2

Models for the Claim Number Process

2.1 The Poisson Process

In this section we consider the most common claim number process: the *Poisson process*. It has very desirable theoretical properties. For example, one can derive its finite-dimensional distributions explicitly. The Poisson process has a long tradition in applied probability and stochastic process theory. In his 1903 thesis, Filip Lundberg already exploited it as a model for the claim number process N. Later on in the 1930s, Harald Cramér, the famous Swedish statistician and probabilist, extensively developed collective risk theory by using the total claim amount process S with arrivals T_i which are generated by a Poisson process. For historical reasons, but also since it has very attractive mathematical properties, the Poisson process plays a central role in insurance mathematics.

Below we will give a definition of the Poisson process, and for this purpose we now introduce some notation. For any real-valued function f on $[0,\infty)$ we write

$$f(s,t] = f(t) - f(s)\,, \quad 0 \le s < t < \infty\,.$$

Recall that an integer-valued random variable M is said to have a Poisson distribution with parameter $\lambda > 0$ ($M \sim \text{Pois}(\lambda)$) if it has distribution

$$P(M = k) = \mathrm{e}^{-\lambda}\,\frac{\lambda^k}{k!}\,, \quad k = 0, 1, \ldots\,.$$

We say that the random variable $M = 0$ a.s. has a Pois(0) distribution. Now we are ready to define the *Poisson process.*

Definition 2.1.1 (Poisson process)
A stochastic process $N = (N(t))_{t\ge 0}$ *is said to be a* Poisson process *if the following conditions hold:*

(1) *The process starts at zero:* $N(0) = 0$ a.s.

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(2) *The process has independent increments: for any* t_i, $i = 0, \ldots, n$, *and* $n \geq 1$ *such that* $0 = t_0 < t_1 < \cdots < t_n$, *the increments* $N(t_{i-1}, t_i]$, $i = 1, \ldots, n$, *are mutually independent.*
(3) *There exists a non-decreasing right-continuous function* $\mu : [0, \infty) \to [0, \infty)$ *with* $\mu(0) = 0$ *such that the increments* $N(s, t]$ *for* $0 < s < t < \infty$ *have a Poisson distribution* $\text{Pois}(\mu(s, t])$. *We call* μ *the* mean value function *of* N.
(4) *With probability* 1, *the sample paths* $(N(t, \omega))_{t \geq 0}$ *of the process* N *are right-continuous for* $t \geq 0$ *and have limits from the left for* $t > 0$. *We say that* N *has* càdlàg (continue à droite, limites à gauche) *sample paths.*

We continue with some comments on this definition and some immediate consequences.

We know that a Poisson random variable M has the rare property that

$$\lambda = EM = \text{var}(M)\,,$$

i.e., it is determined only by its mean value (= variance) if the distribution is specified as Poisson. The definition of the Poisson process essentially says that, in order to determine the distribution of the Poisson process N, it suffices to know its mean value function. The mean value function μ can be considered as an inner clock or *operational time* of the counting process N. Depending on the magnitude of $\mu(s, t]$ in the interval $(s, t]$, $s < t$, it determines how large the random increment $N(s, t]$ is.

Since $N(0) = 0$ a.s. and $\mu(0) = 0$,

$$N(t) = N(t) - N(0) = N(0, t] \sim \text{Pois}(\mu(0, t]) = \text{Pois}(\mu(t))\,.$$

We know that the distribution of a stochastic process (in the sense of Kolmogorov's consistency or existence theorem[1]) is determined by its finite-dimensional distributions. The finite-dimensional distributions of a Poisson process have a rather simple structure: for $0 = t_0 < t_1 < \cdots < t_n < \infty$,

$$(N(t_1), N(t_2), \ldots, N(t_n)) =$$

$$\Big(N(t_1), N(t_1) + N(t_1, t_2], N(t_1) + N(t_1, t_2] + N(t_2, t_3], \ldots, \sum_{i=1}^{n} N(t_{i-1}, t_i]\Big).$$

where any of the random variables on the right-hand side is Poisson distributed. The independent increment property makes it easy to work with the finite-dimensional distributions of N: for any integers $k_i \geq 0$, $i = 1, \ldots, n$,

[1] Two stochastic processes on the real line have the same distribution in the sense of Kolmogorov's consistency theorem (cf. Rogers and Williams [126], p. 123, or Billingsley [18], p. 510) if their finite-dimensional distributions coincide. Here one considers the processes as random elements with values in the product space $\mathbb{R}^{[0,\infty)}$ of real-valued functions on $[0, \infty)$, equipped with the σ-field generated by the cylinder sets of $\mathbb{R}^{[0,\infty)}$.

$$\begin{aligned}
&P(N(t_1) = k_1\,, N(t_2) = k_1 + k_2\,, \ldots, N(t_n) = k_1 + \cdots + k_n) \\
&= P(N(t_1) = k_1\,, N(t_1, t_2] = k_2\,, \ldots\,, N(t_{n-1}, t_n] = k_n) \\
&= \mathrm{e}^{-\mu(t_1)}\,\frac{(\mu(t_1))^{k_1}}{k_1!}\,\mathrm{e}^{-\mu(t_1,t_2]}\,\frac{(\mu(t_1,t_2])^{k_2}}{k_2!}\cdots\mathrm{e}^{-\mu(t_{n-1},t_n]}\,\frac{(\mu(t_{n-1},t_n])^{k_n}}{k_n!} \\
&= \mathrm{e}^{-\mu(t_n)}\,\frac{(\mu(t_1))^{k_1}}{k_1!}\,\frac{(\mu(t_1,t_2])^{k_2}}{k_2!}\cdots\frac{(\mu(t_{n-1},t_n])^{k_n}}{k_n!}\,.
\end{aligned}$$

The càdlàg property is nothing but a standardization property and of purely mathematical interest which, among other things, ensures the measurability property of the stochastic process N in certain function spaces.[2] As a matter of fact, it is possible to show that one can define a process N on $[0, \infty)$ satisfying properties (1)-(3) of the Poisson process and having sample paths which are left-continuous and have limits from the right.[3] Later, in Section 2.1.4, we will give a constructive definition of the Poisson process. That version will automatically be càdlàg.

2.1.1 The Homogeneous Poisson Process, the Intensity Function, the Cramér-Lundberg Model

The most popular Poisson process corresponds to the case of a linear mean value function μ:

$$\mu(t) = \lambda\, t\,, \quad t \ge 0\,,$$

for some $\lambda > 0$. A process with such a mean value function is said to be *homogeneous*, *inhomogeneous* otherwise. The quantity λ is the *intensity* or *rate* of the homogeneous Poisson process. If $\lambda = 1$, N is called *standard homogeneous Poisson process.*

More generally, we say that N has an *intensity function* or *rate function* λ if μ is absolutely continuous, i.e., for any $s < t$ the increment $\mu(s, t]$ has representation

$$\mu(s,t] = \int_s^t \lambda(y)\,dy\,, \quad s < t\,,$$

for some non-negative measurable function λ. A particular consequence is that μ is a continuous function.

We mentioned that μ can be interpreted as operational time or inner clock of the Poisson process. If N is homogeneous, time evolves linearly: $\mu(s, t] = \mu(s + h, t + h]$ for any $h > 0$ and $0 \le s < t < \infty$. Intuitively, this means that

[2] A suitable space is the Skorokhod space $\mathbb{D}$ of càdlàg functions on $[0, \infty)$; cf. Billingsley [17].

[3] See Chapter 2 in Sato [132].

claims arrive roughly uniformly over time. We will see later, in Section 2.1.6, that this intuition is supported by the so-called *order statistics property* of a Poisson process. If N has non-constant intensity function λ time "slows down" or "speeds up" according to the magnitude of $\lambda(t)$. In Figure 2.1.2 we illustrate this effect for different choices of λ. In an insurance context, non-constant λ may refer to seasonal effects or trends. For example, in Denmark more car accidents happen in winter than in summer due to bad weather conditions. Trends can, for example, refer to an increasing frequency of (in particular, large) claims over the last few years. Such an effect has been observed in windstorm insurance in Europe and is sometimes mentioned in the context of climate change. Table 3.2.18 contains the largest insurance losses occurring in the period 1970-2007: it is obvious that the arrivals of the largest claim sizes cluster towards the end of this time period. We also refer to Section 2.1.7 for an illustration of seasonal and trend effects in a real-life claim arrival sequence.

A homogeneous Poisson process with intensity λ has

(1) càdlàg sample paths,
(2) starts at zero,
(3) has independent and *stationary* increments,
(4) $N(t)$ is $\text{Pois}(\lambda t)$ distributed for every $t > 0$.

Stationarity of the increments refers to the fact that for any $0 \leq s < t$ and $h > 0$,

$$N(s,t] \stackrel{d}{=} N(s+h,t+h] \sim \text{Pois}(\lambda\,(t-s))\,,$$

i.e., the Poisson parameter of an increment only depends on the length of the interval, not on its location.

A process on $[0,\infty)$ with properties (1)-(3) is called a *Lévy process.*[4] The homogeneous Poisson process is one of the prime examples of Lévy processes with applications in various areas such as queuing theory, finance, insurance, stochastic networks, to name a few. Another prime example of a Lévy process is *Brownian motion* B. In contrast to the Poisson process, which is a pure jump process, Brownian motion has continuous sample paths with probability 1 and its increments $B(s,t]$ are normally $\text{N}(0,\sigma^2\,(t-s))$ distributed for some $\sigma > 0$. Brownian motion has a multitude of applications in physics and finance, but also in insurance mathematics. Over the last 30 years, Brownian motion has been used to model prices of speculative assets (share prices, foreign exchange rates, composite stock indices, etc.).

Finance and insurance have been merging for many years. Among other things, insurance companies invest in financial derivatives (options, futures, etc.) which are commonly modeled by functions of Brownian motion such as solutions to stochastic differential equations. If one wants to take into account

[4] We refer to Chapter 10 for an introduction to the theory of general Lévy processes and their relation with the Poisson process.

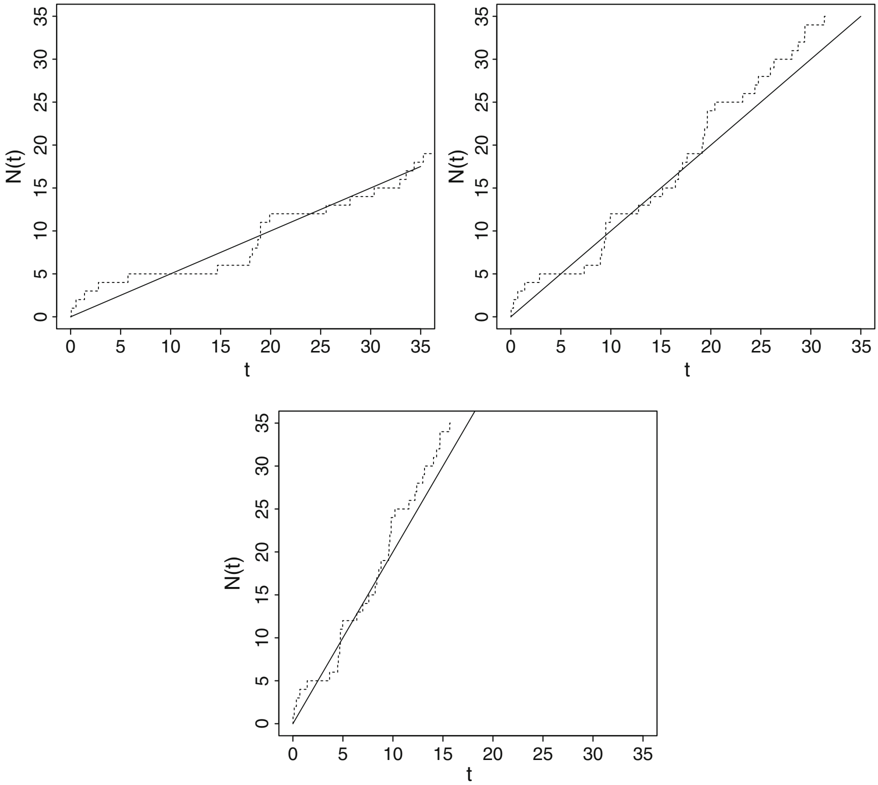

Figure 2.1.2 *One sample path of a Poisson process with intensity* 0.5 (top left)*,* 1 (top right) *and* 2 (bottom)*. The straight lines indicate the corresponding mean value functions. For* $\lambda = 0.5$ *jumps occur less often than for the standard homogeneous Poisson process, whereas they occur more often when* $\lambda = 2$*.*

jump characteristics of real-life financial/insurance phenomena, the Poisson process, or one of its many modifications, in combination with Brownian motion, offers the opportunity to model financial/insurance data more realistically. In this course, we follow the classical tradition of non-life insurance, where Brownian motion plays a less prominent role. This is in contrast to modern life insurance which deals with the inter-relationship of financial and insurance products.[5] For example, unit-linked life insurance can be regarded as classical life insurance which is linked to a financial underlying such as a composite stock index (DAX, S&P 500, Nikkei, CAC40, etc.). Depending on

[5] For a recent treatment of modern life insurance mathematics, see Møller and Steffensen [112].

the performance of the underlying, the policyholder can gain an additional bonus in excess of the cash amount which is guaranteed by the classical life insurance contracts.

Now we introduce one of the models which will be most relevant throughout this text.

Example 2.1.3 (The Cramér-Lundberg model)
The homogeneous Poisson process plays a major role in insurance mathematics. If we specify the claim number process as a homogeneous Poisson process, the resulting model which combines claim sizes and claim arrivals is called *Cramér-Lundberg model*:

- Claims happen at the arrival times $0 \le T_1 \le T_2 \le \cdots$ of a homogeneous Poisson process $N(t) = \#\{i \ge 1 : T_i \le t\}$, $t \ge 0$.
- The ith claim arriving at time T_i causes the claim size X_i. The sequence (X_i) constitutes an iid sequence of non-negative random variables.
- The sequences (T_i) and (X_i) are independent. In particular, N and (X_i) are independent.

The total claim amount process S in the Cramér-Lundberg model is also called a *compound Poisson process*.

The Cramér-Lundberg model is one of the most popular and useful models in non-life insurance mathematics. Despite its simplicity it describes some of the essential features of the total claim amount process which is observed in reality.

We mention in passing that the total claim amount process S in the Cramér-Lundberg setting is a process with independent and stationary increments, starts at zero and has càdlàg sample paths. It is another important example of a Lévy process. Try to show these properties! □

Comments

The reader who wants to learn about Lévy processes is referred to Sato's monograph [132] or the references given in Chapter 10. There we give a short introduction to this class of processes and explain the close relationship with general Poisson processes. For applications of Lévy processes in different areas, see the recent collection of papers edited by Barndorff-Nielsen et al. [12]. Rogers and Williams [126] can be recommended as an introduction to Brownian motion, its properties and related topics such as stochastic differential equations. For an elementary introduction, see Mikosch [107].

2.1.2 The Markov Property

Poisson processes constitute one particular class of *Markov processes* on $[0, \infty)$ with state space $\mathbb{N}_0 = \{0, 1, \ldots\}$. This is a simple consequence of the independent increment property. It is left as an exercise to verify the *Markov* *property*, i.e., for any $0 = t_0 < t_1 < \cdots < t_n$ and non-decreasing natural numbers $k_i \geq 0$, $i = 1, \ldots, n$, $n \geq 2$,

$$P(N(t_n) = k_n \mid N(t_1) = k_1, \ldots, N(t_{n-1}) = k_{n-1})$$
$$= P(N(t_n) = k_n \mid N(t_{n-1}) = k_{n-1}).$$

Markov process theory does not play a prominent role in this course,[6] in contrast to a course on modern life insurance mathematics, where Markov models are fundamental.[7] However, the *intensity function of a Poisson process* N has a nice interpretation as the *intensity function of the Markov process* N. Before we make this statement precise, recall that the quantities

$$p_{k,k+h}(s,t) = P(N(t) = k+h \mid N(s) = k) = P(N(t) - N(s) = h),$$
$$0 \leq s < t, \quad k, h \in \mathbb{N}_0,$$

are called the *transition probabilities* of the Markov process N with state space $\mathbb{N}_0$. Since a.e. path $(N(t,\omega))_{t\geq 0}$ increases (verify this), one only needs to consider transitions of the Markov process N from k to $k+h$ for $h \geq 0$. The transition probabilities are closely related to the *intensities* which are given as the limits

$$\lambda_{k,k+h}(t) = \lim_{s \downarrow 0} \frac{p_{k,k+h}(t, t+s)}{s},$$

provided they and their analogs from the left exist, are finite and coincide. From the theory of stochastic processes, we know that the intensities and the initial distribution of a Markov process determine the distribution of this Markov process.[8]

Proposition 2.1.4 (Relation of the intensity function of the Poisson process and its Markov intensities)
Consider a Poisson process $N = (N(t))_{t\geq 0}$ which has a continuous intensity function λ on $[0,\infty)$. Then, for $k \geq 0$,

$$\lambda_{k,k+h}(t) = \begin{cases} \lambda(t) & \text{if } h = 1, \\ 0 & \text{if } h > 1. \end{cases}$$

In words, the intensity function $\lambda(t)$ of the Poisson process N is nothing but the intensity of the Markov process N for the transition from state k to state $k+1$. The proof of this result is left as an exercise.

[6] It is, however, no contradiction to say that almost all stochastic models in this course have a Markov structure. But we do not emphasize this property.
[7] See for example Koller [87] and Møller and Steffensen [112].
[8] We leave this statement as vague as it is. The interested reader is, for example, referred to Resnick [123] or Rogers and Williams [126] for further reading on Markov processes.

The intensity function of a Markov process is a quantitative measure of the likelihood that the Markov process N jumps in a small time interval. An immediate consequence of Proposition 2.1.4 is that is it is very unlikely that a Poisson process with continuous intensity function λ has jump sizes larger than 1. Indeed, consider the probability that N has a jump greater than 1 in the interval $(t, t+s]$ for some $t \geq 0$, $s > 0$:[9]

$$\begin{aligned} P(N(t,t+s] \geq 2) &= 1 - P(N(t,t+s] = 0) - P(N(t,t+s] = 1) \\ &= 1 - \mathrm{e}^{-\mu(t,t+s]} - \mu(t,t+s]\,\mathrm{e}^{-\mu(t,t+s]}\,. \qquad (2.1.1) \end{aligned}$$

Since λ is continuous,

$$\mu(t,t+s] = \int_t^{t+s} \lambda(y)\,dy = s\,\lambda(t)\,(1+o(1)) \to 0\,, \quad \text{as } s \downarrow 0\,.$$

Moreover, a Taylor expansion yields for $x \to 0$ that $\mathrm{e}^{x} = 1 + x + o(x)$. Thus we may conclude from (2.1.1) that, as $s \downarrow 0$,

$$P(N(t,t+s] \geq 2) = o(\mu(t,t+s]) = o(s)\,. \qquad (2.1.2)$$

It is easily seen that

$$P(N(t,t+s] = 1) = \lambda(t)\,s\,(1+o(1))\,. \qquad (2.1.3)$$

Relations (2.1.2) and (2.1.3) ensure that a Poisson process N with continuous intensity function λ is very unlikely to have jump sizes larger than 1. Indeed, we will see in Section 2.1.4 that N has only upward jumps of size 1 with probability 1.

2.1.3 Relations Between the Homogeneous and the Inhomogeneous Poisson Process

The homogeneous and the inhomogeneous Poisson processes are very closely related: we will show in this section that a deterministic time change transforms a homogeneous Poisson process into an inhomogeneous Poisson process, and vice versa.

Let N be a Poisson process on $[0,\infty)$ with mean value function[10] μ. We start with a standard homogeneous Poisson process $\widetilde{N}$ and define

[9] Here and in what follows, we frequently use the o-notation. Recall that we write for any real-valued function h, $h(x) = o(1)$ as $x \to x_0 \in [-\infty, \infty]$ if $\lim_{x\to x_0} h(x) = 0$ and we write $h(x) = o(g(x))$ as $x \to x_0$ if $h(x) = g(x)\,o(1)$ for any real-valued function $g(x)$.

[10] Recall that the mean value function of a Poisson process starts at zero, is non-decreasing, right-continuous and finite on $[0,\infty)$. In particular, it is a càdlàg function.

$$\widehat{N}(t) = \widetilde{N}(\mu(t)), \quad t \geq 0.$$

It is not difficult to see that $\widehat{N}$ is again a Poisson process on $[0, \infty)$. (Verify this! Notice that the càdlàg property of μ is used to ensure the càdlàg property of the sample paths $\widehat{N}(t, \omega)$.) Since

$$\widehat{\mu}(t) = E\widehat{N}(t) = E\widetilde{N}(\mu(t)) = \mu(t), \quad t \geq 0,$$

and since the distribution of the Poisson process $\widehat{N}$ is determined by its mean value function $\widehat{\mu}$, it follows that $N \stackrel{d}{=} \widehat{N}$, where $\stackrel{d}{=}$ refers to equality of the finite-dimensional distributions of the two processes. Hence the processes $\widehat{N}$ and N are not distinguishable from a probabilistic point of view, in the sense of Kolmogorov's consistency theorem; see the remark on p. 8. Moreover, the sample paths of $\widehat{N}$ are càdlàg as required in the definition of the Poisson process.

