quad \text{and} \quad |\phi| \leftrightarrow |\psi| = |\phi \leftrightarrow \psi|.$$

Lemma 5

(a) $\mathcal{A}_{\approx}$ *is a non-degenerate* (*i.e. at least two-element*) *context algebra.*
(b) *For any* $\phi, \psi \in \mathsf{For}$, $|\phi| \leq_{\approx} |\psi|$ iff $\vdash \phi \rightarrow \psi$.
(c) *For any* $\phi \in \mathsf{For}$, $\vdash \phi$ iff $|\phi| = 1_{\approx}$.
(d) *For any* $\phi \in \mathsf{For}$, $|\neg\phi| \neq \emptyset$ iff not $\vdash \phi$.

Proof: For (a), we show that the Lindenbaum algebra satisfies conditions (AC1)-(AC4). For (AC1) and (AC2) we consider $[\![R]\!]_{\approx}$; the proofs for $[\![S]\!]_{\approx}$ are similar.

For (AC1), $[\![R]\!]_{\approx}(|\phi| \cup |\psi|) = [\![R]\!]_{\approx}(|\phi \vee \psi|) = |[\![R]\!](\phi \vee \psi)| = |[\![R]\!]\phi \wedge [\![R]\!]\psi| = |[\![R]\!]\phi| \cap |[\![R]\!]\psi| = [\![R]\!]_{\approx}|\phi| \cap [\![R]\!]_{\approx}|\psi|$.

For (AC2), $[\![R]\!]_{\approx}0_{\approx} = [\![R]\!]_{\approx}|\phi \wedge \neg\phi| = |[\![R]\!](\phi \wedge \neg\phi)| = |[\![R]\!](0)| = |1| = |\phi \vee \neg\phi| = 1_{\approx}$.

For (AC3), we have to show that for any formula ϕ, $|\phi| \subseteq [\![R]\!]_{\approx}[\![S]\!]_{\approx}|\phi|$. By (LC3) and the definition of the operators in the Lindenbaum algebra $\mathcal{A}_{\approx}$ we have $|\phi| \subseteq |[\![R]\!][\![S]\!]\phi| = [\![R]\!]_{\approx}[\![S]\!]_{\approx}|\phi|$. The proof of (AC4) is similar.

For (b), $|\phi| \leq_{\approx} |\psi|$ iff $|\phi| \cup |\psi| = |\psi|$ iff $|\phi \vee \psi| = |\psi|$ iff $\vdash \phi \vee \psi \leftrightarrow \psi$ iff $\vdash \phi \rightarrow \psi$.

For (c), assume $\vdash \phi$. Since $\vdash \phi \to ((\phi \to \phi) \to \phi)$, by applying modus ponens, we get $\vdash (\phi \to \phi) \to \phi$. By (b), $|\phi \to \phi| \leq_{\approx} |\phi|$, hence $1_{\approx} = |\phi|$. Now assume that $|\phi| = 1_{\approx}$. Since $1_{\approx} = |\phi \to \phi|$, we have $|\phi \to \phi| \leq_{\approx} |\phi|$. By (b), $\vdash (\phi \to \phi) \to \phi$. Since $\vdash \phi \to \phi$, by applying modus ponens we get $\vdash \phi$.

For (d), we use (b) and the fact that $|\phi| = 1_{\approx}$ iff $|\neg\phi| = 0_{\approx} = \emptyset$. □

Consider the canonical frame $(\mathcal{X}(\mathcal{A}_{\approx}), R_{\approx}, S_{\approx})$ of the algebra $\mathcal{A}_{\approx}$ where, for any $F, G \in \mathcal{X}(\mathcal{A}_{\approx})$,

$$FT_{\approx}G \quad \text{iff} \quad [\![T]\!]_{\approx}(G) \cap F \neq \emptyset, \quad \text{for } T = R, S.$$

Definition 6. *The canonical* LC*-model based on the canonical context frame* $(\mathcal{X}(\mathcal{A}_{\approx}), R_{\approx}, S_{\approx})$ *of* $\mathcal{A}_{\approx}$ *is a system* $M_{\approx} = (\mathcal{X}(\mathcal{A}_{\approx}), R_{\approx}, S_{\approx}, m_{\approx})$*, where the meaning function* $m_{\approx} : V \cup \{0,1\} \to 2^{X(\mathcal{A}_{\approx})}$ *is defined, for any* $p \in V \cup \{0,1\}$ *and any* $F \in \mathcal{X}(\mathcal{A}_{\approx})$*, by*

$$F \in m_{\approx}(p) \qquad \text{iff} \qquad |p| \in F.$$

The mapping $m_{\approx}$ *extends homomorphically to all* LC*-formulae, that is, for any* $\phi, \psi \in$ For *and* $T = R, S$*,*

$$m_{\approx}(\neg\phi) = -m_{\approx}(\phi),\ m_{\approx}(\phi \vee \psi) = m_{\approx}(\phi) \cup m_{\approx}(\psi),\ \ m_{\approx}([\![T]\!]\phi) = [\![T]\!]_{\approx}(m_{\approx}(\phi)).$$

