A Propositional Dynamic Logic Approach for Order of Magnitude Reasoning*

A. Burrieza[1], E. Muñoz-Velasco[2], and M. Ojeda-Aciego[2]

[1] Dept. Filosofía. Universidad de Málaga. Spain
burrieza@uma.es
[2] Dept. Matemática Aplicada. Universidad de Málaga. Spain
{emilio,aciego}@ctima.uma.es

Abstract. We introduce a Propositional Dynamic Logic for order of magnitude reasoning in order to formalize qualitative operations of sum and product. This new logic has enough expressive power to consider, for example, the concept of closeness, and to study some interesting properties for the qualitative operations, together with the logical definability of these properties. Finally, we show the applicability of our approach on the basis of some examples.

1 Introduction

Qualitative reasoning (QR) is the area of AI which tries to develop representation and reasoning techniques for dealing with situations in which the information is not sufficiently precise (e.g., exact numerical values are not available) or when numerical models are too complex. QR is somewhere in between heuristic models, based on symbolic manipulations and numerical models, based on numerical calculations.

A form of qualitative reasoning is illustrated by the management of numerical data in terms of orders of magnitude (see, for example, [19, 20, 16]). Order of magnitude reasoning stratifies values according to some notion of scale, for instance, by including hyperreal numbers, numerical thresholds, or logarithmic scales. Three issues faced by all these formalisms are the conditions under which many small effects can combine to produce a significant effect, the soundness of the reasoning supported by the formalism, and the efficiency of using them.

The introduction of a logical approach to QR tries to face the problem about the soundness of the reasoning supported by the formalism, and to give some answers about the efficiency of its use. Several logics have been defined to use QR in different contexts, [2,24,11], e.g., spatial and temporal reasoning. In particular, logics dealing with order of magnitude reasoning have been developed in [8, 7] defining different qualitative relations (*order of magnitude*, *negligibility*, *non-closeness*, etc.) on the basis of qualitative classes obtained from the real line divided in intervals [23].

* Partially supported by projects TIN2006-15455-C03-01 and P6-FQM-02049.

H. Geffner et al. (Eds.): IBERAMIA 2008, LNAI 5290, pp. 11–20, 2008.

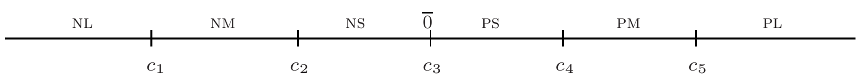
In this paper, we introduce a logic approach in order to formalize qualitative operations of sum and product between our intervals (like "add a small positive number" or "multiply by a negative large number"). We consider that the introduction of this logic represents an important step stone, because, as said in [18], it contributes to model aspects of human knowledge which, together with the intrinsic qualities of the operators, are used in performing qualitative computations. To this end, we present a Propositional Dynamic Logic (henceforth PDL) with constants. The main purpose of extending modal logics to PDL [13,15,4], is to allow not only many fixed accessibility relations (as in multi-modal logic) but also the construction of complex relations from simple ones by using operators as union, composition, transitive closure, etc.

Recent applications of PDL in communication scenarios, message-passing systems, multiagent real-time systems and knowledge acquisition, can be seen in [3, 14, 6, 5], respectively. In our case, we introduce some nominals in order to represent the milestones which divide the different qualitative classes, for this reason we could say that our logic is a *hybrid* PDL, as a part of the Combinatory PDL [1,17]. This *hybrid dynamism* will give us enough expressive power to represent different situations, for example the important concept of closeness [19], and to consider some interesting properties for the qualitative operations, together with the logical definability of these properties. The introduction of closeness is one of the main differences of this approach with respect to the previous works in logic for order of magnitude reasoning. In these works, negligibility, order of magnitude, non-closeness and distance relations where introduced; however, a specific closeness relation was not provided. In our case, PDL will give us theoretical support in order to have, not only complete axiomatic systems to deal with this new relation, but also to study its decidability and complexity.

The paper is organized as follows. In Section 2, the syntax and semantics of the proposed logic is introduced. In Section 3, we study some properties of the qualitative sum and product and its definability, together with an example of application of our logic. Finally, some conclusions and prospects of future work are presented in Section 4.