Now assume that N has a continuous and increasing mean value function μ. This property is satisfied if N has an a.e. positive intensity function λ. Then the inverse μ^{-1} of μ exists. It is left as an exercise to show that the process $\widetilde{N}(t) = N(\mu^{-1}(t))$ is a standard homogeneous Poisson process on $[0, \infty)$ if $\lim_{t \to \infty} \mu(t) = \infty$.[11]

We summarize our findings.

Proposition 2.1.5 (The Poisson process under change of time)
Let μ be the mean value function of a Poisson process N and $\widetilde{N}$ be a standard homogeneous Poisson process. Then the following statements hold:

(1) *The process $(\widetilde{N}(\mu(t)))_{t \geq 0}$ is Poisson with mean value function μ.*
(2) *If μ is continuous, increasing and $\lim_{t \to \infty} \mu(t) = \infty$ then $(N(\mu^{-1}(t)))_{t \geq 0}$ is a standard homogeneous Poisson process.*

This result, which immediately follows from the definition of a Poisson process, allows one in most cases of practical interest to switch from an inhomogeneous Poisson process to a homogeneous one by a simple time change. In particular, it suggests a straightforward way of simulating sample paths of an inhomogeneous Poisson process N from the paths of a homogeneous Poisson process. In an insurance context, one will usually be faced with inhomogeneous claim arrival processes. The above theory allows one to make an "operational time change" to a homogeneous model for which the theory is more accessible. See also Section 2.1.7 for a real-life example.

[11] If $\lim_{t \to \infty} \mu(t) = y_0 < \infty$ for some $y_0 > 0$, μ^{-1} is defined on $[0, y_0)$ and $\widetilde{N}(t) = N(\mu^{-1}(t))$ satisfies the properties of a standard homogeneous Poisson process restricted to the interval $[0, y_0)$. In Section 2.1.8 it is explained that such a process can be interpreted as a Poisson process on $[0, y_0)$.

2.1.4 The Homogeneous Poisson Process as a Renewal Process

In this section we study the sequence of the arrival times $0 \le T_1 \le T_2 \le \cdots$ of a homogeneous Poisson process with intensity $\lambda > 0$. It is our aim to find a constructive way for determining the sequence of arrivals, which in turn can be used as an alternative definition of the homogeneous Poisson process. This characterization is useful for studying the path properties of the Poisson process or for simulating sample paths.

We will show that any homogeneous Poisson process with intensity $\lambda > 0$ has representation

$$N(t) = \#\{i \ge 1 : T_i \le t\}\,, \quad t \ge 0\,, \tag{2.1.4}$$

where

$$T_n = W_1 + \cdots + W_n\,, \quad n \ge 1\,, \tag{2.1.5}$$

and (W_i) is an iid exponential $\mathrm{Exp}(\lambda)$ sequence. In what follows, it will be convenient to write $T_0 = 0$. Since the random walk (T_n) with non-negative step sizes W_n is also referred to as *renewal sequence*, a process N with representation (2.1.4)-(2.1.5) for a general iid sequence (W_i) is called a *renewal* (*counting*) *process*. We will consider general renewal processes in Section 2.2.

Theorem 2.1.6 (The homogeneous Poisson process as a renewal process)

(1) *The process N given by* (2.1.4) *and* (2.1.5) *with an iid exponential* $\mathrm{Exp}(\lambda)$ *sequence* (W_i) *constitutes a homogeneous Poisson process with intensity* $\lambda > 0$.
(2) *Let N be a homogeneous Poisson process with intensity λ and arrival times* $0 \le T_1 \le T_2 \le \cdots$*. Then N has representation* (2.1.4), *and* (T_i) *has representation* (2.1.5) *for an iid exponential* $\mathrm{Exp}(\lambda)$ *sequence* (W_i).

Proof. (1) We start with a renewal sequence (T_n) as in (2.1.5) and set $T_0 = 0$ for convenience. Recall the defining properties of a Poisson process from Definition 2.1.1. The property $N(0) = 0$ a.s. follows since $W_1 > 0$ a.s. By construction, a path $(N(t,\omega))_{t\ge 0}$ assumes the value i in $[T_i, T_{i+1})$ and jumps at T_{i+1} to level $i+1$. Hence the sample paths are càdlàg; cf. p. 8 for a definition.

Next we verify that $N(t)$ is $\mathrm{Pois}(\lambda t)$ distributed. The crucial relationship is given by

$$\{N(t) = n\} = \{T_n \le t < T_{n+1}\}\,, \quad n \ge 0\,. \tag{2.1.6}$$

Since $T_n = W_1 + \cdots + W_n$ is the sum of n iid $\mathrm{Exp}(\lambda)$ random variables it is a well-known property that T_n has a gamma $\Gamma(n, \lambda)$ distribution[12] for $n \ge 1$:

[12] You can easily verify that this is the distribution function of a $\Gamma(n, \lambda)$ distribution by taking the first derivative. The resulting probability density has the well-known gamma form $\lambda\,(\lambda\,x)^{n-1}\mathrm{e}^{-\lambda\,x}/(n-1)!$. The $\Gamma(n, \lambda)$ distribution for $n \in \mathbb{N}$ is also known as the *Erlang distribution* with parameter (n, λ).

$$P(T_n \le x) = 1 - \mathrm{e}^{-\lambda x} \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!}\,, \quad x \ge 0\,.$$

Hence

$$P(N(t) = n) = P(T_n \le t) - P(T_{n+1} \le t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!}\,.$$

This proves the Poisson property of $N(t)$.

Now we switch to the independent stationary increment property. We use a direct "brute force" method to prove this property. A more elegant way via point process techniques is indicated in Resnick [123], Proposition 4.8.1. Since the case of arbitrarily many increments becomes more involved, we focus on the case of two increments in order to illustrate the method. The general case is analogous but requires some bookkeeping. We focus on the adjacent increments $N(t) = N(0,t]$ and $N(t,t+h]$ for $t,h > 0$. We have to show that for any $k,l \in \mathbb{N}_0$,

$$\begin{aligned} q_{k,k+l}(t,t+h) &= P(N(t) = k\,, N(t,t+h] = l) \\ &= P(N(t) = k)\, P(N(t,t+h] = l) \\ &= P(N(t) = k)\, P(N(h) = l) \\ &= \mathrm{e}^{-\lambda\,(t+h)} \frac{(\lambda t)^k\,(\lambda h)^l}{k!\,l!}\,. \end{aligned} \tag{2.1.7}$$

We start with the case $l = 0$, $k \ge 1$; the case $l = k = 0$ being trivial. We make use of the relation

$$\{N(t) = k\,, N(t,t+h] = l\} = \{N(t) = k\,, N(t+h) = k+l\}\,. \tag{2.1.8}$$

Then, by (2.1.6) and (2.1.8) ,

$$\begin{aligned} q_{k,k+l}(t,t+h) &= P(T_k \le t < T_{k+1}\,, T_k \le t+h < T_{k+1}) \\ &= P(T_k \le t\,, t+h < T_k + W_{k+1})\,. \end{aligned}$$

Now we can use the facts that T_k is $\Gamma(k,\lambda)$ distributed with density $\lambda^k x^{k-1}\,\mathrm{e}^{-\lambda x}\,/(k-1)!$ and W_{k+1} is $\mathrm{Exp}(\lambda)$ distributed with density $\lambda\,\mathrm{e}^{-\lambda x}$:

$$\begin{aligned} q_{k,k+l}(t,t+h) &= \int_0^t \mathrm{e}^{-\lambda z} \frac{\lambda\,(\lambda z)^{k-1}}{(k-1)!} \int_{t+h-z}^{\infty} \lambda\,\mathrm{e}^{-\lambda x}\,dx\,dz \\ &= \int_0^t \mathrm{e}^{-\lambda z} \frac{\lambda\,(\lambda z)^{k-1}}{(k-1)!}\,\mathrm{e}^{-\lambda\,(t+h-z)}\,dz \\ &= \mathrm{e}^{-\lambda\,(t+h)} \frac{(\lambda t)^k}{k!}\,. \end{aligned}$$

For $l \geq 1$ we use another conditioning argument and (2.1.6):

$$
\begin{aligned}
& q_{k,k+l}(t,t+h) \\
&= P(T_k \le t < T_{k+1}\,, T_{k+l} \le t+h < T_{k+l+1}) \\
&= E[I_{\{T_k \le t < T_{k+1} \le t+h\}} \\
&\qquad P(T_{k+l} - T_{k+1} \le t+h-T_{k+1} < T_{k+l+1} - T_{k+1} \mid T_k\,, T_{k+1})]\,.
\end{aligned}
$$

Let N' be an independent copy of N, i.e., $N' \stackrel{d}{=} N$. Appealing to (2.1.6) and the independence of T_{k+1} and $(T_{k+l} - T_{k+1}, T_{k+l+1} - T_{k+1})$, we see that

$$
\begin{aligned}
& q_{k,k+l}(t,t+h) \\
&= E[I_{\{T_k \le t < T_{k+1} \le t+h\}}\, P(N'(t+h-T_{k+1}) = l-1 \mid T_{k+1})] \\
&= \int_0^t \mathrm{e}^{-\lambda z} \frac{\lambda (\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda\, \mathrm{e}^{-\lambda x}\, P(N(t+h-z-x) = l-1)\, dx\, dz \\
&= \int_0^t \mathrm{e}^{-\lambda z} \frac{\lambda (\lambda z)^{k-1}}{(k-1)!} \int_{t-z}^{t+h-z} \lambda\, \mathrm{e}^{-\lambda x}\, \mathrm{e}^{-\lambda (t+h-z-x)} \frac{(\lambda (t+h-z-x))^{l-1}}{(l-1)!} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad dx\, dz \\
&= \mathrm{e}^{-\lambda (t+h)} \int_0^t \frac{\lambda (\lambda z)^{k-1}}{(k-1)!}\, dz \int_0^h \frac{\lambda (\lambda x)^{l-1}}{(l-1)!}\, dx \\
&= \mathrm{e}^{-\lambda (t+h)} \frac{(\lambda t)^k}{k!} \frac{(\lambda h)^l}{l!}\,.
\end{aligned}
$$

This is the desired relationship (2.1.7). Since

$$
P(N(t,t+h] = l) = \sum_{k=0}^{\infty} P(N(t) = k\,, N(t,t+h] = l)\,,
$$

it also follows from (2.1.7) that

$$
P(N(t) = k\,, N(t,t+h] = l) = P(N(t) = k)\, P(N(h) = l)\,.
$$

If you have enough patience prove the analog to (2.1.7) for finitely many increments of N.

(2) Consider a homogeneous Poisson process with arrival times $0 \le T_1 \le T_2 \le \cdots$ and intensity $\lambda > 0$. We need to show that there exist iid exponential $\mathrm{Exp}(\lambda)$ random variables W_i such that $T_n = W_1 + \cdots + W_n$, i.e., we need to show that, for any $0 \le x_1 \le x_2 \le \cdots \le x_n$, $n \ge 1$,

$$
\begin{aligned}
& P(T_1 \le x_1\,, \ldots, T_n \le x_n) \\
&= P(W_1 \le x_1\,, \ldots\,, W_1 + \cdots + W_n \le x_n) \\
&= \int_{w_1=0}^{x_1} \lambda\, \mathrm{e}^{-\lambda w_1} \int_{w_2=0}^{x_2 - w_1} \lambda\, \mathrm{e}^{-\lambda w_2} \cdots \int_{w_n=0}^{x_n - w_1 - \cdots - w_{n-1}} \lambda\, \mathrm{e}^{-\lambda w_n}\, dw_n \cdots dw_1.
\end{aligned}
$$

The verification of this relation is left as an exercise. Hint: It is useful to exploit the relationship

$$\{T_1 \le x_1\,,\ldots,T_n \le x_n\} = \{N(x_1) \ge 1\,,\ldots,N(x_n) \ge n\}$$

for $0 \le x_1 \le \cdots \le x_n$, $n \ge 1$. □

An important consequence of Theorem 2.1.6 is that the inter-arrival times

$$W_i = T_i - T_{i-1}\,, \quad i \ge 1\,,$$

of a homogeneous Poisson process with intensity λ are iid $\text{Exp}(\lambda)$. In particular, $T_i < T_{i+1}$ a.s. for $i \ge 1$, i.e., with probability 1 a homogeneous Poisson process does not have jump sizes larger than 1. Since by the strong law of large numbers $T_n/n \stackrel{\text{a.s.}}{\to} EW_1 = \lambda^{-1} > 0$, we may also conclude that T_n grows roughly like n/λ, and therefore there are no limit points in the sequence (T_n) at any finite instant of time. This means that the values $N(t)$ of a homogeneous Poisson process are finite on any finite time interval $[0,t]$.

The Poisson process has many amazing properties. One of them is the following phenomenon which runs in the literature under the name *inspection paradox*.

Example 2.1.7 (The inspection paradox)
Assume that you study claims which arrive in the portfolio according to a homogeneous Poisson process N with intensity λ. We have learned that the inter-arrival times $W_n = T_n - T_{n-1}$, $n \ge 1$, with $T_0 = 0$, constitute an iid $\text{Exp}(\lambda)$ sequence. Observe the portfolio at a fixed instant of time t. The last claim arrived at time $T_{N(t)}$ and the next claim will arrive at time $T_{N(t)+1}$. Three questions arise quite naturally:

(1) What is the distribution of $B(t) = t - T_{N(t)}$, i.e., the length of the period $(T_{N(t)}, t]$ since the last claim occurred?
(2) What is the distribution of $F(t) = T_{N(t)+1} - t$, i.e., the length of the period $(t, T_{N(t)+1}]$ until the next claim arrives?
(3) What can be said about the joint distribution of $B(t)$ and $F(t)$?

The quantity $B(t)$ is often referred to as *backward recurrence time* or *age*, whereas $F(t)$ is called *forward recurrence time*, *excess life* or *residual life*.

Intuitively, since t lies somewhere between two claim arrivals and since the inter-arrival times are iid $\text{Exp}(\lambda)$, we would perhaps expect that $P(B(t) \le x_1) < 1 - \mathrm{e}^{-\lambda\, x_1}$, $x_1 < t$, and $P(F(t) \le x_2) < 1 - \mathrm{e}^{-\lambda\, x_2}$, $x_2 > 0$. However, these conjectures are not confirmed by calculation of the joint distribution function of $B(t)$ and $F(t)$ for $x_1, x_2 \ge 0$:

$$G_{B(t),F(t)}(x_1,x_2) = P(B(t) \le x_1\,, F(t) \le x_2)\,.$$

Since $B(t) \le t$ a.s. we consider the cases $x_1 < t$ and $x_1 \ge t$ separately. We observe for $x_1 < t$ and $x_2 > 0$,

$$\{B(t) \le x_1\} = \big\{t - x_1 \le T_{N(t)} \le t\big\} = \{N(t - x_1, t] \ge 1\}\,,$$

$$\{F(t) \le x_2\} = \big\{t < T_{N(t)+1} \le t + x_2\big\} = \{N(t, t + x_2] \ge 1\}\,.$$

Hence, by the independent stationary increments of N,

$$\begin{aligned} G_{B(t),F(t)}(x_1, x_2) &= P\left(N(t - x_1, t] \ge 1\,, N(t, t + x_2] \ge 1\right) \\ &= P\left(N(t - x_1, t] \ge 1\right)\, P\left(N(t, t + x_2] \ge 1\right) \\ &= \left(1 - \mathrm{e}^{-\lambda\, x_1}\right)\,\left(1 - \mathrm{e}^{-\lambda\, x_2}\right)\,. \end{aligned} \tag{2.1.9}$$

An analogous calculation for $x_1 \ge t$, $x_2 \ge 0$ and (2.1.9) yield

$$G_{B(t),F(t)}(x_1, x_2) = \left[(1 - \mathrm{e}^{-\lambda\, x_1})\, I_{[0,t)}(x_1) + I_{[t,\infty)}(x_1)\right]\,\left(1 - \mathrm{e}^{-\lambda\, x_2}\right)\,.$$

Hence $B(t)$ and $F(t)$ are independent, $F(t)$ is $\mathrm{Exp}(\lambda)$ distributed and $B(t)$ has a truncated exponential distribution with a jump at t:

$$P(B(t) \le x_1) = 1 - \mathrm{e}^{-\lambda\, x_1}\,, \quad x_1 < t\,, \quad \text{and} \quad P(B(t) = t) = \mathrm{e}^{-\lambda\, t}\,.$$

This means in particular that the forward recurrence time $F(t)$ has the same $\mathrm{Exp}(\lambda)$ distribution as the inter-arrival times W_i of the Poisson process N. This property is closely related to the *forgetfulness property* of the exponential distribution:

$$P(W_1 > x + y \mid W_1 > x) = P(W_1 > y)\,, \quad x\,, y \ge 0\,,$$

(Verify the correctness of this relation.) and is also reflected in the independent increment property of the Poisson process. It is interesting to observe that

$$\lim_{t\to\infty} P(B(t) \le x_1) = 1 - \mathrm{e}^{-\lambda\, x_1}\,, \quad x_1 > 0\,.$$

Thus, *in an "asymptotic" sense*, both $B(t)$ and $F(t)$ become independent and are exponentially distributed with parameter λ.

We will return to the forward and backward recurrence times of a general renewal process, i.e., when W_i are not necessarily iid exponential random variables, in Example 2.2.14. □

2.1.5 The Distribution of the Inter-Arrival Times

By virtue of Proposition 2.1.5, an inhomogeneous Poisson process N with mean value function μ can be interpreted as a time changed standard homogeneous Poisson process $\widetilde{N}$:

$$(N(t))_{t\ge 0} \stackrel{d}{=} (\widetilde{N}(\mu(t)))_{t\ge 0}\,.$$

In particular, let $(\widetilde{T}_i)$ be the arrival sequence of $\widetilde{N}$ and μ be increasing and continuous. Then the inverse μ^{-1} exists and

$$N'(t) = \#\{i \geq 1 : \widetilde{T}_i \leq \mu(t)\} = \#\{i \geq 1 : \mu^{-1}(\widetilde{T}_i) \leq t\}\,, \quad t \geq 0\,,$$

is a representation of N in the sense of identity of the finite-dimensional distributions, i.e., $N \stackrel{d}{=} N'$. Therefore and by virtue of Theorem 2.1.6 the arrival times of an inhomogeneous Poisson process with mean value function μ have representation

$$T_n = \mu^{-1}(\widetilde{T}_n)\,, \quad \widetilde{T}_n = \widetilde{W}_1 + \cdots + \widetilde{W}_n\,, \quad n \geq 1\,, \quad \widetilde{W}_i \text{ iid Exp}(1). \tag{2.1.10}$$

Proposition 2.1.8 (Joint distribution of arrival/inter-arrival times)
Assume N is a Poisson process on $[0,\infty)$ with a continuous a.e. positive intensity function λ. Then the following statements hold.

(1) *The vector of the arrival times $(T_1,\ldots,T_n)$ has density*

$$f_{T_1,\ldots,T_n}(x_1,\ldots,x_n) = \mathrm{e}^{-\mu(x_n)} \prod_{i=1}^n \lambda(x_i)\, I_{\{0<x_1<\cdots<x_n\}}\,. \tag{2.1.11}$$

(2) *The vector of inter-arrival times $(W_1,\ldots,W_n) = (T_1, T_2 - T_1, \ldots, T_n - T_{n-1})$ has density*

$$f_{W_1,\ldots,W_n}(x_1,\ldots,x_n) = \mathrm{e}^{-\mu(x_1+\cdots+x_n)} \prod_{i=1}^n \lambda(x_1+\cdots+x_i)\,, \quad x_i \geq 0\,. \tag{2.1.12}$$

Proof. Since the intensity function λ is a.e. positive and continuous, $\mu(t) = \int_0^t \lambda(s)\,ds$ is increasing and μ^{-1} exists. Moreover, μ is differentiable, and $\mu'(t) = \lambda(t)$. We make use of these two facts in what follows.
(1) We start with a standard homogeneous Poisson process. Then its arrivals $\widetilde{T}_n$ have representation $\widetilde{T}_n = \widetilde{W}_1 + \cdots + \widetilde{W}_n$ for an iid standard exponential sequence $(\widetilde{W}_i)$. The joint density of $(\widetilde{T}_1,\ldots,\widetilde{T}_n)$ is obtained from the joint density of $(\widetilde{W}_1,\ldots,\widetilde{W}_n)$ via the transformation:

$$(y_1,\ldots,y_n) \stackrel{S}{\to} (y_1, y_1+y_2, \ldots, y_1+\cdots+y_n)\,,$$

$$(z_1,\ldots,z_n) \stackrel{S^{-1}}{\to} (z_1, z_2-z_1, \ldots, z_n-z_{n-1})\,.$$

Note that $\det(\partial S(\mathbf{y})/\partial \mathbf{y}) = 1$. Standard techniques for density transformations (cf. Billingsley [18], p. 229) yield for $0 < x_1 < \cdots < x_n$,

$$\begin{aligned} f_{\widetilde{T}_1,\ldots,\widetilde{T}_n}(x_1,\ldots,x_n) &= f_{\widetilde{W}_1,\ldots,\widetilde{W}_n}(x_1, x_2-x_1, \ldots, x_n-x_{n-1}) \\ &= \mathrm{e}^{-x_1}\,\mathrm{e}^{-(x_2-x_1)} \cdots \mathrm{e}^{-(x_n-x_{n-1})} = \mathrm{e}^{-x_n}\,. \end{aligned}$$

Since μ^{-1} exists we conclude from (2.1.10) that for $0 < x_1 < \cdots < x_n$,

$$
\begin{aligned}
P(T_1 \le x_1\,,\ldots,T_n \le x_n) &= P(\mu^{-1}(\widetilde{T}_1) \le x_1\,,\ldots,\mu^{-1}(\widetilde{T}_n) \le x_n) \\
&= P(\widetilde{T}_1 \le \mu(x_1)\,,\ldots,\widetilde{T}_n \le \mu(x_n)) \\
&= \int_0^{\mu(x_1)} \cdots \int_0^{\mu(x_n)} f_{\widetilde{T}_1\,,\ldots,\widetilde{T}_n}(y_1,\ldots,y_n)\,dy_n \cdots dy_1 \\
&= \int_0^{\mu(x_1)} \cdots \int_0^{\mu(x_n)} \mathrm{e}^{-y_n}\, I_{\{y_1<\cdots<y_n\}}\, dy_n \cdots dy_1\,.
\end{aligned}
$$

Taking partial derivatives with respect to the variables $x_1,\ldots,x_n$ and noticing that $\mu'(x_i) = \lambda(x_i)$, we obtain the desired density (2.1.11).
(2) Relation (2.1.12) follows by an application of the above transformations S and S^{-1} from the density of $(T_1,\ldots,T_n)$:

$$
f_{W_1,\ldots,W_n}(w_1,\ldots,w_n) = f_{T_1,\ldots,T_n}(w_1, w_1+w_2,\ldots,w_1+\cdots+w_n)\,.
$$

□

From (2.1.12) we may conclude that the joint density of $W_1,\ldots,W_n$ can be written as the product of the densities of the W_i's if and only if $\lambda(\cdot) \equiv \lambda$ for some positive constant λ. This means that only in the case of a homogeneous Poisson process are the inter-arrival times $W_1,\ldots,W_n$ independent (and identically distributed). This fact is another property which distinguishes the homogeneous Poisson process within the class of all Poisson processes on $[0,\infty)$.