Consider the mapping $h_{\approx} : \mathcal{A}_{\approx} \to 2^{\mathcal{X}(\mathcal{A}_{\approx})}$ defined as on page 216, that is, for any $|\phi| \in \mathcal{A}_{\approx}$,

$$h_{\approx}(|\phi|) = \{F \in \mathcal{X}(\mathcal{A}_{\approx}) \mid |\phi| \in F\}.$$

We know from Theorem 5 that $h_{\approx}$ preserves the operations (all operations, not only the sufficiency operator).

Lemma 6. *Let* $\mathcal{M}_{\approx}$ *be the canonical* LC*-model based on the canonical context frame* $(\mathcal{X}(\mathcal{A}_{\approx}), R_{\approx}, S_{\approx})$*. Then, for any* $F \in \mathcal{X}(\mathcal{A}_{\approx})$ *and any* LC *formula* ϕ*,*

$$M_{\approx}, F \models \phi \text{ iff } F \in h_{\approx}(|\phi|).$$

Proof: For this we use structural induction on LC-formulae ϕ. By definition, for any basic formula $p \in V$,

$$\mathcal{M}_{\approx}, F \models p \quad \text{iff} \quad F \in m_{\approx}(p) \quad \text{iff} \quad |p| \in F \text{ iff } F \in h_{\approx}(|p|).$$

Assume as induction hypothesis that the claim holds for $\phi, \psi \in$ For. We consider the cases where θ is $\phi \vee \psi$ and $[\![R]\!]\phi$; the other cases are similar.

$$\begin{aligned} \mathcal{M}_{\approx}, F \models \phi \vee \psi \quad & \text{iff} \quad \mathcal{M}_{\approx}, F \models \phi \text{ or } \mathcal{M}_{\approx}, F \models \psi \\ & \text{iff} \quad F \in h_{\approx}(|\phi|) \text{ or } F \in h_{\approx}(|\psi|) \\ & \text{iff} \quad |\phi| \in F \text{ or } |\psi| \in F \\ & \text{iff} \quad |\phi| \cup |\psi| \in F \quad F \text{ is a prime filter} \\ & \text{iff} \quad |\phi \vee \psi| \in F \\ & \text{iff} \quad F \in h_{\approx}(|\phi \vee \psi|). \end{aligned}$$

$$\begin{aligned}\mathcal{M}_{\approx}, F \models [\![R]\!]\phi \quad &\text{iff} \quad \forall G \in \mathcal{X}(\mathcal{A}_{\approx}),\ \mathcal{M}_{\approx}, G \models \phi \text{ implies } FR_{\approx}G \\ &\text{iff} \quad \forall G \in \mathcal{X}(\mathcal{A}_{\approx}),\ G \in h_{\approx}(|\phi|) \text{ implies } FR_{\approx}G \\ &\text{iff} \quad F \in [\![R_{\approx}]\!](h_{\approx}(|\phi|)) \\ &\text{iff} \quad F \in h_{\approx}([\![R]\!]_{\approx}|\phi|) \quad \text{by Theorem 5} \\ &\text{iff} \quad F \in h_{\approx}(|[\![R]\!]\phi|).\end{aligned}$$

This completes the proof. □

Thus, since $h_{\approx}$ is an embedding that preserves operations $[\![T]\!]$ (for $T = R, S$), we have the following truth lemma.

Lemma 7. *For any $F \in \mathcal{X}(\mathcal{A}_{\approx})$ and any* LC *formula ϕ,*

$$M_{\approx}, F \models \phi \qquad \text{iff} \qquad |\phi| \in F.$$

Theorem 9. (Completeness) *For any* LC*-formula $\phi \in$* For*, if ϕ is* LC*-valid then ϕ is a theorem of* LC*.*

Proof: Take any LC formula ϕ such that $\models \phi$. Suppose that $\vdash \phi$ does not hold. Then, by Lemma 5(d), $|\neg\phi| \neq \emptyset$. So there exists $F \in \mathcal{X}(\mathcal{A}_{\approx})$ such that $|\neg\phi| \in F$. Thus, by Lemma 7, for some $F \in \mathcal{X}(\mathcal{A}_{\approx})$, $M_{\approx}, F \models \neg\phi$, Hence, by definition of $\models$, ϕ is not true in $M_{\approx}$, which contradicts the assumption that ϕ is LC-valid. □