2 Syntax and Semantics

We consider the real line $\mathbb{R}$ divided into seven equivalence classes using five landmarks $c_i \in \mathbb{R}$, chosen depending on the context [22], being $i \in \{1, 2, 3, 4, 5\}$ such that $c_i < c_{i+1}$, for all $i \in \{1, 2, 3, 4\}$:



The labels correspond, respectively, to the qualitative classes representing "negative large", "negative medium", "negative small", "zero", "positive small", "positive medium", and "positive large", being:

$$\text{NL} = (-\infty, c_1), \quad \text{NM} = [c_1, c_2) \quad \text{NS} = [c_2, c_3), \quad \overline{0} = \{c_3\}$$
$$\text{PS} = (c_3, c_4], \quad \text{PM} = (c_4, c_5], \quad \text{PL} = (c_5, +\infty)$$

In order to introduce the language of our logic, we consider a set of formulas Φ and a set of programs Π, which are defined recursively on disjoint sets Φ_0 and Π_0, respectively. Φ_0 is called the set of *atomic formulas* which can be thought as abstractions of properties of states. Similarly, Π_0 is called the set of *atomic programs* which are intended to represent basic instructions.

Formulas

- Φ_0 is a denumerable set of propositional variables which contains the finite set $\mathbb{C} = \{c_i \mid i \in \{1, 2, 3, 4, 5\}\}$ of constants together with $\top$ (true) and $\bot$ (false).
- If φ and ψ are formulas and a is a program, then $\varphi \vee \psi$, $\neg\varphi$, $\langle a \rangle\varphi$ are formulas.

As usual in propositional modal logic, $\varphi \to \psi$ represents $\neg\varphi \vee \psi$, and $[a]$ represents $\neg\langle a \rangle \neg$.

Programs

- $\Pi_0 = \{<, >, +_{\mathsf{Eq}}, \cdot_{\mathsf{Eq}}, \theta\}$, where $\mathsf{Eq} \in \{\mathsf{nl}, \mathsf{nm}, \mathsf{ns}, \mathsf{0}, \mathsf{ps}, \mathsf{pm}, \mathsf{pl}\}$ represents the equivalence classes defined above and θ is the null program.
- If a and b are programs and φ is a formula, then $(a; b)$ ("do a followed by b"), $a \cup b$ ("do a or b, nondeterministically"), a^* ("repeat a a finite, but nondeterministically determined, number of times") and $\varphi?$ ("proceed if φ is true, else fail") are also programs.

As an example of formulas, we can consider $\langle +_{\mathsf{ps}} \cup +_{\mathsf{ns}} \rangle$, $\langle \cdot_{\mathsf{pl}}; \cdot_{\mathsf{pl}} \rangle$ and $\langle \mathsf{ns}?; +_{\mathsf{pl}} \rangle$ in order to represent, respectively, the intuitive meanings of adding a (positive or negative) small number, multiplying twice by a positive large number and adding a positive large number to a negative small number.

We now define the *semantics* of our logic. A *model* $\mathcal{M}$ is a pair (W, m), where W is a non-empty subset[1] of $\mathbb{R}$ and m is a meaning function such that $m(\varphi) \subseteq W$, for all formula φ, and $m(a) \subseteq W \times W$, for all programs a. Moreover, for every formula φ and ψ and for all programs a, b, we have:

- $m(\top) = W$ and $m(\bot) = \varnothing$
- $m(c_i) \in W$, for all $i \in \{1, 2, 3, 4, 5\}$
- $m(\varphi \vee \psi) = m(\varphi) \cup m(\psi)$
- $m(\neg\varphi) = W - m(\varphi)$
- $m(\langle a \rangle\varphi) = \{w \in W : \exists v \in W \text{ such that } (w, v) \in m(a) \text{ and } v \in m(\varphi)\}$
- $m(<)$ is the restriction to W of the usual strict linear ordering of $\mathbb{R}$, such that $(m(c_i), m(c_{i+1})) \in m(<)$, for all $i \in \{1, 2, 3, 4\}$.
- $m(\theta) = \varnothing$
- $m(a \cup b) = m(a) \cup m(b)$

[1] We could use any strict linearly ordered set with two internal operations $+$ and $\cdot$.

- $m(a;b) = m(a);m(b)$ (composition of relations $m(a)$ and $m(b)$)
- $m(a^*) = m(a)^*$ (reflexive and transitive closure of relation $m(a)$).
- $m(\varphi?) = \{(w,w) : w \in m(\varphi)\}$

Given a model $\mathcal{M} = (W, m)$, a formula φ is *true* in $w \in W$ whenever $w \in m(\varphi)$. We say that φ is *valid in a model* $\mathcal{M} = (W, m)$ if φ is true in all $w \in W$, that is, if $m(\varphi) = W$. Finally, φ is *valid* iff φ is valid in all models.