2.1.6 The Order Statistics Property

In this section we study one of the most important properties of the Poisson process which in a sense characterizes the Poisson process. It is the *order statistics property* which it shares only with the mixed Poisson process to be considered in Section 2.3. In order to formulate this property we first give a well-known result on the distribution of the order statistics

$$
X_{(1)} \le \cdots \le X_{(n)}
$$

of an iid sample $X_1,\ldots,X_n$.

Lemma 2.1.9 (Joint density of order statistics)
If the iid X_i's have density f then the density of the vector $(X_{(1)},\ldots,X_{(n)})$ is given by

$$
f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = n! \prod_{i=1}^n f(x_i)\, I_{\{x_1<\cdots<x_n\}}\,.
$$

Remark 2.1.10 By construction of the order statistics, the support of the vector $(X_{(1)}\,,\ldots\,,X_{(n)})$ is the set

$$C_n = \{(x_1, \ldots, x_n) : x_1 \le \cdots \le x_n\} \subset \mathbb{R}^n ,$$

and therefore the density $f_{X_{(1)},\ldots,X_{(n)}}$ vanishes outside C_n. Since the existence of a density of X_i implies that all elements of the iid sample $X_1, \ldots, X_n$ are different a.s., the $\le$'s in the definition of C_n could be replaced by $<$'s. □

Proof. We start by recalling that the iid sample $X_1, \ldots, X_n$ with common density f has *no ties*. This means that the event

$$\widetilde{\Omega} = \{X_{(1)} < \cdots < X_{(n)}\} = \{X_i \ne X_j \text{ for } 1 \le i < j \le n\}$$

has probability 1. It is an immediate consequence of the fact that for $i \ne j$,

$$P(X_i = X_j) = E[P(X_i = X_j \mid X_j)] = \int_{\mathbb{R}} P(X_i = y)\, f(y)\, dy = 0 ,$$

since $P(X_i = y) = \int_{\{y\}} f(z)\, dz = 0$. Then

$$1 - P(\widetilde{\Omega}) = P\left(\bigcup_{1 \le i < j \le n} \{X_i = X_j\}\right) \le \sum_{1 \le i < j \le n} P(X_i = X_j) = 0 .$$

Now we turn to the proof of the statement of the lemma. Let Π_n be the set of the permutations π of $\{1, \ldots, n\}$. Fix the values $x_1 < \cdots < x_n$. Then

$$P\left(X_{(1)} \le x_1 , \ldots , X_{(n)} \le x_n\right) = P\left(\bigcup_{\pi \in \Pi_n} A_\pi\right) , \qquad (2.1.13)$$

where

$$A_\pi = \{X_{\pi(i)} = X_{(i)} , i = 1 , \ldots , n\} \cap \widetilde{\Omega} \cap \{X_{\pi(1)} \le x_1 , \ldots , X_{\pi(n)} \le x_n\} .$$

The identity (2.1.13) means that the ordered sample $X_{(1)} < \cdots < X_{(n)}$ could have come from any of the ordered values $X_{\pi(1)} < \cdots < X_{\pi(n)}$, $\pi \in \Pi_n$, where we also make use of the fact that there are no ties in the sample. Since the A_π's are disjoint,

$$P\left(\bigcup_{\pi \in \Pi_n} A_\pi\right) = \sum_{\pi \in \Pi_n} P(A_\pi) .$$

Moreover, since the X_i's are iid,

$$\begin{aligned} P(A_\pi) &= P\left((X_{\pi(1)} , \ldots , X_{\pi(n)}) \in C_n \cap (-\infty, x_1] \times \cdots \times (-\infty, x_n]\right) \\ &= P((X_1, \ldots, X_n) \in C_n \cap (-\infty, x_1] \times \cdots \times (-\infty, x_n]) \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \prod_{i=1}^{n} f(y_i)\, I_{\{y_1 < \cdots < y_n\}}\, dy_n \cdots dy_1 . \end{aligned}$$

Therefore and since there are $n!$ elements in Π_n,

$$P\left(X_{(1)} \le x_1, \ldots, X_{(n)} \le x_n\right) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} n! \prod_{i=1}^{n} f(y_i)\, I_{\{y_1 < \cdots < y_n\}}\, dy_n \cdots dy_1 \,. \qquad (2.1.14)$$

By Remark 2.1.10 about the support of $(X_{(1)}, \ldots, X_{(n)})$ and by virtue of the Radon-Nikodym theorem, we can read off the density of $(X_{(1)}, \ldots, X_{(n)})$ as the integrand in (2.1.14). Indeed, the Radon-Nikodym theorem ensures that the integrand is the a.e. unique probability density of $(X_{(1)}, \ldots, X_{(n)})$.[13] □

We are now ready to formulate one of the main results of this course.

Theorem 2.1.11 (Order statistics property of the Poisson process)
Consider the Poisson process $N = (N(t))_{t\ge 0}$ with continuous a.e. positive intensity function λ and arrival times $0 < T_1 < T_2 < \cdots$ a.s. Then the conditional distribution of $(T_1, \ldots, T_n)$ given $\{N(t) = n\}$ is the distribution of the ordered sample $(X_{(1)}, \ldots, X_{(n)})$ of an iid sample $X_1, \ldots, X_n$ with common density $\lambda(x)/\mu(t)$, $0 < x \le t$:

$$(T_1, \ldots, T_n \mid N(t) = n) \stackrel{d}{=} (X_{(1)}, \ldots, X_{(n)})\,.$$

In other words, the left-hand vector has conditional density

$$f_{T_1,\ldots,T_n}(x_1, \ldots, x_n \mid N(t) = n) = \frac{n!}{(\mu(t))^n} \prod_{i=1}^{n} \lambda(x_i)\,, \qquad (2.1.15)$$

$$0 < x_1 < \cdots < x_n < t\,.$$

Proof. We show that the limit

$$\lim_{h_i \downarrow 0\,,\, i=1,\ldots,n} \frac{P(T_1 \in (x_1, x_1 + h_1], \ldots, T_n \in (x_n, x_n + h_n] \mid N(t) = n)}{h_1 \cdots h_n} \qquad (2.1.16)$$

exists and is a continuous function of the x_i's. A similar argument (which we omit) proves the analogous statement for the intervals $(x_i - h_i, x_i]$ with the same limit function. The limit can be interpreted as a density for the conditional probability distribution of $(T_1, \ldots, T_n)$, given $\{N(t) = n\}$.

[13] Relation (2.1.14) means that for all rectangles $R = (-\infty, x_1] \times \cdots \times (-\infty, x_n]$ with $0 \le x_1 < \cdots < x_n$ and for $\mathbf{X}_n = (X_{(1)}, \ldots, X_{(n)})$, $P(\mathbf{X}_n \in R) = \int_R f_{\mathbf{X}_n}(\mathbf{x})\, d\mathbf{x}$. By the particular form of the support of $\mathbf{X}_n$, the latter relation remains valid for any rectangles in $\mathbb{R}^n$. An extension argument (cf. Billingsley [18]) ensures that the distribution of $\mathbf{X}_n$ is absolutely continuous with respect to Lebesgue measure with a density which coincides with $f_{\mathbf{X}_n}$ on the rectangles. The Radon-Nikodym theorem ensures the a.e. uniqueness of $f_{\mathbf{X}_n}$.

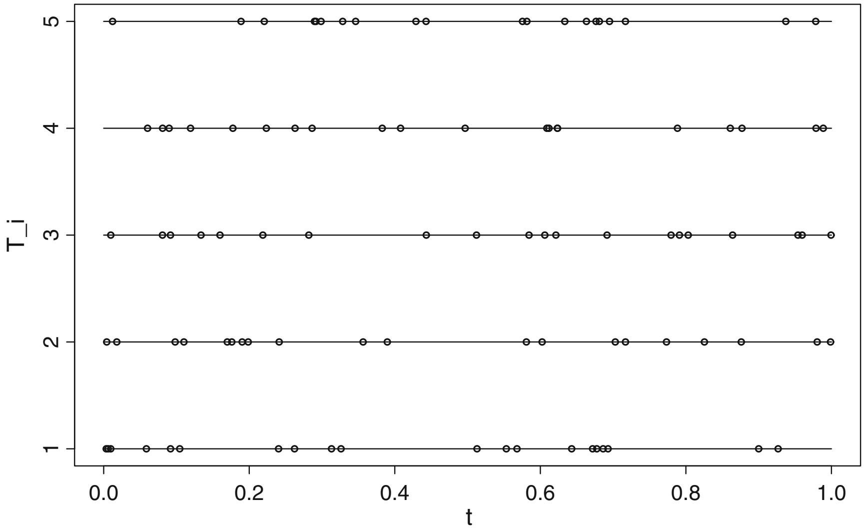

Figure 2.1.12 *Five realizations of the arrival times* T_i *of a standard homogeneous Poisson process conditioned to have* 20 *arrivals in* $[0,1]$*. The arrivals in each row can be interpreted as the ordered sample of an iid* $\mathrm{U}(0,1)$ *sequence.*

Since $0 < x_1 < \cdots < x_n < t$ we can choose the h_i's so small that the intervals $(x_i, x_i + h_i] \subset [0,t]$, $i = 1,\ldots,n$, become disjoint. Then the following identity is immediate:

$$\begin{aligned}&\{T_1 \in (x_1, x_1 + h_1]\,, \ldots\,, T_n \in (x_n, x_n + h_n]\,, N(t) = n\}\\ &= \{N(0,x_1] = 0\,, N(x_1, x_1 + h_1] = 1\,, N(x_1 + h_1, x_2] = 0\,,\\ &\quad N(x_2, x_2 + h_2] = 1\,, \ldots\,, N(x_{n-1} + h_{n-1}, x_n] = 0\,,\\ &\quad N(x_n, x_n + h_n] = 1\,, N(x_n + h_n, t] = 0\}\,.\end{aligned}$$

Taking probabilities on both sides and exploiting the independent increments of the Poisson process N, we obtain

$$\begin{aligned}&P(T_1 \in (x_1, x_1 + h_1]\,, \ldots\,, T_n \in (x_n, x_n + h_n]\,, N(t) = n)\\ &= P(N(0,x_1] = 0)\; P(N(x_1, x_1 + h_1] = 1)\; P(N(x_1 + h_1, x_2] = 0)\\ &\quad P(N(x_2, x_2 + h_2] = 1) \cdots P(N(x_{n-1} + h_{n-1}, x_n] = 0)\\ &\quad P(N(x_n, x_n + h_n] = 1)\; P(N(x_n + h_n, t] = 0)\end{aligned}$$

$$= \mathrm{e}^{-\mu(x_1)} \Big[\mu(x_1, x_1 + h_1]\, \mathrm{e}^{-\mu(x_1,x_1+h_1]}\Big]\; \mathrm{e}^{-\mu(x_1+h_1,x_2]}$$
$$\Big[\mu(x_2, x_2 + h_2]\, \mathrm{e}^{-\mu(x_2,x_2+h_2]}\Big] \cdots \mathrm{e}^{-\mu(x_{n-1}+h_{n-1},x_n]}$$
$$\Big[\mu(x_n, x_n + h_n]\, \mathrm{e}^{-\mu(x_n,x_n+h_n]}\Big]\; \mathrm{e}^{-\mu(x_n+h_n,t]}$$
$$= \mathrm{e}^{-\mu(t)}\; \mu(x_1, x_1 + h_1] \cdots \mu(x_n, x_n + h_n]\,.$$

Dividing by $P(N(t) = n) = \mathrm{e}^{-\mu(t)}(\mu(t))^n/n!$ and $h_1 \cdots h_n$, we obtain the scaled conditional probability

$$\frac{P(T_1 \in (x_1, x_1 + h_1]\,, \ldots , T_n \in (x_n, x_n + h_n] \mid N(t) = n)}{h_1 \cdots h_n}$$
$$= \frac{n!}{(\mu(t))^n}\, \frac{\mu(x_1, x_1 + h_1]}{h_1} \cdots \frac{\mu(x_n, x_n + h_n]}{h_n}$$
$$\to \frac{n!}{(\mu(t))^n}\, \lambda(x_1) \cdots \lambda(x_n)\,, \quad \text{as } h_i \downarrow 0, \quad i = 1, \ldots , n.$$

Keeping in mind (2.1.16), this is the desired relation (2.1.15). In the last step we used the continuity of λ to show that $\mu'(x_i) = \lambda(x_i)$. □

Example 2.1.13 (Order statistics property of the homogeneous Poisson process)
Consider a homogeneous Poisson process with intensity $\lambda > 0$. Then Theorem 2.1.11 yields the joint conditional density of the arrival times T_i:

$$f_{T_1,\ldots,T_n}(x_1, \ldots , x_n \mid N(t) = n) = n!\, t^{-n}\,, \quad 0 < x_1 < \cdots < x_n < t\,.$$

A glance at Lemma 2.1.9 convinces one that this is the joint density of a uniform ordered sample $U_{(1)} < \cdots < U_{(n)}$ of iid $\mathrm{U}(0, t)$ distributed $U_1, \ldots , U_n$. Thus, given there are n arrivals of a homogeneous Poisson process in the interval $[0, t]$, these arrivals constitute the points of a uniform ordered sample in $(0, t)$. In particular, this property is independent of the intensity λ! □

Example 2.1.14 (Symmetric function)
We consider a symmetric measurable function g on $\mathbb{R}^n$, i.e., for any permutation π of $\{1, \ldots , n\}$ we have

$$g(x_1, \ldots , x_n) = g(x_{\pi(1)}, \ldots , x_{\pi(n)})\,.$$

Such functions include products and sums:

$$g_s(x_1, \ldots , x_n) = \sum_{i=1}^{n} x_i\,, \quad g_p(x_1, \ldots , x_n) = \prod_{i=1}^{n} x_i\,.$$

Under the conditions of Theorem 2.1.11 and with the same notation, we conclude that

$$(g(T_1,\ldots,T_n) \mid N(t)=n) \stackrel{d}{=} g(X_{(1)},\ldots,X_{(n)}) = g(X_1,\ldots,X_n)\,.$$

For example, for any measurable function f on $\mathbb{R}$,

$$\left(\sum_{i=1}^n f(T_i) \;\middle|\; N(t)=n\right) \stackrel{d}{=} \sum_{i=1}^n f(X_{(i)}) = \sum_{i=1}^n f(X_i)\,.$$

□

Example 2.1.15 (Shot noise)
This kind of stochastic process was used early on to model an electric current. Electrons arrive according to a homogeneous Poisson process N with rate λ at times T_i. An arriving electron produces an electric current whose time evolution of discharge is described as a deterministic function f with $f(t)=0$ for $t<0$. Shot noise describes the electric current at time t produced by all electrons arrived by time t as a superposition:

$$S(t) = \sum_{i=1}^{N(t)} f(t-T_i)\,.$$

Typical choices for f are exponential functions $f(t) = \mathrm{e}^{-\theta\, t}\, I_{[0,\infty)}(t)$, $\theta > 0$. An extension of classical shot noise processes with various applications is the process

$$S(t) = \sum_{i=1}^{N(t)} X_i\, f(t-T_i)\,, \quad t \ge 0\,, \tag{2.1.17}$$

where

- (X_i) is an iid sequence, independent of (T_i).
- f is a deterministic function with $f(t)=0$ for $t<0$.

For example, if we assume that the X_i's are positive random variables, $S(t)$ is a generalization of the Cramér-Lundberg model, see Example 2.1.3. Indeed, choose $f = I_{[0,\infty)}$, then the shot noise process (2.1.17) is the total claim amount in the Cramér-Lundberg model. In an insurance context, f can also describe delay in claim settlement or some discount factor.

Delay in claim settlement is for example described by a function f satisfying

- $f(t)=0$ for $t<0$,
- $f(t)$ is non-decreasing,
- $\lim_{t\to\infty} f(t) = 1\,.$

In contrast to the Cramér-Lundberg model, where the claim size X_i is paid off at the time T_i when it occurs, a more general payoff function $f(t)$ allows one to delay the payment, and the speed at which this happens depends on the growth of the function f. Delay in claim settlement is advantageous from the point of view of the insurer. In the meantime the amount of money which was not paid for covering the claim could be invested and would perhaps bring some extra gain.

Suppose the amount Y_i is invested at time T_i in a riskless asset (savings account) with constant interest rate $r > 0$, (Y_i) is an iid sequence of positive random variables and the sequences (Y_i) and (T_i) are independent. Continuous compounding yields the amount $\exp\{r(t - T_i)\}\,Y_i$ at time $t > T_i$. For iid amounts Y_i which are invested at the arrival times T_i of a homogeneous Poisson process, the total value of all investments at time t is given by

$$S_1(t) = \sum_{i=1}^{N(t)} \mathrm{e}^{\,r\,(t-T_i)}\,Y_i\,, \quad t \ge 0\,.$$

This is another shot noise process.

Alternatively, one may be interested in the present value of payments Y_i made at times T_i in the future. Then the present value with respect to the time frame $[0,t]$ is given as the *discounted sum*

$$S_2(t) = \sum_{i=1}^{N(t)} \mathrm{e}^{\,-r\,(t-T_i)}\,Y_i\,, \quad t \ge 0\,.$$

A visualization of the sample paths of the processes S_1 and S_2 can be found in Figure 2.1.17. □

The distributional properties of a shot noise process can be treated in the framework of the following general result.