With this logical language we can extend Theorem 7 to a Duality via Truth, in the sense of [13]. Let $\mathsf{Alg}_{\mathsf{LC}}$ denote the class of context algebras, and $\mathsf{Frm}_{\mathsf{LC}}$ denote the class of context frames. As described above the class $\mathsf{Frm}_{\mathsf{LC}}$ of context frames provides a frame semantics for LC. The class $\mathsf{Alg}_{\mathsf{LC}}$ of context algebras provides an algebraic semantics for LC. Let $(W, \vee, \wedge, \neg, 0, 1, e, i)$ be a context algebra. A valuation on W is a function $v : V \to W$ which assigns elements of W to propositional variables and extends homomorphically to all the formulas of LC, that is

$$v(\neg\phi) = \neg v(\phi),\ v(\phi \vee \psi) = v(\phi) \vee v(\psi),\ \ v([\![R]\!]\phi) = e(v(\phi)),\ \ v([\![S]\!]\phi) = i(v(\phi)).$$

The notion of truth determined by this semantics is as follows. A formula ϕ in LC is true in an algebra $(W, \vee, \wedge, \neg, 0, 1, e, i)$ whenever $v(\phi) = 1$ for every v in W. A formula $\phi \in \mathsf{LC}$ is true in the class $\mathsf{Alg}_{\mathsf{LC}}$ of context algebras iff it is true in every context algebra in $\mathsf{Alg}_{\mathsf{LC}}$.

Theorem 10. *A formula $\phi \in$* LC *is true in every model based on a context frame (X, R, S) iff ϕ is true in the complex algebra $(2^X, e^c, i^c)$ defined from that frame.*

Proof: Let (X, R, S) be any context frame. The result is established by taking the meaning function m on any model (X, R, S, m) based on (X, R, S) to coincide with the valuation function on the complex algebra $(2^X, e^c, i^c)$ of (X, R, S). □

Finally, we prove the Duality via Truth theorem between context algebras and context frames.

Theorem 11. *A formula $\phi \in \mathsf{LC}$ is true in every context algebra in $\mathsf{Alg}_{\mathsf{LC}}$ iff ϕ is true in every context frame in $\mathsf{Frm}_{\mathsf{LC}}$.*

Proof: Assume that ϕ is true in all context algebras. In particular, ϕ is true in all the complex algebras of the context frames. Since every context frame has its corresponding complex algebra, by Theorem 10 the required condition follows. For reverse implication, we prove the contrapositive. Assume that for some context algebra W and a valuation v in W $v(\phi) \neq 1$. Consider the canonical frame $\mathcal{X}(W)$ of W. By the representation theorem there is an embedding $h : W \to 2^{\mathcal{X}(W)}$. It follows that $h(v(\phi)) \neq 1$. Consider a model M based on $\mathcal{X}(W)$ such that $m(p) = h(v(p))$. By induction on the complexity of formulas we can show that for every formula ϕ, $m(\phi) = m(h(\phi))$. Hence, ϕ is not true in M. □

4 Applications for Formal Concept Analysis

In this section we show that context algebras and context logic can be used for the specification and verification of various problems concerning contexts and concepts from formal concept analysis.

4.1 Intents, Extents and Operations of Concepts

Let (G, M, I), where $I \subseteq G \times M$, be a context. The Galois connection underlying the notion of context algebra and context frame allows the identification of certain pairs (O, A) where $O \in 2^G$, $A \in 2^M$. Namely, those that are closed in the sense that $i(O) = A$ and $e(A) = O$. That is, given a set O of objects the map i of the Galois connection identifies all the features which they have in common, and given a set A of features the map e of the Galois connection identifies all the objects which they have in common. Such object-feature pairs are called *formal concepts*. The sets $i(O)$ and $e(A)$ are called the *intent* and *extent* of the concept (O, A), respectively.

As explained in [22] care needs to be taken when defining operations of join, meet and complement for concepts in order to ensure the result is again a formal concept. Accordingly, given two formal concepts (O, A) and (O', A'), their join $\vee$ and meet $\wedge$ are defined respectively by:

$$(O, A) \vee (O', A') = (e(i(O \cup O')), A \cap A')$$
$$(O, A) \wedge (O', A') = (O \cap O', i(e(A \cup A'))).$$

The corresponding order $\leq$ on the set of formal concepts is defined, for formal concepts (O, A) and (O', A'), by

$$(O, A) \leq (O', A') \qquad \text{iff} \qquad O \subseteq O' \qquad \text{(or equivalently, iff } A' \subseteq A).$$

With respect to this order, the smallest formal concept is $(e(M), M)$ and the largest is $(G, i(G))$.