The informal meaning of some of our connectives is given as follows:

- $\langle<\rangle\varphi$ is true in w iff there exists w', greater than w, such that φ is true in w'.
- $\langle+_{\mathsf{pm}}\rangle\varphi$ is true in w iff there exists w', obtained by adding a positive medium number to w, such that φ is true in w'.
- $[\cdot_{\mathsf{pl}}]\,\varphi$ is true in w iff for every w', obtained by multiplying w by a positive large number, φ is true in w'.
- $\langle\mathsf{nl}?\rangle\,\varphi$ is true in w iff w is a negative large number and φ is true in w.
- $\langle+^*_{\mathsf{ns}}\rangle\,\varphi$ is true in w iff there exists w', obtained by adding a finite number of small negative numbers to w, such that φ is true in w'.

Notation: In the rest of the paper, we will use the intuitive notation ps as a formula which is true exactly in the set PS, that is, $\mathsf{ps} \equiv (\langle>\rangle c_3 \wedge \langle<\rangle c_4) \vee c_4$, and similarly for the rest of intervals. Moreover, we use the abbreviation $\Diamond\varphi \equiv (\langle>\rangle\varphi \vee \varphi \vee \langle<\rangle\varphi)$, for any formula φ, and similarly for $\Box$.

As said before, one of the main advantages of using PDL is the possibility of constructing complex programs from basic ones. As a consequence, following the ideas presented in [7], we can use our connectives in order to represent some relations as *negligibility* and *distance*. Moreover, we introduce a notion of *closeness* which was not included in the previous approaches of logics for order of magnitude reasoning. Thus, for any formula φ, we have:

$$\langle\mathsf{c}\rangle\,\varphi = \langle+_{\mathsf{ns}} \cup +_{0} \cup +_{\mathsf{ps}}\rangle\varphi \qquad \langle\mathsf{d}\rangle\,\varphi = \langle+_{\mathsf{nl}} \cup +_{\mathsf{pl}}\rangle\varphi$$

$$\langle\mathsf{n}\rangle\,\varphi = \langle c_3?\rangle\Diamond\varphi \vee \langle(\mathsf{ns} \vee \mathsf{ps})?\rangle\Diamond(\langle c_2?;+_{\mathsf{nl}}\rangle\varphi \vee \langle c_4?;+_{\mathsf{pl}}\rangle\varphi)$$

The intuitive interpretation of the closeness relation is that x is close to y if, and only if, y is obtained from x by adding a small number. On the other hand, x is distant from y if and only if y is obtained from x by adding a large number. Moreover, we assume that zero is negligible with respect to any real number and a small number is negligible with respect to any number *sufficiently large*, that is, distant either from c_2 or c_4.

Example 1. In this example, inspired in that given in [10], we show the expressiveness of our logic in order to make different comparisons. Let us consider three computational tasks with different ranges of difficulty. For instance:

(a) Add up a column of 100 numbers.
(b) Sort a list of 10,000 elements.
(c) Invert a 100 × 100 matrix.
(d) Add up a column of 120 numbers.

We can say that the time required by (a) and (d) are much shorter than the others, and task (b) is much shorter than (c). For example, we may assume that the time required by (b) is obtained from the time required by (a) by adding a positive medium number, that the time required by (a) is negligible with respect to the time required by (c) and that the time required by (a) is close to the time required by (d).

If formulas $time\_a, time\_b, time\_c$ and $time\_d$ represent the time required by (a), (b), (c) and (d) respectively, then the following formulas hold:

$$time\_a \to \langle +_{\mathsf{pm}} \rangle\, time\_b, \quad time\_a \to \langle \mathsf{n} \rangle\, time\_c, \quad time\_a \to \langle \mathsf{c} \rangle\, time\_d$$

If we assume also that the time required by (c) is obtained by multiplying by a positive large number the time required by (b), then we have:

$$time\_b \to \langle \cdot_{\mathsf{pl}} \rangle\, time\_c, \quad time\_a \to \langle +_{\mathsf{pm}}; \cdot_{\mathsf{pl}} \rangle\, time\_c$$

3 Some Properties of Qualitative Sum and Product

This section is devoted to study different properties of some classes of our models with respect to the qualitative operations sum and product defined previously. For simplicity, in the rest of the paper, we will consider the class $\mathbb{L}$ of models $\mathcal{M} = (U, m)$ such that:

1. $U \subseteq \mathbb{R}$, such that $0, \alpha, \beta \in U$
2. $m(c_3) = 0$, $-m(c_2) = m(c_4) = \alpha$, $-m(c_1) = m(c_5) = \beta$

To begin with, some elementary results of qualitative arithmetic are recalled; then, some relevant properties regarding sum and product are stated. Finally, we study the definability of these properties and give some examples.