Proposition 2.1.16 *Let (X_i) be an iid sequence, independent of the sequence (T_i) of arrival times of a homogeneous Poisson process N with intensity λ. Then for any measurable function $g : \mathbb{R}^2 \to \mathbb{R}$ the following identity in distribution holds*

$$S(t) = \sum_{i=1}^{N(t)} g(T_i\,, X_i) \stackrel{d}{=} \sum_{i=1}^{N(t)} g(t\,U_i\,, X_i)\,,$$

where (U_i) is an iid $\mathrm{U}(0,1)$ *sequence, independent of (X_i) and (T_i).*

Proof. A conditioning argument together with the order statistics property of Theorem 2.1.11 yields that for $x \in \mathbb{R}$,

$$P\left(\sum_{i=1}^{N(t)} g(T_i, X_i) \le x \,\middle|\, N(t) = n\right) = P\left(\sum_{i=1}^{n} g(t\,U_{(i)}, X_i) \le x\right)\,,$$

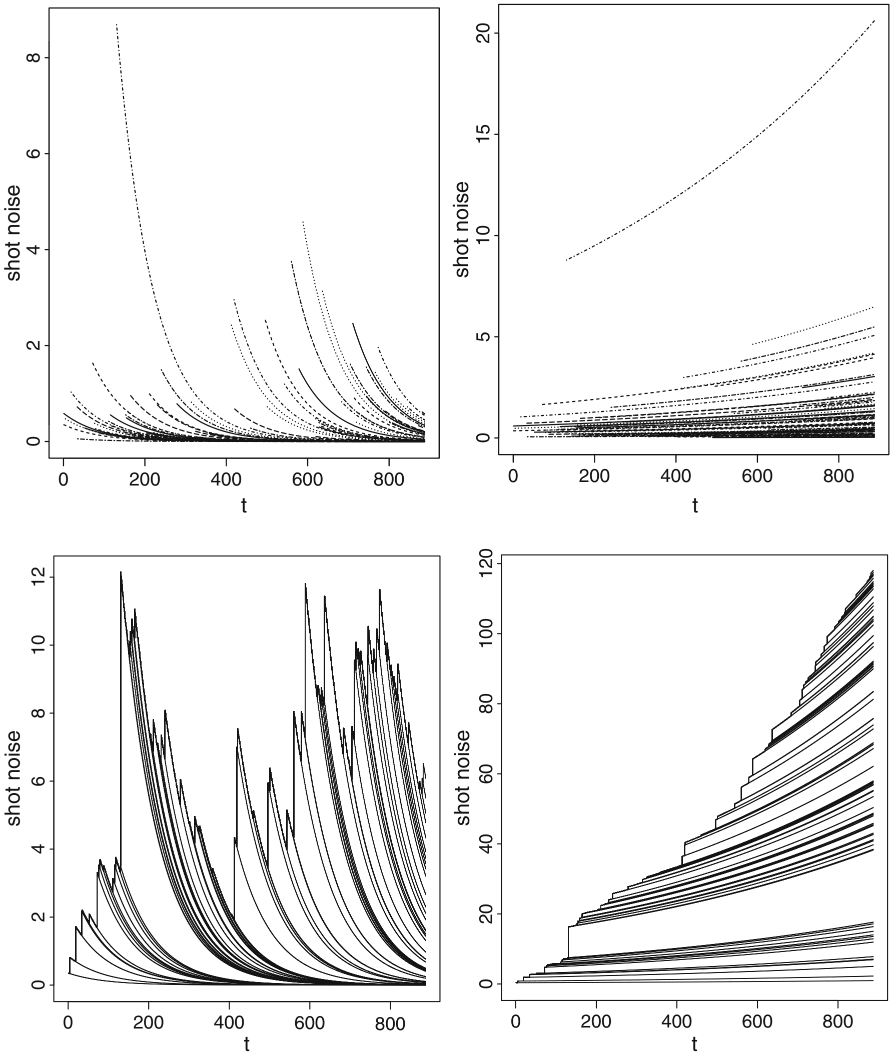

Figure 2.1.17 *Visualization of the paths of a shot noise process.* Top: 80 *paths of the processes* $Y_i \, \mathrm{e}^{\, r \, (t - T_i)}$, $t \ge T_i$, *where* (T_i) *are the point of a Poisson process with intensity* 0.1, (Y_i) *are iid standard exponential,* $r = -0.01$ (left) *and* $r = 0.001$ (right). Bottom: *The corresponding paths of the shot noise process* $S(t) = \sum_{T_i \le t} Y_i \, \mathrm{e}^{\, r \, (t - T_i)}$ *presented as a superposition of the paths in the corresponding top graphs. The graphs show nicely how the interest rate r influences the aggregated value of future claims or payments* Y_i. *We refer to Example 2.1.15 for a more detailed description of these processes.*

where $U_1,\dots,U_n$ is an iid U$(0,1)$ sample, independent of (X_i) and (T_i), and $U_{(1)},\dots,U_{(n)}$ is the corresponding ordered sample. By the iid property of (X_i) and its independence of (U_i), we can permute the order of the X_i's arbitrarily without changing the distribution of $\sum_{i=1}^n g(t\,U_{(i)},X_i)$:

$$\begin{aligned} &P\left(\sum_{i=1}^n g(t\,U_{(i)},X_i)\le x\right)\\ &= E\left[P\left(\sum_{i=1}^n g(t\,U_{(i)},X_i)\le x \,\middle|\, U_1,\dots,U_n\right)\right]\\ &= E\left[P\left(\sum_{i=1}^n g(t\,U_{(i)},X_{\pi(i)})\le x \,\middle|\, U_1,\dots,U_n\right)\right], \end{aligned} \qquad (2.1.18)$$

where π is any permutation of $\{1,\dots,n\}$. In particular, we can choose π such that for given $U_1,\dots,U_n$, $U_{(i)}=U_{\pi(i)}$, $i=1,\dots,n$.[14] Then (2.1.18) turns into

$$\begin{aligned} &E\left[P\left(\sum_{i=1}^n g(t\,U_{\pi(i)},X_{\pi(i)})\le x \,\middle|\, U_1,\dots,U_n\right)\right]\\ &= E\left[P\left(\sum_{i=1}^n g(t\,U_i,X_i)\le x \,\middle|\, U_1,\dots,U_n\right)\right]\\ &= P\left(\sum_{i=1}^n g(t\,U_i,X_i)\le x\right) = P\left(\sum_{i=1}^{N(t)} g(t\,U_i,X_i)\le x \,\middle|\, N(t)=n\right). \end{aligned}$$

Now it remains to take expectations:

$$\begin{aligned} P(S(t)\le x) &= E[P(S(t)\le x\mid N(t))]\\ &= \sum_{n=0}^{\infty} P(N(t)=n)\; P\left(\sum_{i=1}^{N(t)} g(T_i,X_i)\le x \,\middle|\, N(t)=n\right) \end{aligned}$$

---

[14] We give an argument to make this step in the proof more transparent. Since (U_i) and (X_i) are independent, it is possible to define $((U_i),(X_i))$ on the product space $\Omega_1\times\Omega_2$ equipped with suitable σ-fields and probability measures, and such that (U_i) lives on Ω_1 and (X_i) on Ω_2. While conditioning on $u_1=U_1(\omega_1),\dots,u_n=U_n(\omega_1)$, $\omega_1\in\Omega_1$, choose the permutation $\pi=\pi(\omega_1)$ of $\{1,\dots,n\}$ with $u_{\pi(1,\omega_1)}\le\cdots\le u_{\pi(n,\omega_1)}$, and then with probability 1,

$$\begin{aligned} &P(\{\omega_2:(X_1(\omega_2),\dots,X_n(\omega_2))\in A\})=\\ &P(\{\omega_2:(X_{\pi(1,\omega_1)}(\omega_2),\dots,X_{\pi(n,\omega_1)}(\omega_2))\}\in A\mid U_1(\omega_1)=u_1,\dots,U_n(\omega_1)=u_n). \end{aligned}$$

$$= \sum_{n=0}^{\infty} P(N(t) = n)\ P\left(\sum_{i=1}^{N(t)} g(t\,U_i, X_i) \le x \,\middle|\, N(t) = n\right)$$

$$= P\left(\sum_{i=1}^{N(t)} g(t\,U_i, X_i) \le x\right).$$

This proves the proposition. □

It is clear that Proposition 2.1.16 can be extended to the case when (T_i) is the arrival sequence of an inhomogeneous Poisson process. The interested reader is encouraged to go through the steps of the proof in this more general case.

Proposition 2.1.16 has a multitude of applications. We give one of them and consider more in the exercises.

Example 2.1.18 (Continuation of the shot noise Example 2.1.15)
In Example 2.1.15 we considered the stochastically discounted random sums

$$S(t) = \sum_{i=1}^{N(t)} \mathrm{e}^{-r\,(t-T_i)}\,X_i\,. \tag{2.1.19}$$

According to Proposition 2.1.16 , we have

$$S(t) \stackrel{d}{=} \sum_{i=1}^{N(t)} \mathrm{e}^{-r\,(t-tU_i)}\,X_i \stackrel{d}{=} \sum_{i=1}^{N(t)} \mathrm{e}^{-r\,t\,U_i}\,X_i\,, \tag{2.1.20}$$

where (X_i), (U_i) and N are mutually independent. Here we also used the fact that $(1 - U_i)$ and (U_i) have the same distribution. The structure of the random sum (2.1.19) is more complicated than the structure of the right-hand expression in (2.1.20) since in the latter sum the summands are independent of $N(t)$ and iid. For example, it is an easy matter to calculate the mean and variance of the expression on the right-hand side of (2.1.20) whereas it is a rather tedious procedure if one starts with (2.1.19). For example, we calculate

$$ES(t) = E\left(\sum_{i=1}^{N(t)} \mathrm{e}^{-r\,t\,U_i}\,X_i\right) = E\left[E\left(\sum_{i=1}^{N(t)} \mathrm{e}^{-r\,t\,U_i}\,X_i \,\middle|\, N(t)\right)\right]$$

$$= E\left[N(t)E\left(\mathrm{e}^{-r\,t\,U_1}\,X_1\right)\right]$$

$$= EN(t)\,E\mathrm{e}^{-r\,t\,U_1}\,EX_1 = \lambda\,r^{-1}(1 - \mathrm{e}^{-r\,t})\,EX_1\,.$$

Compare with the expectation in the Cramér-Lundberg model ($r = 0$): $ES(t) = \lambda\,t\,EX_1$. □

Comments

The order statistics property of a Poisson process can be generalized to Poisson processes with points in abstract spaces. We give an informal discussion of these processes in Section 2.1.8. In Exercise 20 on p. 52 we indicate how the "order statistics property" can be implemented, for example, in a Poisson process with points in the unit cube of $\mathbb{R}^d$.

In Parts III and IV of this text we continue the discussion of generalized Poisson processes and their applications in a non-life insurance context. For example, in Section 11.3 we study payment processes which describe the settlement of claims arriving at the points of a homogeneous Poisson process on the real line. The combined process of the claim arrivals and payments is again a shot noise process.

2.1.7 A Discussion of the Arrival Times of the Danish Fire Insurance Data 1980-1990

In this section we want to illustrate the theoretical results of the Poisson process by means of the arrival process of a real-life data set: the *Danish fire insurance data* in the period from January 1, 1980, until December 31, 1990. The data were communicated to us by Mette Havning.[15] There is a total of $n = 2\,167$ observations. Here we focus on the arrival process. In Section 3.2, and in particular in Example 3.2.11, we study the corresponding claim sizes.

The arrival and the corresponding inter-arrival times are plotted in Figure 2.1.19. Together with the arrival times we show the straight line $f(t) = 1.85\,t$. The value $\widehat{\lambda} = n/T_n = 1/1.85$ is the maximum likelihood estimator of λ under the hypothesis that the inter-arrival times W_i are iid $\text{Exp}(\lambda)$.

In Table 2.1.21 we summarize some basic statistics of the inter-arrival times for each year and for the whole period. Since the reciprocal of the annual sample mean is an estimator of the intensity, the table gives one the impression that there is a tendency for increasing intensity when time goes by. This phenomenon is supported by the left graph in Figure 2.1.20 where the annual mean inter-arrival times are visualized together with moving average estimates of the intensity function $\lambda(t)$. The estimate of the mean inter-arrival time at $t = i$ is defined as the moving average[16]

[15] In this text we consider two different versions of the Danish fire insurance data. Here we use the data which were reported by December 31, 1990. The claim sizes are expressed in terms of 1985 prices. If a claim was not completely settled on December 31, 1990, the size of this claim might possibly have changed after this date. For this reason the second data set (covering the period 1980-2002) often contains different reported sizes for claims incurred in 1980-1990.

[16] Moving average estimates such as (2.1.21) are proposed in time series analysis in order to estimate a deterministic trend which perturbs a stationary time series. We refer to Brockwell and Davis [24] and Priestley [119] for some theory and properties of the estimator $(\widehat{\lambda}(i))^{-1}$ and related estimates. More sophisticated

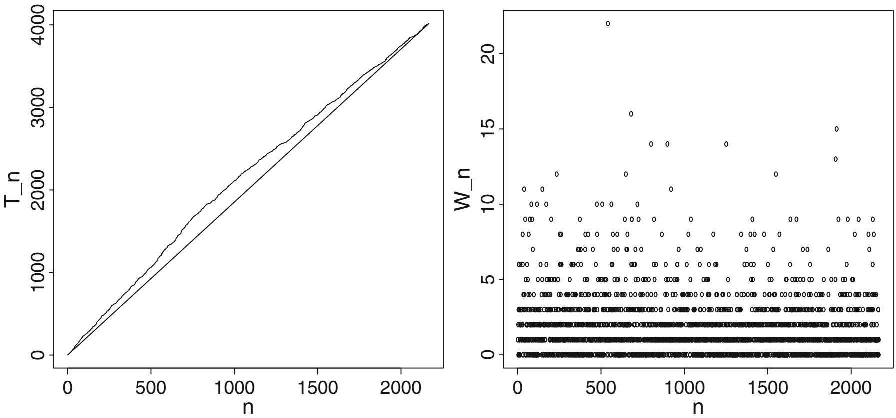

Figure 2.1.19 Left: *The arrival times of the Danish fire insurance data* 1980−1990. *The solid straight line has slope* 1.85 *which is estimated as the overall sample mean of the inter-arrival times. Since the graph of* (T_n) *lies above the straight line an inhomogeneous Poisson process is more appropriate for modeling the claim number in this portfolio.* Right: *The corresponding inter-arrival times. There is a total of* $n = 2\ 167$ *observations.*

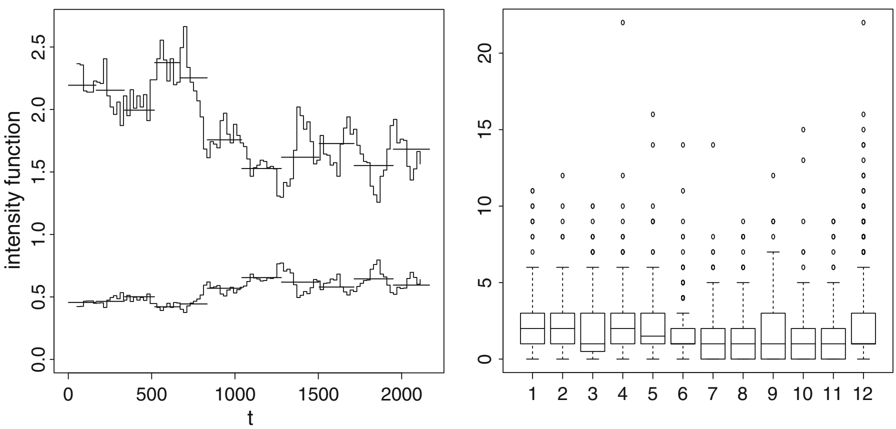

Figure 2.1.20 Left, upper graph: *The piecewise constant function represents the annual expected inter-arrival time between* 1980 *and* 1990. *The length of each constant piece is the claim number in the corresponding year. The annual estimates are supplemented by a moving average estimate* $(\widehat{\lambda}(i))^{-1}$ *defined in* (2.1.21). Left, lower graph: *The reciprocals of the values of the upper graph which can be interpreted as estimates of the Poisson intensity. There is a clear tendency for the intensity to increase over the last years.* Right: *Boxplots for the annual samples of the inter-arrival times* (No 1-11) *and the sample over* 11 *years* (No 12).

$$(\widehat{\lambda}(i))^{-1} = (2m+1)^{-1} \sum_{j=\max(1,i-m)}^{\min(n,i+m)} W_j \quad \text{for } m = 50. \qquad (2.1.21)$$

The corresponding estimates for $\widehat{\lambda}(i)$ can be interpreted as estimates of the intensity function. There is a clear tendency for the intensity to increase over the last years. This tendency can also be seen in the right graph of Figure 2.1.20. Indeed, the boxplots[17] of this figure indicate that the distribution of the inter-arrival times of the claims is less spread towards the end of the 1980s and concentrated around the value 1 in contrast to 2 at the beginning of the 1980s. Moreover, the annual claim number increases.

year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	all
sample size	166	170	181	153	163	207	238	226	210	235	218	2 167
min	0	0	0	0	0	0	0	0	0	0	0	0
1st quartile	1	1	0.75	1	1	1	0	0	0	0	0	1
median	2	2	1	2	1.5	1	1	1	1	1	1	1
mean	2.19	2.15	1.99	2.37	2.25	1.76	1.53	1.62	1.73	1.55	1.68	1.85
$\widehat{\lambda}$ =1/mean	0.46	0.46	0.50	0.42	0.44	0.57	0.65	0.62	0.58	0.64	0.59	0.54
3rd quartile	3	3	3	3	3	2	2	2	3	2	2	3
max	11	12	10	22	16	14	14	9	12	15	9	22

Table 2.1.21 *Basic statistics for the Danish fire inter-arrival times data.*

Since we have gained statistical evidence that the intensity function of the Danish fire insurance data is not constant over 11 years, we assume in Figure 2.1.22 that the arrivals are modeled by an inhomogeneous Poisson process with continuous mean value function. We assume that the intensity is constant for every year, but it may change from year to year. Hence the mean value function $\mu(t)$ of the Poisson process is piecewise linear with possibly different slopes in different years; see the top left graph in Figure 2.1.22. We choose the estimated intensities presented in Table 2.1.21 and in the left graph of Figure 2.1.20. We transform the arrivals T_n into $\mu(T_n)$. According to the theory in Section 2.1.3, one can interpret the points $\mu(T_n)$ as arrivals of a standard homogeneous Poisson process. This is nicely illustrated in the top right graph of Figure 2.1.22, where the sequence $(\mu(T_n))$ is plotted against n. The graph is very close to a straight line, in contrast to the left graph in

estimators can be obtained by using kernel curve estimators in the regression model $W_i = (\lambda(i))^{-1} + \varepsilon_i$ for some smooth deterministic function λ and iid or weakly dependent stationary noise (ε_i). We refer to Fan and Gijbels [49] and Gasser et al. [53] for some standard theory of kernel curve estimation; see also Müller and Stadtmüller [113].

[17] The boxplot of a data set is a means to visualize the empirical distribution of the data. The middle part of the plot (box) indicates the median $x_{0.50}$, the 25% and 75% quantiles ($x_{0.25}$ and $x_{0.75}$) of the data. The "whiskers" of the data are the lines $x_{0.50} \pm 1.5\,(x_{0.75} - x_{0.25})$. Values outside the whiskers ("outliers") are plotted as points.

Figure 2.1.19, where one can clearly see the deviations of the arrivals T_n from a straight line.

In the left middle graph we consider the histogram of the time changed arrival times $\mu(T_n)$. According to the theory in Section 2.1.6, the arrival times of a homogeneous Poisson can be interpreted as a uniform sample on any fixed interval, conditionally on the claim number in this interval. The histogram resembles the histogram of a uniform sample in contrast to the middle right graph, where the histogram of the Danish fire arrival times is presented. However, the left histogram is not perfect either. This is due to the fact that the data T_n are integers, hence the values $\mu(T_n)$ live on a particular discrete set.

The left bottom graph shows a moving average estimate of the intensity function of the arrivals $\mu(T_n)$. Although the function is close to 1 the estimates fluctuate wildly around 1. This is an indication that the process might not be Poisson and that other models for the arrival process could be more appropriate; see for example Section 2.2. The deviation of the distribution of the inter-arrival time $\mu(T_n) - \mu(T_{n-1})$, which according to the theory should be iid standard exponential, can also be seen in the right bottom graph in Figure 2.1.22, where a QQ-plot[18] of these data against the standard exponential distribution is shown. The QQ-plot curves down at the right. This is a clear indication of a right tail of the underlying distribution which is heavier than the tail of the exponential distribution. These observations raise the question as to whether the Poisson process is a suitable model for the whole period of 11 years of claim arrivals.

A homogeneous Poisson process is a suitable model for the arrivals of the Danish fire insurance data for shorter periods of time such as one year. This is illustrated in Figure 2.1.23 for the 166 arrivals in the period January 1 - December $31, 1980$.

As a matter of fact, the data show a clear seasonal component. This can be seen in Figure 2.1.24, where a histogram of all arrivals modulo 366 is given. Hence one receives a distribution on the integers between 1 and 366. Notice for example the peak around day 120 which corresponds to fires in April-May. There is also more activity in summer than in early spring and late fall, and one observes more fires in December and January with the exception of the last week of the year.

2.1.8 An Informal Discussion of Transformed and Generalized Poisson Processes

Consider a Poisson process N with claim arrival times T_i on $[0,\infty)$ and mean value function μ, independent of the iid positive claim sizes X_i with distribution function F. In this section we want to learn about a procedure which allows one to merge the Poisson claim arrival times T_i and the iid claim sizes X_i in one Poisson process with points in $\mathbb{R}^2$.

[18] The reader who is unfamiliar with QQ-plots is referred to Section 3.2.1.