For the complement of a formal concept (O, A), two complements are considered. The one is generated by the set complement $\overline{O}$ of the extent O and the other generated by the set complement $\overline{A}$ of the intent A. Namely, for a formal concept (O, A), its weak negation is defined

$$\neg(O, A) = (e(i(\overline{O})), i(\overline{O}))$$

and its weak opposition is defined by

$$\sim (O, A) = (e(\overline{A}), i(e(\overline{A}))).$$

Note that $\neg$ captures contradictory opposite, in the sense that, for example, positive and negative, and cold and hot are contradictory opposites. On the other hand $\sim$ captures contrary opposite, in the sense that, for example, moist and dry, and cold and warm are contrary opposites.

We now characterize the notions of intent and extent, and the operations of join, meet, weak negation, weak opposition for concepts in terms of LC-formulae. For this, take any $O_i \in 2^G$ (for $i = 1, 2, 3$) and any $A_i \in 2^M$ (for $i = 1, 2, 3$). Suppose that p_i is the propositional variable representing O_i (for $i = 1, 2, 3$), and that q_i is the propositional variable representing A_i (for $i = 1, 2, 3$).

Problems concerning extents:

- $O_1 \subseteq G$ is an extent of some concept
  iff $O_1 = e(i(O_1))$
  iff $p_1 \leftrightarrow [\![R]\!][\![S]\!]p_1$ is true in the models such that the meaning of p_1 is O_1.
- (O_1, A_1) is the unique concept of which O_1 is an extent
  iff $A_1 = i(O_1)$ and $O_1 = e(i(O_1))$
  iff $q_1 \leftrightarrow [\![S]\!]p_1 \wedge p_1 \leftrightarrow [\![R]\!][\![S]\!]p_1$ is true in the models such that the meanings of p_1 and q_1 are O_1 and A_1, respectively.

Problems concerning intents:

- $A_1 \subseteq M$ is an intent of some concept
  iff $A_1 = i(e(A_1))$
  iff $q_1 \leftrightarrow [\![S]\!][\![R]\!]q_1$ is true in the models such that the meaning of q_1 is A_1.
- (O_1, A_1) is the unique concept of which A_1 is an intent
  iff $O_1 = e(A_1)$ and $A_1 = i(e(A_1))$
  iff $p_1 \leftrightarrow [\![R]\!]q_1 \wedge q_1 \leftrightarrow [\![S]\!][\![R]\!]q_1$ is true in the models such that the meanings of p_1 and q_1 are O_1 and A_1, respectively.

Problems concerning operations on concepts:

- A formal concept (O_1, A_1) is the join of two concepts (O_2, A_2) and (O_3, A_3)
  iff $O_1 = e(i(O_2 \cup O_3))$ and $A_1 = A_2 \cap A_3$
  iff $(p_1 \leftrightarrow [\![R]\!][\![S]\!](p_2 \vee p_3)) \wedge (q_1 \leftrightarrow q_2 \wedge q_3)$ is true in the models such that, for $i = 1, 2, 3$, the meanings of p_i and q_i are O_i and A_i, respectively.
- A formal concept (O_1, A_1) is the meet of two concepts (O_2, A_2) and (O_3, A_3)
  iff $O_1 = O_2 \cap O_3$ and $A_1 = i(e(A_2 \cup A_3))$
  iff $(p_1 \leftrightarrow p_2 \wedge p_3) \wedge (q_1 \leftrightarrow [\![S]\!][\![R]\!](q_2 \vee q_3))$ is true in the models such that, for $i = 1, 2, 3$, the meanings of p_i and q_i are O_i and A_i, respectively.

– A formal concept (O_1, A_1) is a weak negation of some concept (O_2, A_2) iff $O_1 = e(i(\overline{O_2}))$ and $A_1 = i(\overline{O_2})$ iff $(p_1 \leftrightarrow [\![R]\!][\![S]\!]\neg p_2) \;\wedge\; (q_1 \leftrightarrow [\![S]\!]\neg p_2)$ is true in the models such that, for $i = 1, 2$, the meanings of p_i and q_i are O_i and A_i, respectively.
– A formal concept (O_1, A_1) is a weak opposition of some concept (O_2, A_2) iff $O_1 = e(\overline{A_2})$ and $A_1 = i(e(\overline{A_2}))$ iff $(p_1 \leftrightarrow [\![R]\!]\neg q_2) \;\wedge\; (q_1 \leftrightarrow [\![S]\!][\![R]\!]\neg q_2)$ is true in the models such that, for $i = 1, 2$, the meanings of p_i and q_i are O_i and A_i, respectively.