Following [19], we consider the sum and product between non-empty sets and between elements and non-empty sets defined as follows:

Definition 1. *Let $X, Y \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ such that $X, Y \neq \varnothing$, we define the sets $x + Y$ and $x \cdot Y$, $X + Y$ and $X \cdot Y$ as follows:*

$$x + Y = \{x + y \mid y \in Y\} \qquad X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$$
$$x \cdot Y = \{xy \mid y \in Y\} \qquad X \cdot Y = \{xy \mid x \in X \text{ and } y \in Y\}$$

In the following proposition, we particularize the definition above to the seven qualitative classes of our order of magnitude model and obtain a number of intuitive properties.

The properties stated in the proposition below can be verified by straightforward inspection:

Proposition 1

1. $0 + \textsc{Eq} = \textsc{Eq}$*, for all* $\textsc{Eq} \in \{\textsc{nl}, \textsc{nm}, \textsc{ns}, 0, \textsc{ps}, \textsc{pm}, \textsc{pl}\}$
2. $\textsc{pl} + (\textsc{ps} \cup \textsc{pm} \cup \textsc{pl}) \subseteq \textsc{pl}$

3. $\textsc{pl} + \textsc{nm} \subseteq (\textsc{ps} \cup \textsc{pm} \cup \textsc{pl})$
4. $\textsc{pm} + (\textsc{ps} \cup \textsc{pm}) \subseteq (\textsc{pm} \cup \textsc{pl})$
5. $\textsc{pm} + \textsc{ns} \subseteq (\textsc{ps} \cup \textsc{pm})$
6. $\textsc{ps} + \textsc{ns} \subseteq (\textsc{ns} \cup 0 \cup \textsc{ps})$
7. If $\beta \geq 2\alpha$, *we have* $\textsc{pl} + \textsc{ns} \subseteq (\textsc{pm} \cup \textsc{pl})$ *and* $\textsc{ps} + \textsc{ps} \subseteq (\textsc{ps} \cup \textsc{pm})$
8. If $\beta \geq 1$, $(\textsc{pl} \cdot \textsc{pl}) \cup (\textsc{nl} \cdot \textsc{nl}) \subseteq \textsc{pl}$; $\quad \textsc{pl} \cdot \textsc{nl} \subseteq \textsc{nl}$
9. If $\alpha \geq 1$, *we have:*
   (a) $(\textsc{pm} \cdot \textsc{pl}) \cup (\textsc{nm} \cdot \textsc{nl}) \subseteq \textsc{pl}$
   (b) $(\textsc{pm} \cdot \textsc{nl}) \cup (\textsc{nm} \cdot \textsc{pl}) \subseteq \textsc{nl}$.
   (c) $(\textsc{nm} \cdot \textsc{nm}) \cup (\textsc{pm} \cdot \textsc{pm}) \subseteq (\textsc{pm} \cup \textsc{pl})$

Note that additional properties are obtained when interchanging negative and positive numbers in the items from 2 to 7.

The following concept of definability based on models gives us a relationship between properties of our models and formulas of our logic.

Definition 2. *Let $\mathbb{K}$ be a class of models and $\mathbb{J} \subseteq \mathbb{K}$ the class of all models in $\mathbb{K}$ having a certain property P. We say that P is definable by a formula φ in the class $\mathbb{K}$ if for every model $\mathcal{M} \in \mathbb{K}$, we have that $\mathcal{M} \in \mathbb{J}$ iff φ is valid in $\mathcal{M}$.*

In our case, we will say that P is definable in the class $\mathbb{K}$ if there exists a formula φ such that P is definable by φ in $\mathbb{K}$.

Proposition 2. *All the properties in Table 1 are definable in the class $\mathbb{L}$ defined above, but the exceptions stated below:*

– *Property 7 is definable in the class $\mathbb{L}_1$ of models in $\mathbb{L}$ such that $\beta \geq 2\alpha$.*
– *Property 8 is definable in the class $\mathbb{L}_2$ of models in $\mathbb{L}$ such that $\beta \geq 1$.*
– *Property 9 is definable in the class $\mathbb{L}_3$ of models in $\mathbb{L}$ such that $\alpha \geq 1$.*

Proof. We only show the definability of the first property in 7, that is, $\textsc{pl} + \textsc{ns} \subseteq (\textsc{pm} \cup \textsc{pl})$. The proof in the rest of the cases is similar.