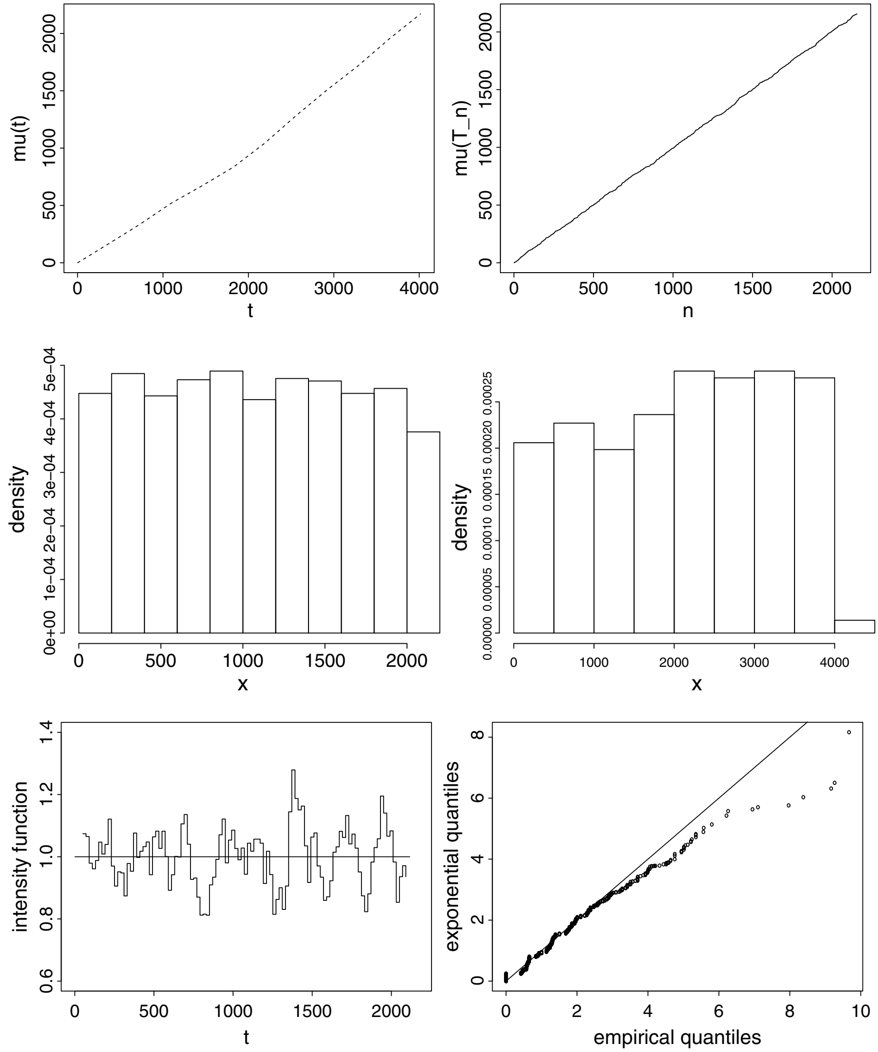

Figure 2.1.22 Top left: *The estimated mean value function* $\mu(t)$ *of the Danish fire insurance arrivals. The function is piecewise linear. The slopes are the estimated intensities from Table 2.1.21.* Top right: *The transformed arrivals* $\mu(T_n)$*. Compare with Figure 2.1.19. The histogram of the values* $\mu(T_n)$ (middle left) *resembles a uniform density, whereas the histogram of the* T_n*'s shows clear deviations from it* (middle right). Bottom left: *Moving average estimate of the intensity function corresponding to the transformed sequence* $(\mu(T_n))$*. The estimates fluctuate around the value* 1. Bottom right: *QQ-plot of the values* $\mu(T_n) - \mu(T_{n-1})$ *against the standard exponential distribution. The plot curves down at the right end indicating that the values come from a distribution with tails heavier than exponential.*

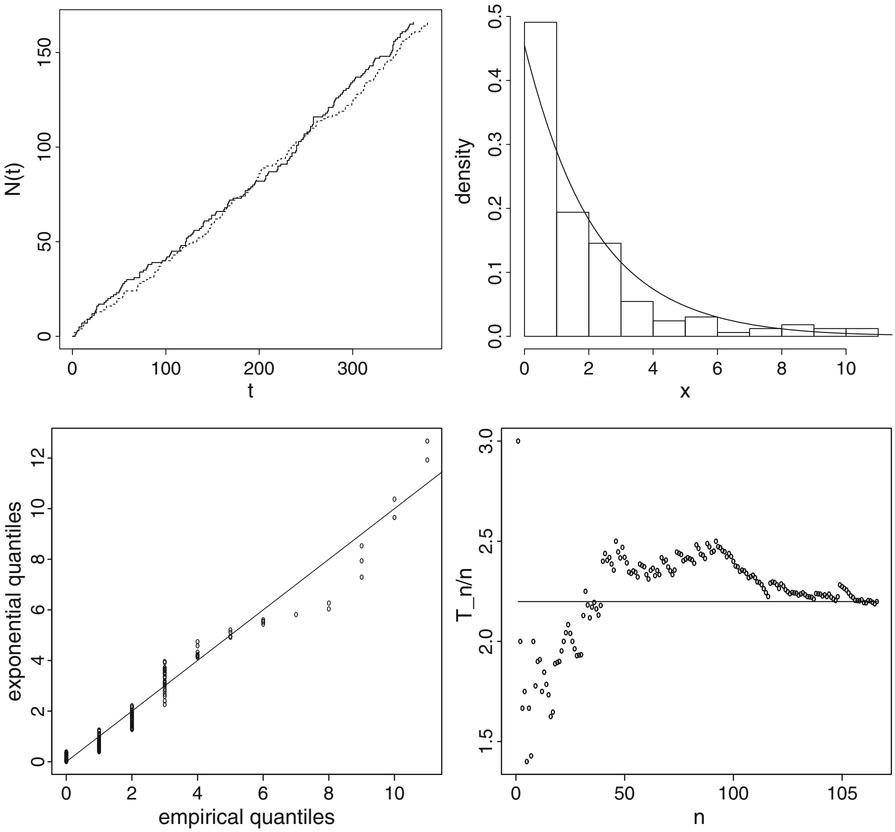

Figure 2.1.23 *The Danish fire insurance arrivals from January* 1, 1980, *until December* 31, 1980. *The inter-arrival times have sample mean* $\widehat{\lambda}^{-1} = 2.19$. Top left: *The renewal process* $N(t)$ *generated by the arrivals (solid boldface curve). For comparison, one sample path of a homogeneous Poisson process with intensity* $\lambda = (2.19)^{-1}$ *is drawn.* Top right: *The histogram of the inter-arrival times. For comparison, the density of the* $\text{Exp}(\lambda)$ *distribution is drawn.* Bottom left: *QQ-plot for the inter-arrival sample against the quantiles of the* $\text{Exp}(\lambda)$ *distribution. The fit of the data by an exponential* $\text{Exp}(\lambda)$ *is not unreasonable. However, the QQ-plot indicates a clear difference to exponential inter-arrival times: the data come from an integer-valued distribution. This deficiency could be overcome if one knew the exact claim times.* Bottom right: *The ratio* T_n/n *as a function of time. The values cluster around* $\widehat{\lambda}^{-1} = 2.19$ *which is indicated by the constant line. For a homogeneous Poisson process,* $T_n/n \stackrel{\text{a.s.}}{\to} \lambda^{-1}$ *by virtue of the strong law of large numbers. For an iid* $\text{Exp}(\lambda)$ *sample* $W_1, \ldots, W_n$, $\widehat{\lambda} = n/T_n$ *is the maximum likelihood estimator of* λ. *If one accepts the hypothesis that the arrivals in* 1980 *come from a homogeneous Poisson process with intensity* $\lambda = (2.19)^{-1}$, *one would have an expected inter-arrival time of* 2.19, *i.e., roughly every second day a claim occurs.*

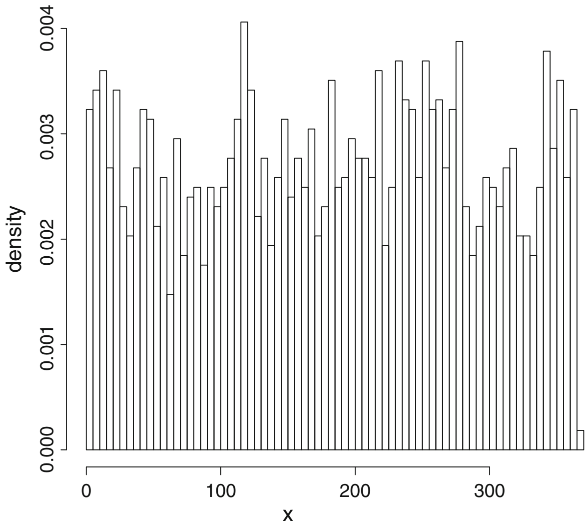

Figure 2.1.24 *Histogram of all arrival times of the Danish fire insurance claims considered as a distribution on the integers between* 1 *and* 366*. The bars of the histogram correspond to periods of* 5 *days. There is a clear indication of seasonality in the data.*

Define the counting process

$$M(a,b) = \#\{i \geq 1 : \, X_i \leq a \, , T_i \leq b\} = \sum_{i=1}^{N(b)} I_{(0,a]}(X_i) \, , \quad a \, , b \geq 0 \, .$$

We want to determine the distribution of $M(a,b)$. For this reason, recall the characteristic function[19] of a Poisson random variable $M \sim \text{Pois}(\gamma)$:

$$E\mathrm{e}^{\,itM} = \sum_{n=0}^{\infty} \mathrm{e}^{\,itn} \, P(M=n) = \sum_{n=0}^{\infty} \mathrm{e}^{\,itn} \, \mathrm{e}^{-\gamma} \, \frac{\gamma^n}{n!} = \mathrm{e}^{-\gamma\,(1-\mathrm{e}^{\,it})} \, , \quad t \in \mathbb{R} \, . \tag{2.1.22}$$

We know that the characteristic function of a random variable M determines its distribution and vice versa. Therefore we calculate the characteristic function of $M(a,b)$. A similar argument as the one leading to (2.1.22) yields

[19] In what follows we work with characteristic functions because this notion is defined for all distributions on $\mathbb{R}$. Alternatively, we could replace the characteristic functions by moment generating functions. However, the moment generating function of a random variable is well-defined only if this random variable has certain finite exponential moments. This would restrict the class of distributions we consider.

$$
\begin{aligned}
E e^{itM(a,b)} &= E\left[E \exp\left\{ i\,t \sum_{j=1}^{N(b)} I_{(0,a]}(X_j) \right\} \middle|\, N(b) \right] \\
&= E\left[\left(E \exp\left\{ i\,t\, I_{(0,a]}(X_1) \right\} \right)^{N(b)} \right] \\
&= E\left(\left[1 - F(a) + F(a)\, e^{it} \right]^{N(b)} \right) \\
&= e^{-\mu(b)\, F(a)\, (1 - e^{it})} \,. \qquad (2.1.23)
\end{aligned}
$$

We conclude from (2.1.22) and (2.1.23) that $M(a,b) \sim \text{Pois}(F(a)\,\mu(b))$. Using similar characteristic function arguments, one can show that

- The increments

$$
\begin{aligned}
&M((x, x+h] \times (t, t+s]) \\
&= \#\{ i \ge 1 : \, (X_i, T_i) \in (x, x+h] \times (t, t+s] \} \,, \quad x, t \ge 0 \,, \; h, s > 0 \,,
\end{aligned}
$$

  are $\text{Pois}(F(x, x+h]\, \mu(t, t+s])$ distributed.
- For disjoint intervals $\Delta_i = (x_i, x_i + h_i] \times (t_i, t_i + s_i]$, $i = 1, \ldots, n$, the increments $M(\Delta_i)$, $i = 1, \ldots, n$, are independent.

From measure theory, we know that the quantities $F(x, x+h]\, \mu(t, t+s]$ determine the product measure $\gamma = F \times \mu$ on the Borel σ-field of $[0, \infty)^2$, where F denotes the distribution function as well as the distribution of X_i and μ is the measure generated by the values $\mu(a, b]$, $0 \le a < b < \infty$. This is a consequence of the extension theorem for measures; cf. Billingsley [18]. In the case of a homogeneous Poisson process, $\mu = \lambda\,\text{Leb}$, where Leb denotes Lebesgue measure on $[0, \infty)$.

In analogy to the extension theorem for deterministic measures, one can find an extension M of the random counting variables $M(\Delta)$, $\Delta = (x, x+h] \times (t, t+s]$, such that for any Borel set[20] $A \subset [0, \infty)^2$,

$$
M(A) = \#\{ i \ge 1 : (X_i, T_i) \in A \} \sim \text{Pois}(\gamma(A)) \,,
$$

and for disjoint Borel sets $A_1, \ldots, A_n \subset [0, \infty)^2$, $M(A_1), \ldots, M(A_n)$ are independent. We call $\gamma = F \times \mu$ the *mean measure* of M, and M is called a *Poisson process* or a *Poisson random measure with mean measure* γ, denoted $M \sim \text{PRM}(\gamma)$. Notice that M is indeed a random counting measure on the Borel σ-field of $[0, \infty)^2$.

The embedding of the claim arrival times and the claim sizes in a Poisson process with two-dimensional points gives one a precise answer as to how many claim sizes of a given magnitude occur in a fixed time interval. For example, the number of claims exceeding a high threshold u, say, in the period $(a, b]$ of time is given by

[20] For A with mean measure $\gamma(A) = \infty$, we write $M(A) = \infty$.

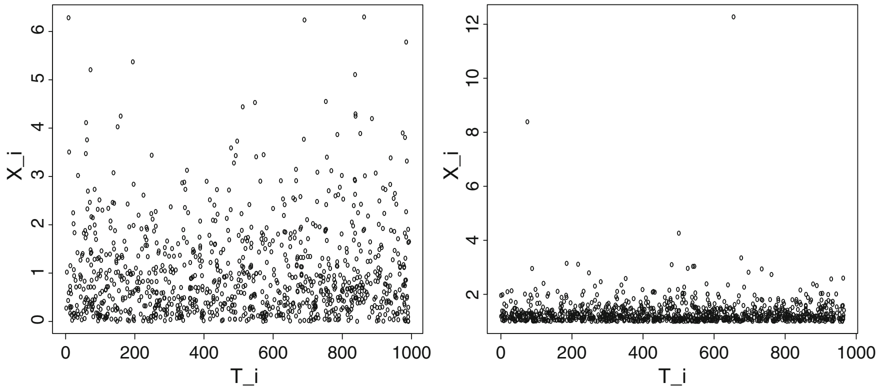

Figure 2.1.25 1 000 *points* (T_i, X_i) *of a two-dimensional Poisson process, where* (T_i) *is the sequence of the the arrival times of a homogeneous Poisson process with intensity* 1 *and* (X_i) *is a sequence of iid claim sizes, independent of* (T_i)*.* Left: *Standard exponential claim sizes.* Right: *Pareto distributed claim sizes with* $P(X_i > x) = x^{-4}$, $x \geq 1$. *Notice the difference in scale of the claim sizes!*

$$M((u,\infty) \times (a,b]) = \#\{i \geq 1 : X_i > u \,, T_i \in (a,b]\} \,.$$

This is a $\text{Pois}((1-F(u))\,\mu(a,b])$ distributed random variable. It is independent of the number of claims below the threshold u occurring in the same time interval. Indeed, the sets $(u,\infty) \times (a,b]$ and $[0,u] \times (a,b]$ are disjoint and therefore $M((u,\infty) \times (a,b])$ and $M([0,u] \times (a,b])$ are independent Poisson distributed random variables.

In the previous sections[21] we used various transformations of the arrival times T_i of a Poisson process N on $[0,\infty)$ with mean measure ν, say, to derive other Poisson processes on the interval $[0,\infty)$. The restriction of processes to $[0,\infty)$ can be relaxed. Consider a measurable set $E \subset \mathbb{R}$ and equip E with the σ-field $\mathcal{E}$ of the Borel sets. Then

$$N(A) = \#\{i \geq 1 : T_i \in A\} \,, \quad A \in \mathcal{E} \,,$$

defines a *random measure* on the measurable space $(E, \mathcal{E})$. Indeed, $N(A) = N(A, \omega)$ depends on $\omega \in \Omega$ and for fixed ω, $N(\cdot, \omega)$ is a counting measure on $\mathcal{E}$. The set E is called the *state space* of the random measure N. It is again called a *Poisson random measure* or *Poisson process* with mean measure ν restricted to E since one can show that $N(A) \sim \text{Pois}(\nu(A))$ for $A \in \mathcal{E}$, and $N(A_i)$, $i = 1, \ldots, n$, are mutually independent for disjoint $A_i \in \mathcal{E}$. The notion of Poisson random measure is very general and can be extended to abstract state spaces E. At the beginning of the section we considered a particular

[21] See, for example, Section 2.1.3.

example in $E = [0, \infty)^2$. The Poisson processes we considered in the previous sections are examples of Poisson processes with state space $E = [0, \infty)$.

One of the strengths of this general notion of Poisson process is the fact that *Poisson random measures remain Poisson random measures under measurable transformations*. Indeed, let $\psi : E \to \widetilde{E}$ be such a transformation and $\widetilde{E}$ be equipped with the σ-field $\widetilde{\mathcal{E}}$. Assume N is PRM(ν) on E with points T_i. Then the points $\psi(T_i)$ are in $\widetilde{E}$ and, for $A \in \widetilde{\mathcal{E}}$,

$$N_\psi(A) = \#\{i \geq 1 : \psi(T_i) \in A\} = \#\{i \geq 1 : T_i \in \psi^{-1}(A)\} = N(\psi^{-1}(A)),$$

where $\psi^{-1}(A) = \{x \in E : \psi(x) \in A\}$ denotes the inverse image of A which belongs to $\mathcal{E}$ since ψ is measurable. Then we also have that $N_\psi(A) \sim \text{Pois}(\nu(\psi^{-1}(A)))$ since $EN_\psi(A) = EN(\psi^{-1}(A)) = \nu(\psi^{-1}(A))$. Moreover, since disjointness of $A_1, \ldots, A_n$ in $\widetilde{\mathcal{E}}$ implies disjointness of $\psi^{-1}(A_1), \ldots, \psi^{-1}(A_n)$ in $\mathcal{E}$, it follows that $N_\psi(A_1), \ldots, N_\psi(A_n)$ are independent, by the corresponding property of the PRM N. We conclude that $N_\psi \sim \text{PRM}(\nu(\psi^{-1}))$.

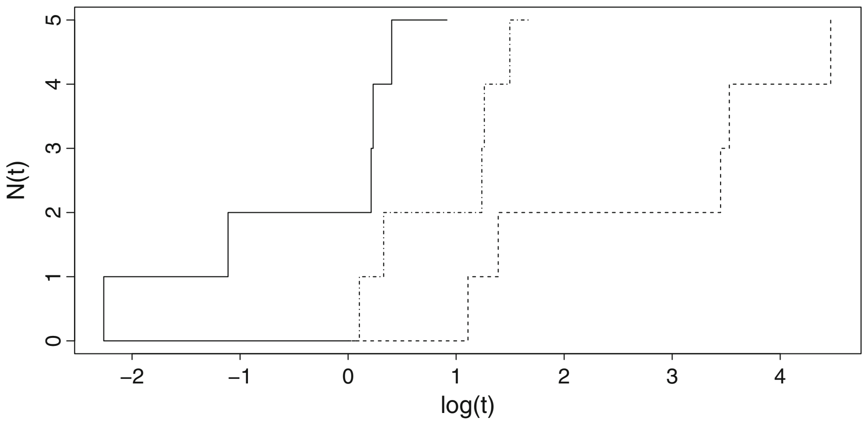

Figure 2.1.26 *Sample paths of the Poisson processes with arrival times* $\exp\{T_i\}$ (bottom dashed curve), T_i (middle dashed curve) *and* $\log T_i$ (top solid curve). *The T_i's are the arrival times of a standard homogeneous Poisson process. Time is on logarithmic scale in order to visualize the three paths in one graph.*

Example 2.1.27 (Measurable transformations of Poisson processes remain Poisson processes)

(1) Let $\widetilde{N}$ be a Poisson process on $[0, \infty)$ with mean value function $\widetilde{\mu}$ and arrival times $0 < T_1 < T_2 < \cdots$. Consider the transformed process

$$N(t) = \#\{i \geq 1 : 0 \leq T_i - a \leq t\}, \quad 0 \leq t \leq b - a,$$

for some interval $[a,b] \subset [0,\infty)$, where $\psi(x) = x-a$ is clearly measurable. This construction implies that $N(A) = \#\{i \ge 1 : \psi(T_i) \in A\} = 0$ for $A \subset [0, b-a]^c$, the complement of $[0, b-a]$. Therefore it suffices to consider N on the Borel sets of $[0, b-a]$. This defines a *Poisson process on* $[a,b]$ with mean value function $\mu(t) = \widetilde{\mu}(t) - \widetilde{\mu}(a)$, $t \in [a,b]$.
(2) Consider a standard homogeneous Poisson process on $[0,\infty)$ with arrival times $0 < T_1 < T_2 < \cdots$. We transform the arrival times with the measurable function $\psi(x) = \log x$. Then the points $(\log T_i)$ constitute a Poisson process N on $\mathbb{R}$. The Poisson measure of the interval $(a,b]$ for $a < b$ is given by

$$N(a,b] = \#\{i \ge 1 : \log(T_i) \in (a,b]\} = \#\{i \ge 1 : T_i \in (\mathrm{e}^{\,a}, \mathrm{e}^{\,b}]\}\,.$$

This is a $\mathrm{Pois}(\mathrm{e}^{\,b} - \mathrm{e}^{\,a})$ distributed random variable, i.e., the mean measure of the interval $(a,b]$ is given by $\mathrm{e}^{\,b} - \mathrm{e}^{\,a}$.
Alternatively, transform the arrival times T_i by the exponential function. The resulting Poisson process M is defined on $[1,\infty)$. The Poisson measure of the interval $(a,b] \subset [1,\infty)$ is given by

$$M(a,b] = \#\{i \ge 1 :\, \mathrm{e}^{\,T_i} \in (a,b]\} = \#\{i \ge 1 :\, T_i \in (\log a, \log b]\}\,.$$

This is a $\mathrm{Pois}(\log(b/a))$ distributed random variable, i.e., the mean measure of the interval $(a,b]$ is given by $\log(b/a)$. Notice that this Poisson process has the remarkable property that $M(ca, cb]$ for any $c \ge 1$ has the same $\mathrm{Pois}(\log(b/a))$ distribution as $M(a,b]$. In particular, the expected number of points $\exp\{T_i\}$ falling into the interval $(ca, cb]$ is independent of the value $c \ge 1$. This is somewhat counterintuitive since the length of the interval $(ca, cb]$ can be arbitrarily large. However, the larger the value c the higher the threshold ca which prevents sufficiently many points $\exp\{T_i\}$ from falling into the interval $(ca, cb]$, and on average there are as many points in $(ca, cb]$ as in $(a,b]$. □

Example 2.1.28 (Construction of transformed planar PRM)
Let (T_i) be the arrival sequence of a standard homogeneous Poisson process on $[0,\infty)$, independent of the iid sequence (X_i) with common distribution function F. Then the points (T_i, X_i) constitute a $\mathrm{PRM}(\nu)$ N with state space $E = [0,\infty) \times \mathbb{R}$ and mean measure $\nu = \mathrm{Leb} \times F$; see the discussion on p. 39.