It follows that the reasoning tools of the context logic LC can be used for verification of the properties of concepts listed above, among others. A dual tableau deduction system for the logic LC is presented in [8].

4.2 Dependencies of Attributes

Discovering dependencies in sets of data is an important issue addressed in various theories, in particular in rough set theory [17] and in formal concept analysis [7]. Typically, in an information system objects are described in terms of some attributes and their values. The queries to an information system often have the form of a request for finding a set of objects whose sets of attribute values satisfy some conditions. This leads to the notion of information relation determined by a set of attributes. Let $a(x)$ and $a(y)$ be sets of values of an attribute a of the objects x and y, respectively. We may want to know a set of those objects from an information system whose sets of values of all (or some) of the attributes from a subset A of attributes are equal (or disjoint, or overlap etc.). To represent such queries we define, first, information relations on the set of objects. Some examples, defined in [5], include similarity relation, indiscernibility relations, forward inclusion, backward inclusion, negative similarity, incomplementarity relation. In the rough set-based approach the most fundamental information relation is indiscernibility and its weaker version, namely, similarity.

Let M be a set of attributes and G a set of objects. Given an attribute $a \in M$, the similarity relation $\mathsf{sim}(a) \subseteq G \times G$ is defined, for any $x, y \in G$, by

$$(x, y) \in \mathsf{sim}(a) \quad \text{iff} \quad a(x) \cap a(y) \neq \emptyset$$

and the indiscernibility relation $\mathsf{ind}(a) \subseteq G \times G$ is defined, for any $x, y \in G$, by

$$(x, y) \in \mathsf{ind}(a) \quad \text{iff} \quad a(x) = a(y).$$

These relations can be extended to any subset A of attributes by quantifying over A:

$$\begin{array}{llll} (x, y) \in \mathsf{sim}(A) & \text{iff} & a(x) \cap a(y) \neq \emptyset & \text{for all (some) } a \in A. \\ (x, y) \in \mathsf{ind}(A) & \text{iff} & a(x) = a(y) & \text{for all (some) } a \in A. \end{array}$$

Relations defined with the universal (existential) quantifier are referred to as strong (weak) relations.

Attribute dependencies, introduced in [2], express a constraint between two sets of attributes. Such constraints have been used to exclude from an information system data inappropriate for a particular application. An example of an attribute dependency involving single sets A and B of attributes is a *functional dependency* $A \to B$. Typically, a functional dependency is based on an information relation, for example,

$$A \to_{\mathsf{sim}} B \quad \text{iff} \quad \mathsf{sim}(A) \subseteq \mathsf{sim}(B)$$
$$A \to_{\mathsf{ind}} B \quad \text{iff} \quad \mathsf{ind}(A) \subseteq \mathsf{ind}(B).$$

Some attribute dependencies involve combinations of attributes. For example, a *multi-valued dependency* $A \to\to B$ between sets A and B of attributes is defined by

$$A \to\to B \qquad \text{iff} \qquad \mathsf{ind}(A) \subseteq \mathsf{ind}(A \cup B); \mathsf{ind}(M/(A \cup B))),$$

where for relations R and S over a universe U their composition $R;S$ is defined, for any $x, y \in U$, by $xR;Sy$ iff for some $z \in U$ xRz and zSy.

A representation of attribute dependencies in terms of relations generated by equivalence relations $\mathsf{ind}(A)$ is presented in [3].

In the next two theorems we will characterise these notions within the framework of context algebras. For this we need the following observations. For each $A \subseteq M$, $\mathsf{ind}(A) \subseteq G \times G$ and ind may be viewed as a binary relation of type $M \times (G \times G)$. Then $[\![\mathsf{ind}(A)]\!] : 2^G \to 2^G$ is given, for any $Q \subseteq G$, by

$$[\![\mathsf{ind}(A)]\!](Q) = \{x \in G \mid \forall y \in G,\ y \in Q \ \Rightarrow\ (x, y) \in \mathsf{ind}(A)\},$$

and $[\![\mathsf{ind}]\!] : 2^{G \times G} \to 2^M$ is given, for any $R \subseteq G \times G$, by

$$[\![\mathsf{ind}]\!](R) = \{a \in M \mid \forall (x, y) \in G \times G,\ (x, y) \in R \ \Rightarrow\ (x, y) \in \mathsf{ind}(a)\}.$$

Also, $\mathsf{ind}^{-1} \subseteq (G \times G) \times M$, so $[\![\mathsf{ind}^{-1}]\!] : 2^M \to 2^{G \times G}$ is given, for any $A \subseteq M$, by

$$[\![\mathsf{ind}^{-1}]\!](A) = \{(x, y) \mid \forall a \in M,\ a \in A \ \Rightarrow\ (x, y) \in \mathsf{ind}(a)\}.$$

Similarly, for sim.