To obtain the validity of the formula $\varphi = \langle \mathsf{pl}? \rangle [+_{\mathsf{ns}}] (\mathsf{pm} \vee \mathsf{pl})$, let (W, m) be a model in $\mathbb{L}_1$ and suppose $(\textsc{pl} + \textsc{ns}) \subseteq (\textsc{pm} \cup \textsc{pl})$. If $w \in W$ such that $w \in m(\mathsf{pl})$, this means that $w \in \textsc{pl}$. By hypothesis, this implies that, for every $w' \in W$ such that $(w, w') \in m(+_{\textsc{ns}})$, we have that $w' \in m(\mathsf{pm} \cup \mathsf{pl})$, hence $w \in m(\langle \mathsf{pl}? \rangle [+_{\mathsf{ns}}] (\mathsf{pm} \cup \mathsf{pl}))$. This means that φ is valid.

Conversely, consider a model (W, m) in which $(\textsc{pl} + \textsc{ns}) \not\subseteq (\textsc{pm} \cup \textsc{pl})$ holds. Then, there exists $w' \in \textsc{pl} + \textsc{ns}$, such that $w' \notin \textsc{pm} \cup \textsc{pl}$. Thus, $w' = w + w''$, being $w \in \textsc{pl}$ and $w'' \in \textsc{ns}$, hence $w \notin m(\langle \mathsf{pl}? \rangle [+_{\mathsf{ns}}] (\mathsf{pm} \vee \mathsf{pl}))$. This means that formula φ is not true in w, that is, φ is not valid in (W, m).

We now present some examples which show up the expressiveness of our logic and the applicability of the previous properties.

Example 2. We continue with the Example 1, by assuming now that the time required by tasks (a), (b) and (c) are positive small, medium and large numbers, respectively. This means that:

Table 1. Definability of Properties

	PROPERTY	DEFINED BY
1	$0 + \mathrm{EQ} = \mathrm{EQ}$	$\overline{0} \to [+_{\mathsf{Eq}}]\mathsf{Eq}$
2	$\mathrm{PL} + (\mathrm{PS} \cup \mathrm{PM} \cup \mathrm{PL}) \subseteq \mathrm{PL}$	$\langle \mathsf{pl}? \rangle [+_{\mathsf{ps}} \cup +_{\mathsf{pm}} \cup +_{\mathsf{pm}}]\mathsf{pl}$
3	$(\mathrm{PL} + \mathrm{NM}) \subseteq (\mathrm{PS} \cup \mathrm{PM} \cup \mathrm{PL})$	$\langle \mathsf{pl}? \rangle [+_{\mathsf{nm}}] (\mathsf{ps} \vee \mathsf{pm} \vee \mathsf{pl})$
4	$\mathrm{PM} + (\mathrm{PS} \cup \mathrm{PM}) \subseteq (\mathrm{PM} \cup \mathrm{PL})$	$\langle \mathsf{pm}? \rangle [+_{\mathsf{ps}} \cup +_{\mathsf{pm}}](\mathsf{pm} \vee \mathsf{pl})$
5	$(\mathrm{PM} + \mathrm{NS}) \subseteq (\mathrm{PS} \cup \mathrm{PM})$	$\langle \mathsf{pm}? \rangle [+_{\mathsf{ns}}] (\mathsf{ps} \vee \mathsf{pm})$
6	$(\mathrm{PS} + \mathrm{NS}) \subseteq (\mathrm{NS} \cup 0 \cup \mathrm{PS})$	$\langle \mathsf{ps}? \rangle [+_{\mathsf{ns}}] (\mathsf{ns} \vee \overline{0} \vee \mathsf{ps})$
7	$\mathrm{PL} + \mathrm{NS} \subseteq (\mathrm{PM} \cup \mathrm{PL})$ $\mathrm{PS} + \mathrm{PS} \subseteq (\mathrm{PS} \cup \mathrm{PM})$	$\langle \mathsf{pl}? \rangle [+_{\mathsf{ns}}] (\mathsf{pm} \vee \mathsf{pl})$ $\langle \mathsf{ps}? \rangle [+_{\mathsf{ps}}] (\mathsf{ps} \vee \mathsf{pm})$
8	$(\mathrm{PL} \cdot \mathrm{PL}) \cup (\mathrm{NL} \cdot \mathrm{NL}) \subseteq \mathrm{PL}$	$(\langle \mathsf{pl}? \rangle [\cdot_{\mathsf{pl}}] \mathsf{pl}) \wedge (\langle \mathsf{nl}? \rangle [\cdot_{\mathsf{nl}}] \mathsf{pl})$
9	$(\mathrm{PM} \cdot \mathrm{PL}) \cup (\mathrm{NM} \cdot \mathrm{NL}) \subseteq \mathrm{PL}$ $(\mathrm{PM} \cdot \mathrm{NL}) \cup (\mathrm{NM} \cdot \mathrm{PL}) \subseteq \mathrm{NL}$ $(\mathrm{NM} \cdot \mathrm{NM}) \cup (\mathrm{PM} \cdot \mathrm{PM}) \subseteq (\mathrm{PM} \cup \mathrm{PL})$	$(\langle \mathsf{pl}? \rangle [\cdot_{\mathsf{pm}}] \mathsf{pl}) \wedge (\langle \mathsf{nl}? \rangle [\cdot_{\mathsf{nm}}] \mathsf{pl})$ $(\langle \mathsf{nl}? \rangle [\cdot_{\mathsf{pm}}] \mathsf{nl}) \wedge (\langle \mathsf{pl}? \rangle [\cdot_{\mathsf{nm}}] \mathsf{nl})$ $(\langle \mathsf{nm}? \rangle [\cdot_{\mathsf{nm}}] (\mathsf{pm} \vee \mathsf{pl})) \wedge (\langle \mathsf{pm}? \rangle [\cdot_{\mathsf{pm}}] (\mathsf{pm} \vee \mathsf{pl}))$