After a measurable transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ the points $\psi(T_i, X_i)$ constitute a PRM N_ψ with state space $E_\psi = \{\psi(t,x) : (t,x) \in E\}$ and mean measure $\nu_\psi(A) = \nu(\psi^{-1}(A))$ for any Borel set $A \subset E_\psi$. We choose $\widetilde{\psi}(t,x) = t^{-1/\alpha}\,(\cos(2\,\pi\,x), \sin(2\,\pi\,x))$ for some $\alpha \ne 0$, i.e., the PRM $N_{\widetilde{\psi}}$ has points $\mathbf{Y}_i = T_i^{-1/\alpha}(\cos(2\,\pi\,X_i), \sin(2\,\pi\,X_i))$. In Figure 2.1.30 we visualize the points $\mathbf{Y}_i$ of the resulting PRM for different choices of α and distribution functions F of X_1.

Planar PRMs such as the ones described above are used, among others, in spatial statistics (see Cressie [37]) in order to describe the distribution of random configurations of points in the plane such as the distribution of minerals, locations of highly polluted spots or trees in a forest. The particular

PRM $N_{\widetilde{\psi}}$ and its modifications are major models in multivariate extreme value theory. It describes the dependence of extremes in the plane and in space. In particular, it is suitable for modeling clustering behavior of points $\mathbf{Y}_i$ far away from the origin. See Resnick [122] for the theoretical background on multivariate extreme value theory and Mikosch [108] for a recent attempt to use $N_{\widetilde{\psi}}$ for modeling multivariate financial time series. □

Example 2.1.29 (Modeling arrivals of Incurred But Not Reported (IBNR) claims)
In a portfolio, the claims are not reported at their arrival times T_i, but with a certain delay. This delay may be due to the fact that the policyholder is not aware of the claim and only realizes it later (for example, a damage in his/her house), or that the policyholder was injured in a car accident and did not have the opportunity to call his agent immediately, or the policyholder's flat burnt down over Christmas, but the agent was on a skiing vacation in Switzerland and could not receive the report about the fire, etc.

We consider a simple model for the reporting times of IBNR claims: the arrival times T_i of the claims are modeled by a Poisson process N with mean value function μ and the delays in reporting by an iid sequence (V_i) of positive random variables with common distribution F. Then the sequence $(T_i + V_i)$ constitutes the reporting times of the claims to the insurance business. We assume that (V_i) and (T_i) are independent. Then the points (T_i, V_i) constitute a PRM(ν) with mean measure $\nu = \mu \times F$. By time t, $N(t)$ claims have occurred, but only

$$N_{\mathrm{IBNR}}(t) = \sum_{i=1}^{N(t)} I_{[0,t]}(T_i + V_i) = \#\{i \geq 1 : T_i + V_i \leq t\}$$

have been reported. The mapping $\psi(t, v) = t + v$ is measurable. It transforms the points (T_i, V_i) of the PRM(ν) into the points $T_i + V_i$ of the PRM N_ψ with mean measure of a set A given by $\nu_\psi(A) = \nu(\psi^{-1}(A))$. In particular, $N_{\mathrm{IBNR}}(s) = N_\psi([0, s])$ is Pois$(\nu_\psi([0, s]))$ distributed. We calculate the mean value $\nu_\psi([0, s])$ in Example 7.3.9 below. There we further discuss this IBNR model in the context of point processes. □

Comments

The Poisson process is one of the most important stochastic processes. For the abstract understanding of this process one would have to consider it as a *point process*, i.e., as a random counting measure. We have indicated in Section 2.1.8 how one has to approach this problem. In Chapters 7 and 8 we give a more advanced treatment of the theory of point processes. There we focus on *generalized Poisson processes* or *Poisson random measures* and their use in non-life insurance applications.

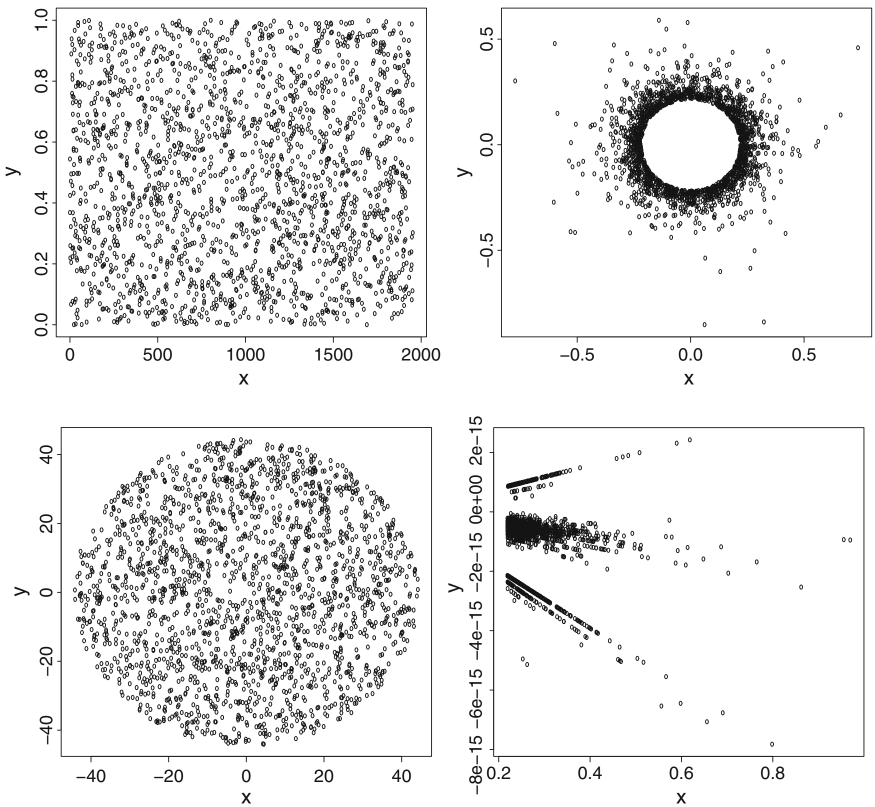

Figure 2.1.30 *Poisson random measures in the plane.*

Top left: 2 000 *points of a Poisson random measure with points* (T_i, X_i), *where* (T_i) *is the arrival sequence of a standard homogeneous Poisson process on* $[0,\infty)$, *independent of the iid sequence* (X_i) *with* $X_1 \sim \mathrm{U}(0,1)$. *The PRM has mean measure* $\nu = \mathrm{Leb} \times \mathrm{Leb}$ *on* $[0,\infty) \times (0,1)$.

After the measurable transformation $\widetilde{\psi}(t,x) = t^{-1/\alpha}\,(\cos(2\,\pi\,x), \sin(2\,\pi\,x))$ *for some* $\alpha \neq 0$ *the resulting PRM* $N_{\widetilde{\psi}}$ *has points* $\mathbf{Y}_i = T_i^{-1/\alpha}(\cos(2\,\pi\,X_i), \sin(2\,\pi\,X_i))$.

Top right: *The points of the process* $N_{\widetilde{\psi}}$ *for* $\alpha = 5$ *and iid* $\mathrm{U}(0,1)$ *uniform* X_i*'s. Notice that the spherical part* $(\cos(2\,\pi\,X_i), \sin(2\,\pi\,X_i))$ *of* $\mathbf{Y}_i$ *is uniformly distributed on the unit circle.*

Bottom left: *The points of the process* $N_{\widetilde{\psi}}$ *with* $\alpha = -5$ *and iid* $\mathrm{U}(0,1)$ *uniform* X_i*'s.*

Bottom right: *The points of the process* $N_{\widetilde{\psi}}$ *for* $\alpha = 5$ *with iid* $X_i \sim \mathrm{Pois}(10)$.

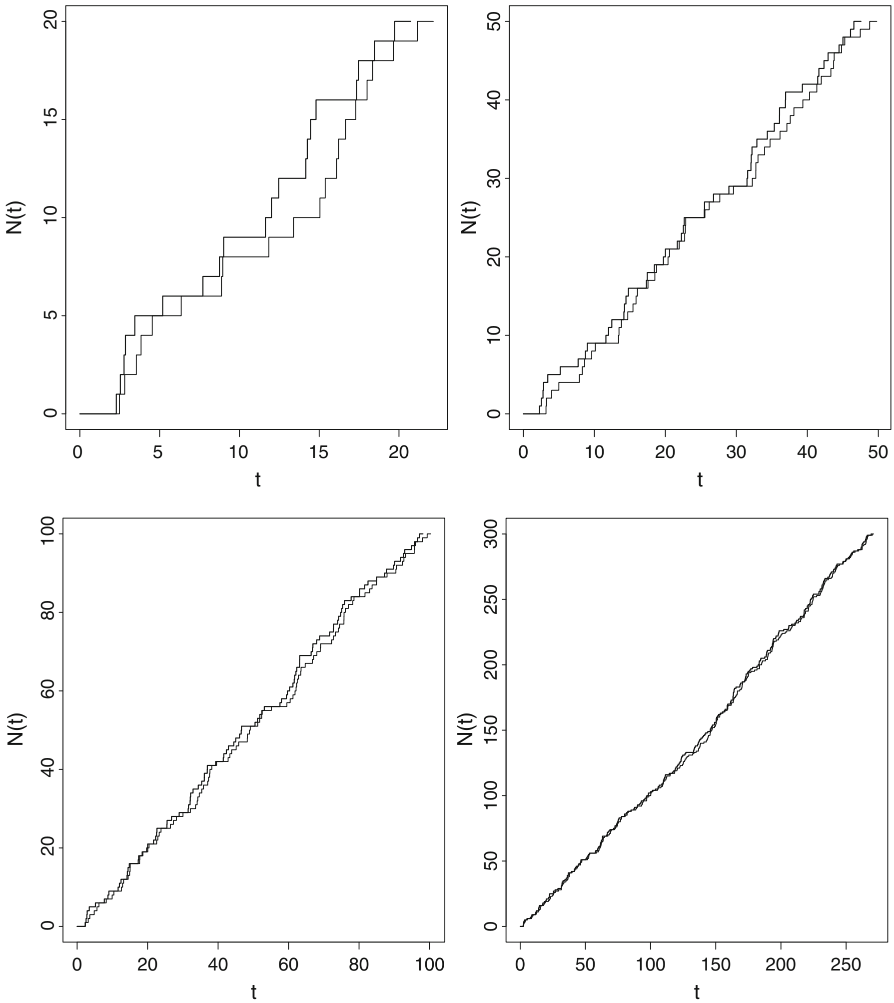

Figure 2.1.31 *Incurred But Not Reported claims. We visualize one sample of a standard homogeneous Poisson process with* n *arrivals* T_i *(top boldface graph) and the corresponding claim number process for the delayed process with arrivals* $T_i + V_i$*, where the* V_i*'s are iid Pareto distributed with distribution* $P(V_1 > x) = x^{-2}$*,* $x \geq 1$*, independent of* (T_i)*.* Top: $n = 30$ (left) *and* $n = 50$ (right). Bottom: $n = 100$ (left) *and* $n = 300$ (right). *As explained in Example 2.1.29, the sample paths of the claim number processes differ from each other approximately by the constant value* EV_1*. For sufficiently large* t*, the difference is negligible compared to the expected claim number.*

As a matter of fact, various other counting processes such as the renewal process treated in Section 2.2 are approximated by suitable Poisson processes in the sense of convergence in distribution. Therefore the Poisson process with nice mathematical properties is also a good approximation to various real-life counting processes such as the claim number process in an insurance portfolio. In Chapter 9 we develop the theory of convergence in distribution of point processes. The convergence to a Poisson process is of particular interest. We show how these asymptotic relations can be used to determine the distribution of extremely large claim sizes.

The treatment of general Poisson processes requires more sophisticated tools and techniques from the theory of stochastic processes. For a gentle introduction to point processes and generalized Poisson processes we refer to Embrechts et al. [46], Chapter 5; for a rigorous treatment at a moderate level, Resnick's monograph [123] or Kingman's book [85] are good references. Resnick [122] is an advanced text on the Poisson process with applications to extreme value theory. See also Daley and Vere-Jones [38] or Kallenberg [79] for rigorous treatments of the general point process theory.

Exercises

Sections 2.1.1-2.1.2

(1) Let $N = (N(t))_{t\geq 0}$ be a Poisson process with continuous intensity function $(\lambda(t))_{t\geq 0}$.
   (a) Show that the intensities $\lambda_{n,n+k}(t)$, $n \geq 0$, $k \geq 1$ and $t > 0$, of the Markov process N with transition probabilities $p_{n,n+k}(s,t)$ exist, i.e.,

$$\lambda_{n,n+k}(t) = \lim_{h\downarrow 0} \frac{p_{n,n+k}(t,t+h)}{h}\,, \quad n \geq 0\,, k \geq 1\,,$$

   and that they are given by

$$\lambda_{n,n+k}(t) = \begin{cases} \lambda(t)\,, & k = 1\,, \\ 0\,, & k \geq 2\,. \end{cases} \qquad (2.1.24)$$

   (b) What can you conclude from $p_{n,n+k}(t,t+h)$ for h small about the short term jump behavior of the Markov process N?
   (c) Show by counterexample that (2.1.24) is in general not valid if one gives up the assumption of continuity of the intensity function $\lambda(t)$.

(2) Let $N = (N(t))_{t\geq 0}$ be a Poisson process with continuous intensity function $(\lambda(t))_{t\geq 0}$. By using the properties of N given in Definition 2.1.1, show that the following properties hold:
   (a) The sample paths of N are non-decreasing.
   (b) The process N does not have a jump at zero with probability 1.
   (c) For every fixed t, the process N does not have a jump at t with probability 1. Does this mean that the sample paths do not have jumps?

(3) Let N be a homogeneous Poisson process on $[0,\infty)$ with intensity $\lambda > 0$. Show that for $0 < t_1 < t < t_2$,

$$\lim_{h\downarrow 0} P(N(t_1-h\,,t-h]=0\,,N(t-h,t]=1\,,N(t,t_2]=0\mid N(t-h\,,t]>0)$$

$$= \mathrm{e}^{-\lambda\,(t-t_1)}\,\mathrm{e}^{-\lambda\,(t_2-t)}\,.$$

Give an intuitive interpretation of this property.

(4) Let $N_1,\ldots,N_n$ be independent Poisson processes on $[0,\infty)$ defined on the same probability space. Show that $N_1+\cdots+N_n$ is a Poisson process and determine its mean value function.

This property extends the well-known property that the sum M_1+M_2 of two independent Poisson random variables $M_1 \sim \mathrm{Pois}(\lambda_1)$ and $M_2 \sim \mathrm{Pois}(\lambda_2)$ is $\mathrm{Pois}(\lambda_1+\lambda_2)$. We also mention that a converse to this result holds. Indeed, suppose $M = M_1 + M_2$, $M \sim \mathrm{Pois}(\lambda)$ for some $\lambda > 0$ and M_1, M_2 are independent non-negative random variables. Then both M_1 and M_2 are necessarily Poisson random variables. This phenomenon is referred to as *Raikov's theorem*; see Lukacs [97], Theorem 8.2.2. An analogous theorem can be shown for so-called *point processes* which are counting processes on $[0,\infty)$, including the Poisson process and the renewal process, see Chapter 7 for an introduction to the theory of point processes. Indeed, if the Poisson process N has representation $N \stackrel{d}{=} N_1 + N_2$ for independent point processes N_1, N_2, then N_1 and N_2 are necessarily Poisson processes.

(5) Consider the total claim amount process S in the Cramér-Lundberg model.
  (a) Show that the total claim amount $S(s,t]$ in $(s,t]$ for $s < t$, i.e., $S(s,t] = S(t)-S(s)$, has the same distribution as the total claim amount in $[0,t-s]$, i.e., $S(t-s)$.
  (b) Show that, for every $0 = t_0 < t_1 < \cdots < t_n$ and $n \ge 1$, the random variables $S(t_1)$, $S(t_1,t_2]\,,\ldots\,,S(t_{n-1},t_n]$ are independent. Hint: Calculate the joint characteristic function of the latter random variables.

(6) For a homogeneous Poisson process N on $[0,\infty)$ show that for $0 < s < t$,

$$P(N(s)=k\mid N(t)) = \begin{cases} \binom{N(t)}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{N(t)-k} & \text{if } k \le N(t)\,,\\ 0 & \text{if } k > N(t)\,. \end{cases}$$

Section 2.1.3

(7) Let $\widetilde{N}$ be a standard homogeneous Poisson process on $[0,\infty)$ and N a Poisson process on $[0,\infty)$ with mean value function μ.
  (a) Show that $N_1 = (\widetilde{N}(\mu(t)))_{t\ge 0}$ is a Poisson process on $[0,\infty)$ with mean value function μ.
  (b) Assume that the inverse μ^{-1} of μ exists, is continuous and $\lim_{t\to\infty}\mu(t)=\infty$. Show that $\widetilde{N}_1(t) = N(\mu^{-1}(t))$ defines a standard homogeneous Poisson process on $[0,\infty)$.
  (c) Assume that the Poisson process N has an intensity function λ. Which condition on λ ensures that $\mu^{-1}(t)$ exists for $t \ge 0$?
  (d) Let $f : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function with $f(0) = 0$. Show that

$$N_f(t) = N(f(t)), \quad t \geq 0,$$

is again a Poisson process on $[0, \infty)$. Determine its mean value function.

Sections 2.1.4-2.1.5

(8) Recall from Theorem 2.1.6 that the homogeneous Poisson process $\widetilde{N}$ with intensity $\widetilde{\lambda} > 0$ can be written as a renewal process

$$\widetilde{N}(t) = \#\{i \geq 1 : \widetilde{T}_i \leq t\}, \quad t \geq 0,$$

where $\widetilde{T}_i = \widetilde{W}_1 + \cdots + \widetilde{W}_i$ and $(\widetilde{W}_n)$ is an iid $\text{Exp}(\widetilde{\lambda})$ sequence.

Let N be a Poisson process with mean value function μ which has an a.e. positive continuous intensity function λ. Let $0 \leq T_1 \leq T_2 \leq \cdots$ be the arrival times of the process N.

(a) Show that the random variables $\int_{T_n}^{T_{n+1}} \lambda(s)\,ds$ are iid exponentially distributed.

(b) Show that, with probability 1, no multiple claims can occur, i.e., at an arrival time T_i of a claim, $N(T_i) - N(T_i-) = 1$ a.s. and $P(N(T_i) - N(T_i-) > 1 \text{ for some } i) = 0$.

(9) Consider a homogeneous Poisson process N with intensity $\lambda > 0$ and arrival times T_i.

(a) Assume the renewal representation $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$, for N, i.e., $T_0 = 0$, $W_i = T_i - T_{i-1}$ are iid $\text{Exp}(\lambda)$ inter-arrival times. Calculate for $0 \leq t_1 < t_2$,

$$P(T_1 \leq t_1) \quad \text{and} \quad P(T_1 \leq t_1, T_2 \leq t_2). \tag{2.1.25}$$

(b) Assume the properties of Definition 2.1.1 for N. Calculate for $0 \leq t_1 < t_2$,

$$P(N(t_1) \geq 1) \quad \text{and} \quad P(N(t_1) \geq 1, N(t_2) \geq 2). \tag{2.1.26}$$

(c) Give reasons why you get the same probabilities in (2.1.25) and (2.1.26).

(10) Consider a homogeneous Poisson process on $[0, \infty)$ with arrival time sequence (T_i) and set $T_0 = 0$. The inter-arrival times are defined as $W_i = T_i - T_{i-1}$, $i \geq 1$.

(a) Show that T_1 has the *forgetfulness property*, i.e., $P(T_1 > t + s \mid T_1 > t) = P(T_1 > s)$, $t, s \geq 0$.

(b) Another version of the forgetfulness property is as follows. Let $Y \geq 0$ be independent of T_1 and Z be a random variable whose distribution is given by

$$P(Z > z) = P(T_1 > Y + z \mid T_1 > Y), \quad z \geq 0.$$

Then Z and T_1 have the same distribution. Verify this.

(c) Show that the events $\{W_1 < W_2\}$ and $\{\min(W_1, W_2) > x\}$ are independent.

(d) Determine the distribution of $m_n = \min(T_1, T_2 - T_1, \ldots, T_n - T_{n-1})$.

(11) Suppose you want to simulate sample paths of a Poisson process.

(a) How can you exploit the renewal representation to simulate paths of a homogeneous Poisson process?

(b) How can you use the renewal representation of a homogeneous Poisson N to simulate paths of an inhomogeneous Poisson process?

Sections 2.1.6

(12) Let $U_1, \ldots, U_n$ be an iid U(0, 1) sample with the corresponding order statistics $U_{(1)} < \cdots < U_{(n)}$ a.s. Let $(\widetilde{W}_i)$ be an iid sequence of Exp(λ) distributed random variables and $\widetilde{T}_n = \widetilde{W}_1 + \cdots + \widetilde{W}_n$ the corresponding arrival times of a homogeneous Poisson process with intensity λ.