Theorem 12. *For any* $A, B \in 2^M$,

$$A \to_{\mathsf{ind}} B \quad \text{iff} \quad B \subseteq [\![\mathsf{ind}]\!][\![\mathsf{ind}^{-1}]\!](A)$$
$$A \to_{\mathsf{sim}} B \quad \text{iff} \quad B \subseteq [\![\mathsf{sim}]\!][\![\mathsf{sim}^{-1}]\!](A)$$

Proof: For any $A \in 2^M$ and any $b \in M$,

$$\begin{aligned}
A \to_{\mathsf{ind}} b \quad & \text{iff} \quad \mathsf{ind}(A) \subseteq \mathsf{ind}(b) \\
& \text{iff} \quad \forall x, y,\ A \subseteq \mathsf{ind}^{-1}((x, y)) \ \Rightarrow\ (x, y) \in \mathsf{ind}(b) \\
& \text{iff} \quad \forall x, y,\ (x, y) \in [\![\mathsf{ind}^{-1}]\!](A) \ \Rightarrow\ (x, y) \in \mathsf{ind}(b) \\
& \text{iff} \quad [\![\mathsf{ind}^{-1}]\!](A) \subseteq \mathsf{ind}(b) \\
& \text{iff} \quad b \in [\![\mathsf{ind}]\!][\![\mathsf{ind}^{-1}]\!](A)
\end{aligned}$$

Now $A \rightarrow_{\mathsf{ind}} B$ holds if, for all $b \in B$, $A \rightarrow_{\mathsf{ind}} b$ holds, hence the result follows. Similarly, for sim. □

For any binary relation $R \subseteq X \times Y$, operators $[R] : 2^Y \rightarrow 2^X$ and $\langle R \rangle : 2^Y \rightarrow 2^X$ may be defined in terms of the sufficiency operator as follows:

$$[R]Q = [\![-R]\!](-Q) \qquad \langle R \rangle Q = -[\![-R]\!]Q, \quad \text{for any } Q \in 2^Y.$$

Theorem 13. *For any $A, B \in 2^M$,*

$$A \rightarrow\!\!\rightarrow B \quad \text{iff} \quad \forall y \in G,\ y \in [(\mathsf{ind}(A))^{-1}]\, \langle \mathsf{ind}(A \cup B) \rangle\, [\![\mathsf{ind}]\!](M/\ (A \cup B))(\{y\}).$$

Proof: For this it suffices to show that for any $A, B, C \in 2^M$,

$$\mathsf{ind}(A) \subseteq \mathsf{ind}(B); \mathsf{ind}(C) \quad \text{iff} \quad \forall y \in G,\ y \in [(\mathsf{ind}(A))^{-1}]\, \langle \mathsf{ind}(B) \rangle\, [\![\mathsf{ind}(C)]\!](\{y\}).$$

$$\begin{array}{ll} & \mathsf{ind}(A) \subseteq \mathsf{ind}(B); \mathsf{ind}(C) \\ \text{iff} & \forall x, \forall y,\ (x, y) \in \mathsf{ind}(A) \ \Rightarrow\ \exists z, (x, z) \in \mathsf{ind}(B) \ \wedge\ (z, y) \in \mathsf{ind}(C) \\ \text{iff} & \forall x, \forall y,\ (x, y) \in \mathsf{ind}(A) \ \Rightarrow\ \exists z, (x, z) \in \mathsf{ind}(B) \ \wedge\ z \in [\![\mathsf{ind}(C)]\!](\{y\}) \\ \text{iff} & \forall y, \forall x,\ (x, y) \in \mathsf{ind}(A) \ \Rightarrow\ x \in \langle \mathsf{ind}(B) \rangle [\![\mathsf{ind}(C)]\!](\{y\}) \\ \text{iff} & \forall y,\ y \in [(\mathsf{ind}(A))^{-1}] \langle \mathsf{ind}(B) \rangle [\![\mathsf{ind}(C)]\!](\{y\}). \end{array}$$

□

Relationships between Galois connections and dependencies of attributes are also studied in [9]. It is shown there that every Galois connection between two complete lattices determines an Armstrong system of functional dependencies.