$$time\_a \to \mathsf{ps} \quad time\_b \to \mathsf{pm}, \quad time\_c \to \mathsf{pl}$$

If we assume that $\beta \geq 2\alpha$, properties 4 and 7 of Proposition 1 can be applied to obtain, respectively:

$$time\_b \to [+_{\mathsf{ps}}]\,(\mathsf{pm} \vee \mathsf{pl}), \quad time\_a \to [+_{\mathsf{ps}}]\,(\mathsf{ps} \vee \mathsf{pm})$$

The intuitive reading of the formula $time\_b \to [+_{\mathsf{ps}}]\,(\mathsf{pm} \vee \mathsf{pl})$, is "if we add the time required by task (b) to the required one by any task with the same order of magnitude to task (a) (in this case ps), the time obtained has the same order of magnitude than the one required by task (b) or by task (c), i.e., pm or pl." In the same way, formula $time\_a \to [+_{\mathsf{ps}}]\,(\mathsf{ps} \vee \mathsf{pm})$ can be interpreted as "if we add the time required by task (a) to the time required by any task with the same order of magnitude than (a), the time obtained has the same order of magnitude than the one required by any task (a) or (b)".

Example 3. We consider now the heat exchanger studied, for example, in [20]. Let DTH be the temperature drop of the hot stream, DTC is the temperature rise of the cold stream, and DT1, DT2 are the driving force in the left and right ends of the device, respectively. Moreover, FH and KH are, respectively, the molar-flowrate and molar-heat of the hot stream and FC, KC the molar-flowrate and molar-heat of the cold stream. Notice that DTH, DTC, DT1, DT2, FH, KH, FC and KC are all positive real numbers.

The following equations are consequence of the previous definition and energy conservation:

$$\mathsf{DTH} - \mathsf{DT1} - \mathsf{DTC} + \mathsf{DT2} = 0 \tag{1}$$

$$\mathsf{DTH} \cdot \mathsf{KH} \cdot \mathsf{FH} = \mathsf{DTC} \cdot \mathsf{KC} \cdot \mathsf{FC} \tag{2}$$

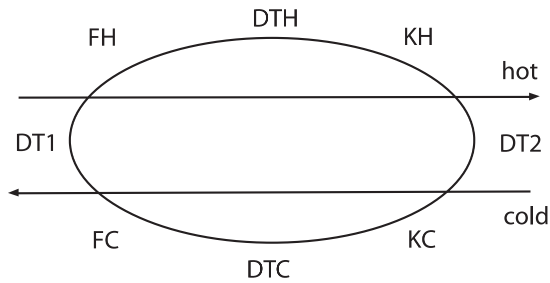


Fig. 1. The heat exchanger

In this example, we use models such that every value of $\mathsf{DTH}, \mathsf{DTC}, \ldots$ is represented as a current state in the real line, being the milestones α and β such that $\beta \geq 2\alpha$ and $\alpha \geq 1$.