(a) Show that the following identity in distribution holds for every fixed $n \geq 1$:

$$\left(U_{(1)}, \ldots, U_{(n)}\right) \stackrel{d}{=} \left(\frac{\widetilde{T}_1}{\widetilde{T}_{n+1}}, \ldots, \frac{\widetilde{T}_n}{\widetilde{T}_{n+1}}\right). \tag{2.1.27}$$

Hint: Calculate the densities of the vectors on both sides of (2.1.27). The density of the vector

$$[(\widetilde{T}_1, \ldots, \widetilde{T}_n)/\widetilde{T}_{n+1}, \widetilde{T}_{n+1}]$$

can be obtained from the known density of the vector $(\widetilde{T}_1, \ldots, \widetilde{T}_{n+1})$.

(b) Why is the distribution of the right-hand vector in (2.1.27) independent of λ?

(c) Let T_i be the arrivals of a Poisson process on $[0, \infty)$ with a.e. positive intensity function λ and mean value function μ. Show that the following identity in distribution holds for every fixed $n \geq 1$:

$$\left(U_{(1)}, \ldots, U_{(n)}\right) \stackrel{d}{=} \left(\frac{\mu(T_1)}{\mu(T_{n+1})}, \ldots, \frac{\mu(T_n)}{\mu(T_{n+1})}\right).$$

(13) Let $W_1, \ldots, W_n$ be an iid Exp(λ) sample for some $\lambda > 0$. Show that the ordered sample $W_{(1)} < \cdots < W_{(n)}$ has representation in distribution:

$$\begin{aligned}
&\left(W_{(1)}, \ldots, W_{(n)}\right) \\
&\stackrel{d}{=} \left(\frac{W_n}{n}, \frac{W_n}{n} + \frac{W_{n-1}}{n-1}, \ldots, \frac{W_n}{n} + \frac{W_{n-1}}{n-1} + \cdots + \frac{W_2}{2},\right. \\
&\qquad \left.\frac{W_n}{n} + \frac{W_{n-1}}{n-1} + \cdots + \frac{W_1}{1}\right).
\end{aligned}$$

Hint: Use a density transformation starting with the joint density of $W_1, \ldots, W_n$ to determine the density of the right-hand expression.

(14) Consider the stochastically discounted total claim amount

$$S(t) = \sum_{i=1}^{N(t)} \mathrm{e}^{-rT_i} X_i,$$

where $r > 0$ is an interest rate, $0 < T_1 < T_2 < \cdots$ are the claim arrival times, defining the homogeneous Poisson process $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$, with intensity $\lambda > 0$, and (X_i) is an iid sequence of positive claim sizes, independent of (T_i).

(a) Calculate the mean and the variance of $S(t)$ by using the order statistics property of the Poisson process N. Specify the mean and the variance in the case when $r = 0$ (Cramér-Lundberg model).

(b) Show that $S(t)$ has the same distribution as

$$e^{-rt} \sum_{i=1}^{N(t)} \mathrm{e}^{\,rT_i}\, X_i \,.$$

(15) Suppose you want to simulate sample paths of a Poisson process on $[0,T]$ for $T>0$ and a given continuous intensity function λ, by using the order statistics property.
(a) How should you proceed if you are interested in one path with exactly n jumps in $[0,T]$?
(b) How would you simulate several paths of a homogeneous Poisson process with (possibly) different jump numbers in $[0,T]$?
(c) How could you use the simulated paths of a homogeneous Poisson process to obtain the paths of an inhomogeneous one with given intensity function?

(16) Let (T_i) be the arrival sequence of a standard homogeneous Poisson process N and $\alpha \in (0,1)$.
(a) Show that the infinite series

$$X_\alpha = \sum_{i=1}^{\infty} T_i^{-1/\alpha} \tag{2.1.28}$$

converges a.s. Hint: Use the strong law of large numbers for (T_n).
(b) Show that

$$X_{N(t)} = \sum_{i=1}^{N(t)} T_i^{-1/\alpha} \stackrel{\text{a.s.}}{\rightarrow} X_\alpha \quad \text{as } t \to \infty.$$

Hint: Use Lemma 2.2.6.
(c) It follows from standard limit theory for sums of iid random variables (see Feller [51], Theorem 1 in Chapter XVII.5) that for iid $\mathrm{U}(0,1)$ random variables U_i,

$$n^{-1/\alpha} \sum_{i=1}^{n} U_i^{-1/\alpha} \stackrel{d}{\rightarrow} Z_\alpha \,, \tag{2.1.29}$$

where Z_α is a positive random variable with an *α-stable distribution* determined by its Laplace-Stieltjes transform $E\exp\{-s\,Z_\alpha\} = \exp\{-c\,s^\alpha\}$ for some $c>0$, all $s \ge 0$. See p. 178 for some information about Laplace-Stieltjes transforms. Show that $X_\alpha \stackrel{d}{=} c'\,Z_\alpha$ for some positive constant $c'>0$.
Hints: (i) Apply the order statistics property of the homogeneous Poisson process to $X_{N(t)}$ to conclude that

$$X_{N(t)} \stackrel{d}{=} t^{-1/\alpha} \sum_{i=1}^{N(t)} U_i^{-1/\alpha} \,,$$

where (U_i) is an iid $\mathrm{U}(0,1)$ sequence, independent of $N(t)$.
(ii) Prove that

$$(N(t))^{-1/\alpha} \sum_{i=1}^{N(t)} U_i^{-1/\alpha} \xrightarrow{d} Z_\alpha \quad \text{as } t \to \infty\,.$$

Hint: Condition on $N(t)$ and exploit (2.1.29).
(iii) Use the strong law of large numbers $N(t)/t \stackrel{\text{a.s.}}{\rightarrow} 1$ as $t \to \infty$ (Theorem 2.2.5) and the continuous mapping theorem to conclude the proof.

(d) Show that $EX_\alpha = \infty$.

(e) Let $Z_1, \ldots, Z_n$ be iid copies of the α-stable random variable Z_α with Laplace-Stieltjes transform $E\mathrm{e}^{-s\,Z_\alpha} = \mathrm{e}^{-c\,s^\alpha}$, $s \ge 0$, for some $\alpha \in (0,1)$ and $c > 0$. Show that for every $n \ge 1$ the relation

$$Z_1 + \cdots + Z_n \stackrel{d}{=} n^{1/\alpha}\, Z_\alpha$$

holds. It is due to this "stability condition" that the distribution gained its name.
Hint: Use the properties of Laplace-Stieltjes transforms (see p. 178) to show this property.

(f) Consider Z_α from (e) for some $\alpha \in (0,1)$.
(i) Show the relation

$$E\mathrm{e}^{\,i\,t\,A\,Z_\alpha^{1/2}} = \mathrm{e}^{-c\,|t|^{2\alpha}}\,, \quad t \in \mathbb{R}\,, \tag{2.1.30}$$

where $A \sim \mathrm{N}(0,2)$ is independent of Z_α. A random variable Y with characteristic function given by the right-hand side of (2.1.30) and its distribution are said to be *symmetric 2α-stable.*
(ii) Let $Y_1, \ldots, Y_n$ be iid copies of Y from (i). Show the stability relation

$$Y_1 + \cdots + Y_n \stackrel{d}{=} n^{1/(2\alpha)}\, Y\,.$$

(iii) Conclude that Y must have infinite variance. Hint: Suppose that Y has finite variance and try to apply the central limit theorem.

The interested reader who wants to learn more about the exciting class of stable distributions and stable processes is referred to Samorodnitsky and Taqqu [131].

Section 2.1.8

(17) Let $(N(t))_{t\ge 0}$ be a standard homogeneous Poisson process with claim arrival times T_i.

(a) Show that the sequences of arrival times $(\sqrt{T_i})$ and (T_i^2) define two Poisson processes N_1 and N_2, respectively, on $[0,\infty)$. Determine their mean measures by calculating $EN_i(s,t]$ for any $s < t$, $i = 1, 2$.

(b) Let N_3 and N_4 be Poisson processes on $[0,\infty)$ with mean value functions $\mu_3(t) = \sqrt{t}$ and $\mu_4(t) = t^2$ and arrival time sequences $(T_i^{(3)})$ and $(T_i^{(4)})$, respectively. Show that the processes $(N_3(t^2))_{t\ge 0}$ and $(N_4(\sqrt{t}))_{t\ge 0}$ are Poisson on $[0,\infty)$ and have the same distribution.

(c) Show that the process

$$N_5(t) = \#\{i \ge 1 : \mathrm{e}^{\,T_i} \le t+1\}\,, \quad t \ge 0\,,$$

is a Poisson process and determine its mean value function.

(d) Let N_6 be a Poisson process on $[0,\infty)$ with mean value function $\mu_6(t) = \log(1+t)$. Show that N_6 has the property that, for $1 \le s < t$ and $a \ge 1$, the distribution of $N_6(at-1) - N_6(as-1)$ does not depend on a.

(18) Let (T_i) be the arrival times of a homogeneous Poisson process N on $[0,\infty)$ with intensity $\lambda > 0$, independent of the iid claim size sequence (X_i) with $X_i > 0$ and distribution function F.

(a) Show that for $s < t$ and $a < b$ the counting random variable

$$M((s,t] \times (a,b]) = \#\{i \ge 1 : T_i \in (s,t]\,, X_i \in (a,b]\}$$

is $\mathrm{Pois}(\lambda\,(t-s)F(a,b])$ distributed.

(b) Let $\Delta_i = (s_i,t_i] \times (a_i,b_i]$ for $s_i < t_i$ and $a_i < b_i$, $i = 1,2$, be disjoint. Show that $M(\Delta_1)$ and $M(\Delta_2)$ are independent.

(19) Consider the two-dimensional PRM $N_{\widetilde{\psi}}$ from Figure 2.1.30 with $\alpha > 0$.

(a) Calculate the mean measure of the set $A(r,S) = \{\mathbf{x} : |\mathbf{x}| > r\,, \mathbf{x}/|\mathbf{x}| \in S\}$, where $r > 0$ and S is any Borel subset of the unit circle.

(b) Show that $EN_{\widetilde{\psi}}(A(rt,S)) = t^{-\alpha}\,EN_{\widetilde{\psi}}(A(r,S))$ for any $t > 0$.

(c) Let $\mathbf{Y} = R\,(\cos(2\,\pi\,X)\,, \sin(2\,\pi\,X))$, where $P(R > x) = x^{-\alpha}$, $x \ge 1$, X is uniformly distributed on $(0,1)$ and independent of R. Show that for $r \ge 1$,

$$EN_{\widetilde{\psi}}(A(r,S)) = P(\mathbf{Y} \in A(r,S))\,.$$

(20) Let $(E,\mathcal{E},\mu)$ be a measure space such that $0 < \mu(E) < \infty$ and τ be $\mathrm{Pois}(\mu(E))$ distributed. Assume that τ is independent of the iid sequence (X_i) with distribution given by

$$F_{X_1}(A) = P(X_1 \in A) = \mu(A)/\mu(E)\,, \quad A \in \mathcal{E}\,.$$

(a) Show that the counting process

$$N(A) = \sum_{i=1}^{\tau} I_A(X_i)\,, \quad A \in \mathcal{E}\,,$$

is $\mathrm{PRM}(\mu)$ on E. Hint: Calculate the joint characteristic function of the random variables $N(A_1),\ldots,N(A_m)$ for any disjoint $A_1,\ldots,A_m \in \mathcal{E}$.

(b) Specify the construction of (a) in the case that $E = [0,1]$ equipped with the Borel σ-field, when μ has an a.e. positive density λ. What is the relation with the order statistics property of the Poisson process N?

(c) Specify the construction of (a) in the case that $E = [0,1]^d$ equipped with the Borel σ-field for some integer $d \ge 1$ when $\mu = \lambda\,\mathrm{Leb}$ for some constant $\lambda > 0$. Propose how one could define an "order statistics property" for this (homogeneous) Poisson process with points in E.

(21) Let τ be a $\mathrm{Pois}(1)$ random variable, independent of the iid sequence (X_i) with common distribution function F and a positive density on $(0,\infty)$.

(a) Show that

$$N(t) = \sum_{i=1}^{\tau} I_{(0,t]}(X_i)\,, \quad t \ge 0\,,$$

defines a Poisson process on $[0,\infty)$ in the sense of Definition 2.1.1.

(b) Determine the mean value function of N.

(c) Find a function $f : [0,\infty) \to [0,\infty)$ such that the time changed process $(N(f(t)))_{t\ge 0}$ becomes a standard homogeneous Poisson process.

(22) For an iid sequence (X_i) with common continuous distribution function F define the sequence of partial maxima $M_n = \max(X_1,\dots,X_n)$, $n \ge 1$. Define $L(1) = 1$ and, for $n \ge 1$,

$$L(n+1) = \inf\{k > L(n) : X_k > X_{L(n)}\}\,.$$

The sequence $(X_{L(n)})$ is called the *record value sequence* and $(L(n))$ is the sequence of the *record times*.

It is well-known that for an iid standard exponential sequence (W_i) with record time sequence $(\widetilde{L}(n))$, $(W_{\widetilde{L}(n)})$ constitute the arrivals of a standard homogeneous Poisson process on $[0,\infty)$; see Example 7.2.4.

(a) Let $R(x) = -\log \overline{F}(x)$, where $\overline{F} = 1 - F$ and $x \in (x_l, x_r)$, $x_l = \inf\{x : F(x) > 0\}$ and $x_r = \sup\{x : F(x) < 1\}$. Show that $(X_{L(n)}) \stackrel{d}{=} (R^{\leftarrow}(W_{\widetilde{L}(n)}))$, where $R^{\leftarrow}(t) = \inf\{x \in (x_l, x_r) : R(x) \ge t\}$ is the *generalized inverse of* R.

(b) Conclude from (a) that $(X_{L(n)})$ is the arrival sequence of a Poisson process on (x_l, x_r) with mean measure of $(a,b] \subset (x_l, x_r)$ given by $R(a,b]$.

2.2 The Renewal Process

2.2.1 Basic Properties

In Section 2.1.4 we learned that the homogeneous Poisson process is a particular renewal process. In this section we want to study this model. We start with a formal definition.

Definition 2.2.1 (Renewal process)
Let (W_i) be an iid sequence of a.s. positive random variables. Then the random walk

$$T_0 = 0\,,\quad T_n = W_1 + \cdots + W_n\,,\quad n \ge 1\,,$$

is said to be a renewal sequence *and the counting process*

$$N(t) = \#\{i \ge 1 : T_i \le t\} \quad t \ge 0\,,$$

is the corresponding renewal (counting) process.

We also refer to (T_n) and (W_n) as the sequences of the arrival and inter-arrival times of the renewal process N, respectively.

Example 2.2.2 (Homogeneous Poisson process)
It follows from Theorem 2.1.6 that a homogeneous Poisson process with intensity λ is a renewal process with iid exponential $\mathrm{Exp}(\lambda)$ inter-arrival times W_i. □

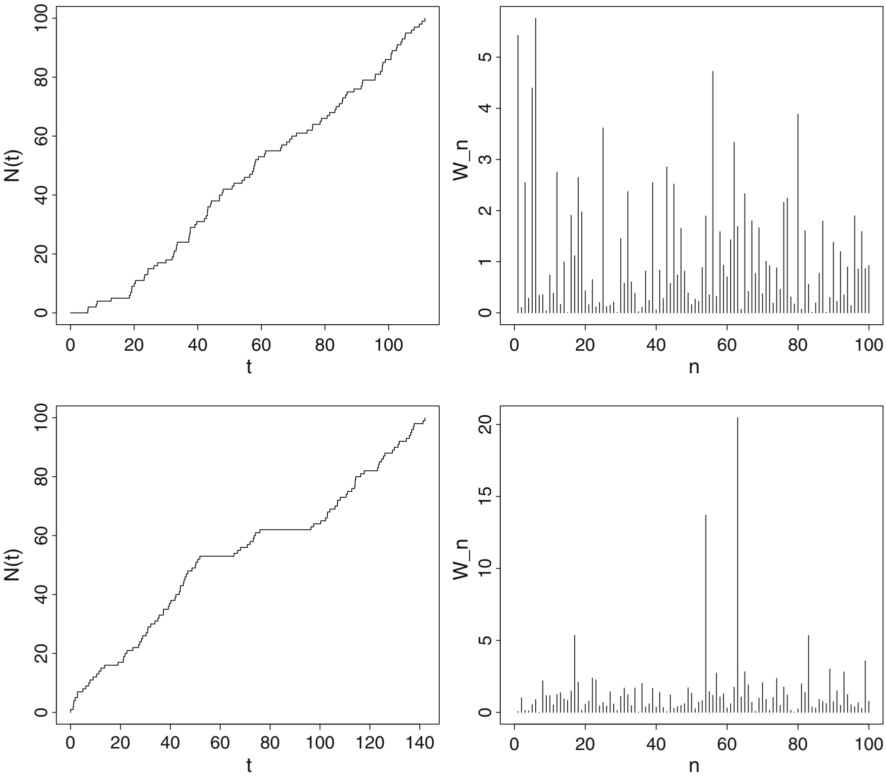

Figure 2.2.3 *One path of a renewal process* (left graphs) *and the corresponding inter-arrival times* (right graphs). Top: *Standard homogeneous Poisson process with iid standard exponential inter-arrival times.* Bottom: *The renewal process has iid Pareto distributed inter-arrival times with* $P(W_i > x) = x^{-4}$, $x \geq 1$. *Both renewal paths have* 100 *jumps. Notice the extreme lengths of some inter-arrival times in the bottom graph; they are atypical for a homogeneous Poisson process.*

A main motivation for introducing the renewal process is that the (homogeneous) Poisson process does not always describe claim arrivals in an adequate way. There can be large gaps between arrivals of claims. For example, it is unlikely that windstorm claims arrive according to a homogeneous Poisson process. They happen now and then, sometimes with years in between. In this case it is more natural to assume that the inter-arrival times have a distribution which allows for modeling these large time intervals. The log-normal or the Pareto distributions would do this job since their tails are much heavier than those of the exponential distribution; see Section 3.2.We have also seen

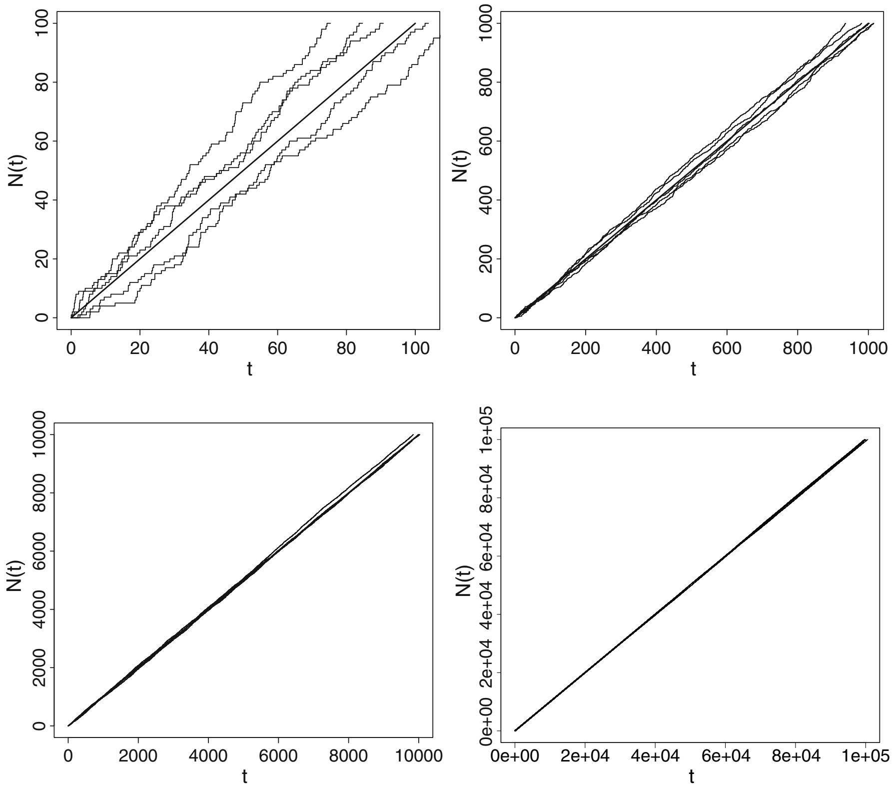

Figure 2.2.4 *Five paths of a renewal process with* $\lambda = 1$ *and* $n = 10^i$ *jumps,* $i = 2, 3, 4, 5$. *The mean value function* $EN(t) = t$ *is also indicated (solid straight line). The approximation of* $N(t)$ *by* $EN(t)$ *for increasing* t *is nicely illustrated; on a large time scale* $N(t)$ *and* $EN(t)$ *can hardly be distinguished.*

in Section 2.1.7 that the Poisson process is not always a realistic model for real-life claim arrivals, in particular if one considers long periods of time.

On the other hand, if we give up the hypothesis of a Poisson process we lose most of the nice properties of this process which are closely related to the exponential distribution of the W_i's. For example, it is in general unknown which distribution $N(t)$ has and what the exact values of $EN(t)$ or $\text{var}(N(t))$ are. We will, however, see that the renewal processes and the homogeneous Poisson process have various *asymptotic* properties in common.

The first result of this kind is a strong law of large numbers for the renewal counting process.