4.3 Implications

In the representation of data of an information system as a formal context, (many-valued) attributes are refined into several (single-valued) features which are essentially attribute-value pairs. For example, the attribute colour may be refined to the attribute-value pair (colour, green) which corresponds to the feature being of colour green. Each object determines an object-concept (O, A) where A is the set of features of the given object, and O is the set of all objects having features in A. On the other hand, each feature determines a feature-concept (O, A) where O is the set of objects having the given feature, and A is the set of all features of objects in O.

Constraints between two sets of features are usually called *implications*. An *implication* $A \rightarrow B$ between sets A and B of features holds in a context (G, M, I) iff $e(A) \subseteq e(B)$, meaning that each object in G having all the features from A has all the features from B. An implication $A \rightarrow B$ between sets A and B of attributes is *trivial*, if $B \subseteq A$. As a consequence of the connections established in Section 2 between context algebras and formal contexts, we have

Theorem 14. *Let (G, M, I) be a context. For any $A, B \in 2^M$,*

$$A \rightarrow B \ \text{ iff } \ [\![I]\!](A) \subseteq [\![I]\!](B).$$

Hence an implication $A \to B$ holds in a context (G, M, I) iff

$$\forall g \in G,\ (\forall a \in A,\ gIa) \Rightarrow (\forall b \in B,\ gIb).$$

This is the definition of an association rule [1] used in data mining and therefore within our framework we have established a connection between implications in formal concept analysis and association rules. Moreover, the above provides an alternative to the relational characterisation, considered in [11], in terms of a so-called association relation.

Taking into account support and confidence, an association rule is defined, in [1], to be a constraint, denoted by $r : A \to B$, between sets A and B of attributes where $A, B \neq \emptyset$, $A \cap B = \emptyset$, and the support and the confidence of $r : A \to B$ are defined respectively to be

$$\mathsf{supp}(r) = \frac{|e(A \cup B)|}{|G|} \qquad \text{and} \qquad \mathsf{conf}(r) = \frac{|e(A \cup B)|}{|e(A)|},$$

where $|X|$ denotes the cardinality of a set X.

The set of association rules holding in a formal context (G, M, R) given $\mathsf{minsupp}$ and $\mathsf{minconf}$ is

$$AR = \{r : A \to B/A \mid A \subset B \subseteq M \ \wedge\ \mathsf{supp}(r) \geq \mathsf{minsupp} \ \wedge\ \mathsf{conf}(r) \geq \mathsf{minconf}\}.$$

If $\mathsf{conf}(r) = 1$ then r is called an *exact*. If $\mathsf{supp}(r) = \mathsf{supp}(A \cup B) = \mathsf{supp}(A)$ and $\mathsf{conf}(r) = 1$ then r is called a *deterministic* association rule. Otherwise it is an *approximate association rule*.

Since the notions of support and confidence are defined in terms of the extent operator e which can be characterised in terms of a sufficiency operator, we have the following characterisation within our framework of this notion of deterministic rule.

Theorem 15. *Let (G, M, I) be a context. If $r : A \to B/A$ is a deterministic association rule then $[\![I]\!](A \cup B) = G = [\![I]\!](A) = [\![I]\!](B)$.*

Proof: Assume $A, B \in 2^M$ are non-empty. If $r : A \to B/A$ is a deterministic association rule then $A \subset B$ and $|[\![I]\!](B)| = |[\![I]\!](A)|$ and $|[\![I]\!](A \cup B)| = |G|$. Hence $A \subset B$ and $|[\![I]\!](A)| = |[\![I]\!](B)| = |[\![I]\!](A \cup B)| = |G|$. Thus $[\![I]\!](A) = [\![I]\!](B) = [\![I]\!](A \cup B) = G$. □

Therefore, this notion of deterministic association rule is a special type of implication arising in formal concept analysis.

5 Conclusion

The aim of this paper has been to present a framework, based on discrete duality, for representing contexts from formal concept analysis. For contexts we established a discrete duality between context algebras and context frames, the latter being the frame semantics for context logic. In addition, we motivate the usefulness of the associated context logic for reasoning about properties of formal concepts, of attribute dependencies, and of implications.

This paper builds on earlier work in a number of ways. First, the discrete duality between context algebras and context frames extends an observation that intent and extent operators of a context are sufficiency operators on Boolean algebras, and provides another application of the duality via truth framework of [13]. Second, the uniform characterizations of often independently studied attribute dependencies and implications are new and allow for their comparison with the notion of association rule [1] used in data mining. As a consequence, the associated context logical techniques may be used for verifying typical problems, such as satisfaction and logical implication, of attribute dependencies and/or association rules.