We define dth as the formula which is true iff the current value is DTH, and similarly for the rest of values dtc, $\mathsf{dt1}$, etc. Assume that DTH is a positive large number and $\mathsf{DT1}, \mathsf{DT2}$ are positive small numbers, that is, the following formulas hold:

$$\mathsf{dth} \to \mathsf{pl} \qquad (\mathsf{dt1} \vee \mathsf{dt2}) \to \mathsf{ps}$$

From (1) and our previous assumptions, we have that $\mathsf{DTC} \in (\mathsf{DTH} + \textsc{ps}) + \textsc{ns}$, which implies that the following formula is true:

$$\mathsf{dth} \to \langle +_{\mathsf{ps}}; +_{\mathsf{ns}} \rangle\, \mathsf{dtc}$$

Thus[2], we have $\mathsf{DTC} \in (\textsc{pl} + \textsc{ps}) + \textsc{ns}$. Now, from Proposition 1(2), we have that $\textsc{pl} + \textsc{ps} \subseteq \textsc{pl}$, and from Proposition 1(7), we obtain $\textsc{pl} + \textsc{ns} \subseteq (\textsc{pm} \cup \textsc{pl})$. As a consequence, $(\textsc{pl} + \textsc{ps}) + \textsc{ns} \subseteq (\textsc{pm} \cup \textsc{pl})$. This means that $\mathsf{DTC} \in (\textsc{pm} \cup \textsc{pl})$. Hence, the following formula is true:

$$\mathsf{dtc} \to (\mathsf{pm} \vee \mathsf{pl})$$

This means that if DTH is *large*, then DTC is *medium* or *large*, that is, if we want that the device decreases very much the temperature of the hot stream, we have to accept that the temperature of the cold stream increases much.

On the other hand, from (1) again and our previous assumptions, we have that $\mathsf{DTC} - \mathsf{DTH} \in \textsc{ps} + \textsc{ns}$. Now, Proposition 1(6), we have that $\textsc{ps} + \textsc{ns} \subseteq (\textsc{ns} \cup 0 \cup \textsc{ps})$.

As a consequence, the difference between DTC and DTH is a small number while DTH is a large number.

Finally, if we assume also that KH and KC are positive medium numbers, by Proposition 1(9a,c), we have: $(\textsc{pm} \cdot \textsc{pl}) \subseteq \textsc{pl}$ and $(\textsc{pm} \cdot \textsc{pm}) \subseteq (\textsc{pm} \cup \textsc{pl})$, from (2), we have:

$$\langle \mathsf{fh}? \rangle \langle \cdot_{\mathsf{pl}} \rangle\, \mathsf{pl} \to \Box \langle \mathsf{fc}? \rangle \langle \cdot_{\mathsf{pm}} \cup \cdot_{\mathsf{pl}} \rangle\, \mathsf{pl}$$

[2] Notice that although qualitative sum is not necessarily associative, in this case we could eliminate the brackets.

The meaning of this formula is: if we make the product of FH by a large number (as the product $\mathsf{DTH} \cdot \mathsf{KH}$) and we obtain, for example, a positive large number, then we have the same order of magnitude if we make the product of FC by a medium or large number (as the product $\mathsf{DTC} \cdot \mathsf{KC}$).

4 Conclusions and Future Work

A PDL for order of magnitude reasoning has been introduced, which gives us enough expressive power to introduce different qualitative operations of sum and product. As a consequence, we are able to define notions as negligibility, closeness and distance. Moreover, we have studied the definability of interesting properties of sum and product of real numbers and its applicability has been shown on the basis of an example.

As a future work, the introduction of PDL will give us the theoretical support in order to give a complete axiomatization of this logic and to study its decidability and complexity [13, 17]. Last, but not least, we want to give a relational proof system based on dual tableaux for this extension in the line of [9, 12].

Acknowledgements

We thank Prof. Andrzej Salwicki for suggesting the subject of this work.