A number of challenges remain. For example: to extend the connections established in Subsection 4.3 and develop an approach based on the presented framework for mining association rules. Perhaps further questions will occur to the reader.

Acknowledgement. The authors would like to thank the anonymous referees for helpful comments and suggestions.

References

1. Agrawal, R., Imielinski, T., Swami, A.N.: Mining association rules between sets of items in large databases. In: Buneman, P., Jajodia, S. (eds.) Proceedings of the 1993 ACM SIGMOD International Conference on Managements of Data, pp. 207–216 (1993)
2. Armstrong, W.W.: Dependency structures of database relationships. Information Processing Letters 7, 580–583 (1974)
3. Buszkowski, W., Orłowska, E.: Indiscernibility based formalization of dependencies in information systems. In: Orłowska, E. (ed.) Incomplete Information: Rough Set Analysis, pp. 293–315. Physica-Verlag, Heidelberg (1997)
4. Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press, Cambridge (1990)
5. Demri, S.P., Orłowska, E.S.: Incomplete Information: Structure, Inference, Complexity. EATCS Monographs in Theoretical Computer Science. Springer, Heidelberg (2002)
6. Düntsch, I., Orłowska, E.: Beyond modalities: Sufficiency and mixed algebras. In: Orłowska, E., Szałas, A. (eds.) Relational Methods in Algebra, Logic, and Computer Science, pp. 277–299. Physica-Verlag, Heidelberg (2001)
7. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer, Heidelberg (1999)
8. Golińska-Pilarek, J., Orłowska, E.: Relational Reasoning in Formal Concept Analysis. In: Proceedings of the 2007 IEEE International Conference on Fuzzy Systems, London, pp. 1048–1053. IEEE Press, Los Alamitos (2007)
9. Järvinen, J.: Pawlak's Information Systems in Terms of Galois Connections and Functional Dependencies. Fundamenta Informaticae 75, 315–330 (2007)

10. Jónson, B., Tarski, A.: Boolean algebras with operators, Part II. American Journal of Mathematics 74, 127–162 (1952)
11. MacCaul, W.: A tableaux for the implication problem for association rules. In: Orłowska, E., Szalas, A. (eds.) Relational Methods in Computer Science Applications, pp. 73–91. Springer-Physica Verlag, Heidelberg (2001)
12. Orłowska, E., Rewitzky, I., Düntsch, I.: Relational semantics through duality. In: Proceedings of the 8th International Conference on Relational Methods in Computer Science and 3rd International Conference on Applications of Kleene Algebras. LNCS, vol. 2939, pp. 17–32. Springer, Heidelberg (2006)
13. Orłowska, E., Rewitzky, I.: Duality via Truth: Semantic frameworks for lattice-based logics. Logic Journal of the IGPL 13(4), 467–490 (2005)
14. Orłowska, E., Rewitzky, I.: Discrete duality and its application to reasoning with incomplete information. In: Kryszkiewicz, M., Peters, J.F., Rybinski, H., Skowron, A. (eds.) RSEISP 2007. LNCS (LNAI), vol. 4585, pp. 51–56. Springer, Heidelberg (2007)
15. Orłowska, E., Rewitzky, I.: Algebras for Galois-style connections and their discrete duality. Fuzzy Sets and Systems (submitted, 2008)
16. Rasiowa, H., Sikorski, R.: The Mathematics of Metamathematics. Polish Scientific Publishers, Warsaw (1963)
17. Pawlak, Z.: Rough Sets - Theoretical Aspects of Reasoning about Data. Kluwer, Dordrecht (1991)
18. Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bulletin of the London Mathematical Society 2, 186–190 (1970)
19. Stone, M.H.: The theory of representations for Boolean algebras. Transactions of the American Mathematical Society 40, 37–111 (1936)
20. Von Karger, B.: Temporal algebra. Mathematical Structures in Computer Science 8, 277–320 (1995)
21. Wille, R.: Restructuring lattice theory: An approach based on hierarchies of concepts. In: Rival, I. (ed.) Ordered sets, NATO Advanced Studies Institute, vol. 83, pp. 445–470. Reidel, Dordrecht (1982)
22. Wille, R.: Boolean concept logic. In: Ganter, B., Mineau, G.W. (eds.) ICCS 2000. LNCS (LNAI), vol. 1867, pp. 317–331. Springer, Heidelberg (2000)