References

1. Areces, C., ten Cate, B.: Hybrid Logics. In: Blackburn, P., Van Benthem, J., Wolter, F. (eds.) Handbook of Modal Logic. Studies in Logic and Practical Reasoning, vol. 3, pp. 821–868. Elsevier, Amsterdam (2007)
2. Bennett, B., Cohn, A.G., Wolter, F., Zakharyaschev, M.: Multi-Dimensional Modal Logic as a Framework for Spatio-Temporal Reasoning. Applied Intelligence 17(3), 239–251 (2002)
3. Benthem, J., Eijck, J., Kooi, B.: Logics of communication and change. Information and Computation 204(11), 1620–1662 (2006)
4. Blackburn, P., Van Benthem, J.: Modal Logic: A semantic perspective. In: Blackburn, P., Van Benthem, J., Wolter, F. (eds.) Handbook of Modal Logic. Studies in Logic and Practical Reasoning, vol. 3, pp. 58–61. Elsevier, Amsterdam (2007)
5. Bollig, B., Kuske, D., Meinecke, I.: Propositional dynamic logic for message-passing systems. In: Arvind, V., Prasad, S. (eds.) FSTTCS 2007. LNCS, vol. 4855, pp. 303–315. Springer, Heidelberg (2007)
6. Bugaychenko, D., Soloviev, I.: MASL: A logic for the specification of multiagent real-time systems. In: Burkhard, H.-D., Lindemann, G., Verbrugge, R., Varga, L.Z. (eds.) CEEMAS 2007. LNCS (LNAI), vol. 4696, pp. 183–192. Springer, Heidelberg (2007)
7. Burrieza, A., Muñoz-Velasco, E., Ojeda-Aciego, M.: A Logic for Order of Magnitude Reasoning with Negligibility, Non-closeness and Distance. In: Borrajo, D., Castillo, L., Corchado, J.M. (eds.) CAEPIA 2007. LNCS (LNAI), vol. 4788, pp. 210–219. Springer, Heidelberg (2007)

8. Burrieza, A., Ojeda-Aciego, M.: A multimodal logic approach to order of magnitude qualitative reasoning with comparability and negligibility relations. Fundamenta Informaticae 68, 21–46 (2005)
9. Burrieza, A., Ojeda-Aciego, M., Orłowska, E.: Relational approach to order of magnitude reasoning. In: de Swart, H., Orłowska, E., Schmidt, G., Roubens, M. (eds.) TARSKI 2006. LNCS (LNAI), vol. 4342, pp. 105–124. Springer, Heidelberg (2006)
10. Davis, E.: Order of Magnitude Comparisons of Distance. Journal of Artificial Intelligence Research 10, 1–38 (1999)
11. Duckham, M., Lingham, J., Mason, K., Worboys, M.: Qualitative reasoning about consistency in geographic information. Information Sciences 176(6,22), 601–627 (2006)
12. Golińska-Pilarek, J., Muñoz-Velasco, E.: Relational approach for a logic for order of magnitude qualitative reasoning with negligibility, non-closeness and distance. Technical Report (2008)
13. Harel, D., Kozen, D., Tiury: Dynamic logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn., vol. 4, pp. 99–218 (2002)
14. Heinemann, B.: A PDL-like logic of knowledge acquisition. In: Diekert, V., Volkov, M.V., Voronkov, A. (eds.) CSR 2007. LNCS, vol. 4649, pp. 146–157. Springer, Heidelberg (2007)
15. Mirkowska, C., Salwicki, A.: Algorithmic Logic. Kluwer Academic Publishers, Norwell (1987)
16. Nayak, P.: Causal Approximations. Artificial Intelligence 70, 277–334 (1994)
17. Passy, S., Tinchev, T.: An essay in combinatory dynamic logic. Information and Computation 93(2), 263–332 (1991)
18. Piera, N., Agell, N.: Binary Relations for Qualitative Reasoning. In: IEEE International Conference on Systems, Man and Cybernetics, vol. 1, pp. 267–271 (1992)
19. Raiman, O.: Order of magnitude reasoning. Artificial Intelligence 51, 11–38 (1991)
20. Sanchez, M., Prats, F., Piera, N.: Una formalización de relaciones de comparabilidad en modelos cualitativos. Boletín de la AEPIA (Bulletin of the Spanish Association for AI) 6, 15–22 (1996)
21. Shults, B., Kuipers, B.J.: Proving properties of continuous systems: qualitative simulation and temporal logic. Artificial Intelligence 92, 91–129 (1997)
22. Travé-Massuyès, L., Ironi, L., Dague, P.: Mathematical Foundations of Qualitative Reasoning. AI Magazine, American Asociation for Artificial Intelligence, 91–106 (2003)
23. Travé-Massuyès, L., Prats, F., Sánchez, M., Agell, N.: Relative and absolute order-of-magnitude models unified. Annals of Mathematics and Artificial Intelligence 45, 323–341 (2005)
24. Wolter, F., Zakharyaschev, M.: Qualitative spatio-temporal representation and reasoning: a computational perspective. In: Lakemeyer, G., Nebel, B. (eds.) Exploring Artificial Intelligence in the New Millenium. Morgan Kaufmann, San Francisco (2002)