Theorem 2.2.5 (Strong law of large numbers for the renewal process)
If the expectation $EW_1 = \lambda^{-1}$ of the inter-arrival times W_i is finite, N satisfies the strong law of large numbers:

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \quad \text{a.s.}$$

Proof. We need a simple auxiliary result.

Lemma 2.2.6 *Let (Z_n) be a sequence of random variables such that $Z_n \stackrel{\text{a.s.}}{\to} Z$ as $n \to \infty$ for some random variable Z, and let $(M(t))_{t\ge 0}$ be a stochastic process of integer-valued random variables such that $M(t) \stackrel{\text{a.s.}}{\to} \infty$ as $t \to \infty$. If M and (Z_n) are defined on the same probability space Ω, then*

$$Z_{M(t)} \to Z \quad \text{a.s.} \qquad \text{as } t \to \infty.$$

Proof. Write

$$\Omega_1 = \{\omega \in \Omega : \, M(t,\omega) \to \infty\} \quad \text{and} \quad \Omega_2 = \{\omega \in \Omega : \, Z_n(\omega) \to Z(\omega)\}\,.$$

By assumption, $P(\Omega_1) = P(\Omega_2) = 1$, hence $P(\Omega_1 \cap \Omega_2) = 1$ and therefore

$$P(\{\omega : Z_{M(t,\omega)}(\omega) \to Z(\omega)\}) \ge P(\Omega_1 \cap \Omega_2) = 1\,.$$

This proves the lemma. □

Recall the following basic relation of a renewal process:

$$\{N(t) = n\} = \{T_n \le t < T_{n+1}\}\,, \quad n \in \mathbb{N}_0\,.$$

Then it is immediate that the following sandwich inequalities hold:

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{T_{N(t)+1}}{N(t)+1} \, \frac{N(t)+1}{N(t)}\,. \tag{2.2.31}$$

By the strong law of large numbers for the iid sequence (W_n) we have

$$n^{-1}\, T_n \stackrel{\text{a.s.}}{\to} \lambda^{-1}\,.$$

In particular, $N(t) \to \infty$ a.s. as $t \to \infty$. Now apply Lemma 2.2.6 with $Z_n = T_n/n$ and $M = N$ to obtain

$$\frac{T_{N(t)}}{N(t)} \stackrel{\text{a.s.}}{\to} \lambda^{-1}\,. \tag{2.2.32}$$

The statement of the theorem follows by a combination of (2.2.31) and (2.2.32). □

In the case of a homogeneous Poisson process we know the exact value of the expected renewal process: $EN(t) = \lambda\, t$. In the case of a general renewal process N the strong law of large numbers $N(t)/t \stackrel{\text{a.s.}}{\to} \lambda = (EW_1)^{-1}$ suggests that the expectation $EN(t)$ of the renewal process is approximately of the order $\lambda\, t$. A lower bound for $EN(t)/t$ is easily achieved. By an application of Fatou's lemma (see for example Williams [145])) and the strong law of large numbers for $N(t)$,

$$\lambda = E \liminf_{t\to\infty} \frac{N(t)}{t} \le \liminf_{t\to\infty} \frac{EN(t)}{t}\,. \tag{2.2.33}$$

This lower bound can be complemented by the corresponding upper one which leads to the following standard result.

Theorem 2.2.7 (Elementary renewal theorem)
If the expectation $EW_1 = \lambda^{-1}$ of the inter-arrival times is finite, the following relation holds:

$$\lim_{t\to\infty} \frac{EN(t)}{t} = \lambda\,.$$

Proof. By virtue of (2.2.33) it remains to prove that

$$\limsup_{t\to\infty} \frac{EN(t)}{t} \le \lambda\,. \tag{2.2.34}$$

We use a truncation argument which we borrow from Resnick [123], p. 191. Write for any $b > 0$,

$$W_i^{(b)} = \min(W_i, b)\,, \quad T_i^{(b)} = W_1^{(b)} + \cdots + W_i^{(b)}\,, \quad i \ge 1\,.$$

Obviously, $(T_n^{(b)})$ is a renewal sequence and $T_n \ge T_n^{(b)}$ which implies $N_b(t) \ge N(t)$ for the corresponding renewal process

$$N_b(t) = \#\{i \ge 1 : T_i^{(b)} \le t\}\,, \quad t \ge 0\,.$$

Hence

$$\limsup_{t\to\infty} \frac{EN(t)}{t} \le \limsup_{t\to\infty} \frac{EN_b(t)}{t}\,. \tag{2.2.35}$$

We observe that, by definition of N_b,

$$T^{(b)}_{N_b(t)} = W_1^{(b)} + \cdots + W^{(b)}_{N_b(t)} \le t\,.$$

The following result is due to the fact that $N_b(t)+1$ is a so-called *stopping time*[22] *with respect to the natural filtration* generated by the sequence $(W_i^{(b)})$.

[22] Let $\mathcal{F}_n = \sigma(W_i^{(b)}, i \le n)$ be the σ-field generated by $W_1^{(b)}, \ldots, W_n^{(b)}$. Then $(\mathcal{F}_n)$ is the natural filtration generated by the sequence $(W_n^{(b)})$. An integer-valued random variable τ is a stopping time with respect to $(\mathcal{F}_n)$ if $\{\tau = n\} \in \mathcal{F}_n$. If $E\tau < \infty$ Wald's identity yields $E\left(\sum_{i=1}^{\tau} W_i^{(b)}\right) = E\tau\, EW_1^{(b)}$. Notice that $\{N_b(t) = n\} = \{T_n^{(b)} \le t < T_{n+1}^{(b)}\}$. Hence $N_b(t)$ is not a stopping time. However, the same argument shows that $N_b(t)+1$ is a stopping time with respect to $(\mathcal{F}_n)$. The interested reader is referred to Williams's textbook [145] which gives a concise introduction to discrete-time martingales, filtrations and stopping times.

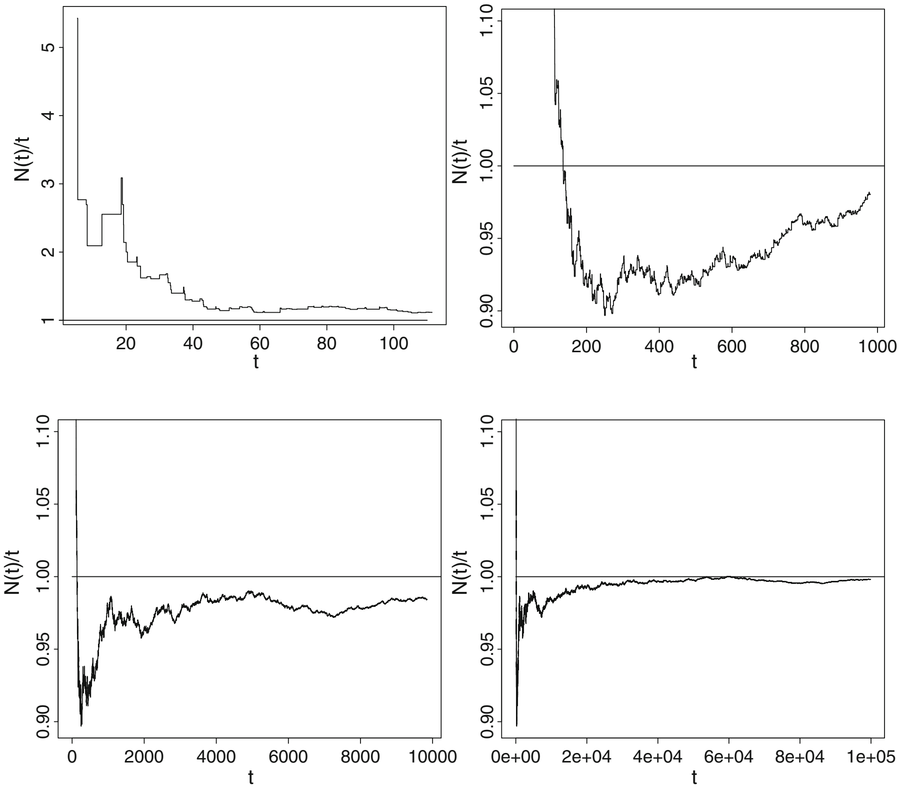

Figure 2.2.8 *The ratio $N(t)/t$ for a renewal process with $n = 10^i$ jumps, $i = 2,3,4,5$, and $\lambda = 1$. The strong law of large numbers forces $N(t)/t$ towards* 1 *for large t.*

Then the relation

$$E(T^{(b)}_{N_b(t)+1}) = E(N_b(t)+1)\, EW_1^{(b)} \tag{2.2.36}$$

holds by virtue of *Wald's identity.* Combining (2.2.35)-(2.2.36), we conclude that

$$\limsup_{t\to\infty} \frac{EN(t)}{t} \le \limsup_{t\to\infty} \frac{E(T^{(b)}_{N_b(t)+1})}{t\, EW_1^{(b)}} \le \limsup_{t\to\infty} \frac{t+b}{t\, EW_1^{(b)}} = (EW_1^{(b)})^{-1}\,.$$

Since by the monotone convergence theorem (see for example Williams [145]), letting $b \uparrow \infty$,

$$EW_1^{(b)} = E(\min(b, W_1)) \uparrow EW_1 = \lambda^{-1}\,,$$

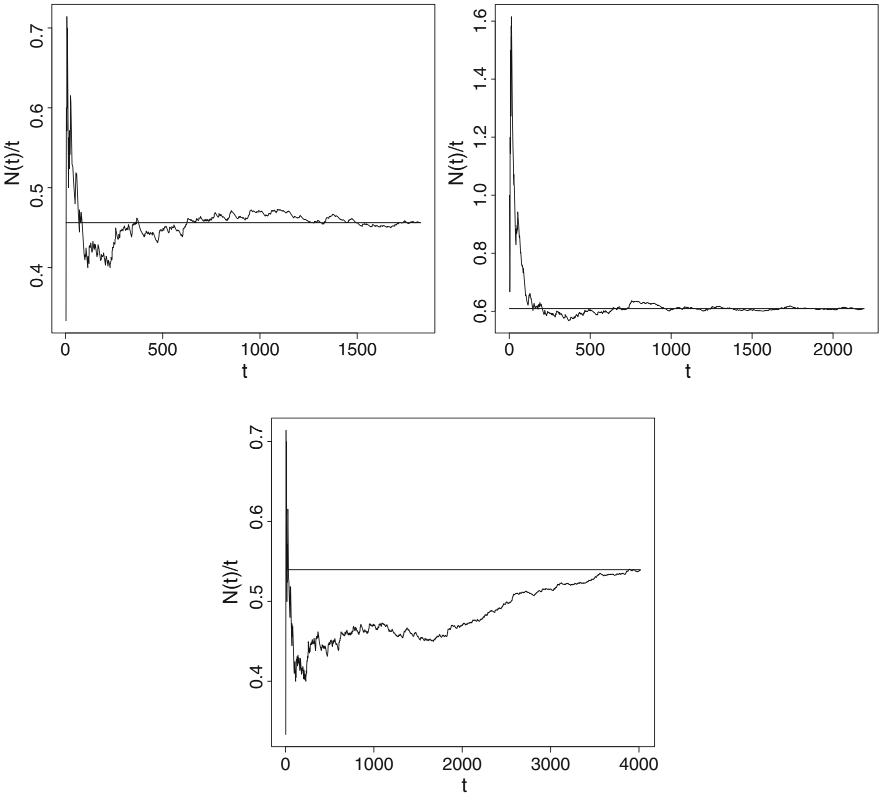

Figure 2.2.9 *Visualization of the validity of the strong law of large numbers for the arrivals of the Danish fire insurance data* $1980-1990$*; see Section 2.1.7 for a description of the data.* Top left: *The ratio* $N(t)/t$ *for* $1980-1984$*, where* $N(t)$ *is the claim number at day* t *in this period. The values cluster around the value* 0.46 *which is indicated by the constant line.* Top right: *The ratio* $N(t)/t$ *for* $1985-1990$*, where* $N(t)$ *is the claim number at day* t *in this period. The values cluster around the value* 0.61 *which is indicated by the constant line.* Bottom: *The ratio* $N(t)/t$ *for the whole period* $1980-1990$*, where* $N(t)$ *is the claim number at day* t *in this period. The graph gives evidence about the fact that the strong law of large numbers does not apply to* N *for the whole period. This is caused by an increase of the annual intensity in* $1985-1990$ *which can be observed in Figure 2.1.20. This fact makes the assumption of iid inter-arrival times over the whole period of* 11 *years questionable. We do, however, see in the top graphs that the strong law of large numbers works satisfactorily in the two distinct periods.*

the desired relation (2.2.34) follows. This concludes the proof. □

For further reference we include a result about the asymptotic behavior of $\mathrm{var}(N(t))$. The proof can be found in Gut [65], Theorem 5.2.

Proposition 2.2.10 (The asymptotic behavior of the variance of the renewal process)
Assume $\mathrm{var}(W_1) < \infty$. *Then*

$$\lim_{t\to\infty} \frac{\mathrm{var}(N(t))}{t} = \frac{\mathrm{var}(W_1)}{(EW_1)^3}\,.$$

Finally, we mention that $N(t)$ satisfies the central limit theorem; see Embrechts et al. [46], Theorem 2.5.13, for a proof.

Theorem 2.2.11 (The central limit theorem for the renewal process)
Assume that $\mathrm{var}(W_1) < \infty$. *Then the central limit theorem*

$$\left(\mathrm{var}(W_1)\,(EW_1)^{-3}\,t\right)^{-1/2}\,(N(t) - \lambda\,t) \stackrel{d}{\to} Y \sim \mathrm{N}(0,1)\,. \qquad (2.2.37)$$

holds as $t \to \infty$.

By virtue of Proposition 2.2.10, the normalizing constants $\sqrt{\mathrm{var}(W_1)(EW_1)^{-3}t}$ in (2.2.37) can be replaced by the standard deviation $\sqrt{\mathrm{var}(N(t))}$.

2.2.2 An Informal Discussion of Renewal Theory

Renewal processes model occurrences of events happening at random instants of time, where the inter-arrival times are approximately iid. In the context of non-life insurance these instants were interpreted as the arrival times of claims. Renewal processes play a major role in applied probability. Complex stochastic systems can often be described by one or several renewal processes as building blocks. For example, the Internet can be understood as the superposition of a huge number of ON/OFF processes. Each of these processes corresponds to one "source" (computer) which communicates with other sources. ON refers to an active period of the source, OFF to a period of silence. The ON/OFF periods of each source constitute two sequences of iid positive random variables, both defining renewal processes.[23] A renewal process is also defined by the sequence of renewals (times of replacement) of a technical device or tool, say the light bulbs in a lamp or the fuel in a nuclear power station. From these elementary applications the process gained its name.

Because of their theoretical importance renewal processes are among the best studied processes in applied probability theory. The object of main interest in renewal theory is the *renewal function*[24]

[23] The approach to tele-traffic via superpositions of ON/OFF processes became popular in the 1990s; see Willinger et al. [146].

[24] The addition of one unit to the mean $EN(t)$ refers to the fact that $T_0 = 0$ is often considered as the first renewal time. This definition often leads to more elegant

$$m(t) = EN(t) + 1\,, \quad t \geq 0\,.$$

It describes the average behavior of the renewal counting process. In the insurance context, this is the expected number of claim arrivals in a portfolio. This number certainly plays an important role in the insurance business and its theoretical understanding is therefore essential. The iid assumption of the inter-arrival times is perhaps not the most realistic but is convenient for building up a theory.

The *elementary renewal theorem* (Theorem 2.2.7) is a simple but not very precise result about the average behavior of renewals: $m(t) = \lambda\, t\,(1 + o(1))$ as $t \to \infty$, provided $EW_1 = \lambda^{-1} < \infty$. Much more precise information is gained by *Blackwell's renewal theorem*. It says that for $h > 0$,

$$m(t, t+h] = EN(t, t+h] \to \lambda\, h\,, \quad t \to \infty\,.$$

(For Blackwell's renewal theorem and the further statements of this section we assume that the inter-arrival times W_i have a density.) Thus, for sufficiently large t, the expected number of renewals in the interval $(t, t+h]$ becomes independent of t and is proportional to the length of the interval. Since m is a non-decreasing function on $[0,\infty)$ it defines a measure m (we use the same symbol for convenience) on the Borel σ-field of $[0,\infty)$, the so-called *renewal measure*.

A special calculus has been developed for integrals with respect to the renewal measure. In this context, the crucial condition on the integrands is called *direct Riemann integrability*. Directly Riemann integrable functions on $[0,\infty)$ constitute quite a sophisticated class of integrands; it includes Riemann integrable functions on $[0,\infty)$ which have compact support (the function vanishes outside a certain finite interval) or which are non-increasing and non-negative. The *key renewal theorem* states that for a directly Riemann integrable function f,

$$\int_0^t f(t-s)\, dm(s) \to \lambda \int_0^\infty f(s)\, ds\,. \tag{2.2.38}$$

Under general conditions, it is equivalent to Blackwell's renewal theorem which, in a sense, is a special case of (2.2.38) for indicator functions $f(x) = I_{(0,h]}(x)$ with $h > 0$ and for $t > h$:

$$\int_0^t f(t-s)\, dm(s) = \int_{t-h}^t I_{(0,h]}(t-s)\, dm(s) = m(t-h, t]$$
$$\to \lambda \int_0^\infty f(s)\, ds = \lambda\, h\,.$$

---

theoretical formulations. Alternatively, we have learned on p. 57 that the process $N(t) + 1$ has the desirable theoretical property of a stopping time, which $N(t)$ does not have.

An important part of renewal theory is devoted to the *renewal equation.* It is a convolution equation of the form

$$U(t) = u(t) + \int_0^t U(t-y)\, dF_{T_1}(y)\,, \tag{2.2.39}$$

where all functions are defined on $[0,\infty)$. The function U is *unknown*, u is a *known* function and F_{T_1} is the distribution function of the iid positive inter-arrival times $W_i = T_i - T_{i-1}$. The main goal is to find a solution U to (2.2.39). It is provided by the following general result which can be found in Resnick [123], p. 202.

Theorem 2.2.12 (W. Smith's key renewal theorem)

(1) *If u is bounded on every finite interval then*

$$U(t) = \int_0^t u(t-s)\, dm(s)\,, \quad t \ge 0\,, \tag{2.2.40}$$

is the unique solution of the renewal equation (2.2.39) *in the class of all functions on* $(0,\infty)$ *which are bounded on finite intervals. Here the right-hand integral has to be interpreted as* $\int_{(-\infty,t]} u(t-s)\, dm(s)$ *with the convention that* $m(s) = u(s) = 0$ *for* $s < 0$*.*

(2) *If, in addition, u is directly Riemann integrable, then*

$$\lim_{t\to\infty} U(t) = \lambda \int_0^\infty u(s)\, ds\,.$$

Part (2) of the theorem is immediate from Blackwell's renewal theorem.

The renewal function itself satisfies the renewal equation with $u = I_{[0,\infty)}$. From this fact the general equation (2.2.39) gained its name.

Example 2.2.13 (The renewal function satisfies the renewal equation)
Observe that for $t \ge 0$,

$$\begin{aligned} m(t) = EN(t) + 1 = 1 + E\left(\sum_{n=1}^\infty I_{[0,t]}(T_n)\right) &= 1 + \sum_{n=1}^\infty P(T_n \le t) \\ &= I_{[0,\infty)}(t) + \sum_{n=1}^\infty \int_0^t P(y + (T_n - T_1) \le t)\, dF_{T_1}(y) \\ &= I_{[0,\infty)}(t) + \int_0^t \sum_{n=1}^\infty P(T_{n-1} \le t-y)\, dF_{T_1}(y) \\ &= I_{[0,\infty)}(t) + \int_0^t m(t-y)\, dF_{T_1}(y)\,. \end{aligned}$$

This is a renewal equation with $U(t) = m(t)$ and $u(t) = I_{[0,\infty)}(t)$. □

The usefulness of the renewal equation is illustrated in the following example.

Example 2.2.14 (Recurrence times of a renewal process)
In our presentation we closely follow Section 3.5 in Resnick [123]. Consider a renewal sequence (T_n) with $T_0 = 0$ and $W_n > 0$ a.s. Recall that

$$\{N(t) = n\} = \{T_n \le t < T_{n+1}\}\,.$$

In particular, $T_{N(t)} \le t < T_{N(t)+1}$. For $t \ge 0$, the quantities

$$F(t) = T_{N(t)+1} - t \quad \text{and} \quad B(t) = t - T_{N(t)}$$

are the *forward* and *backward recurrence times* of the renewal process, respectively. For obvious reasons, $F(t)$ is also called the *excess life* or *residual life*, i.e., it is the time until the next renewal, and $B(t)$ is called the *age process*. In an insurance context, $F(t)$ is the time until the next claim arrives, and $B(t)$ is the time which has evolved since the last claim arrived.

It is our aim to show that the function $P(B(t) \le x)$ for fixed $0 \le x < t$ satisfies a renewal equation. It suffices to consider the values $x < t$ since $B(t) \le t$ a.s., hence $P(B(t) \le x) = 1$ for $x \ge t$. We start with the identity

$$P(B(t) \le x) = P(B(t) \le x\,, T_1 \le t) + P(B(t) \le x\,, T_1 > t)\,, \quad x > 0\,. \tag{2.2.41}$$

If $T_1 > t$, no jump has occurred by time t, hence $N(t) = 0$ and therefore $B(t) = t$. We conclude that

$$P(B(t) \le x\,, T_1 > t) = (1 - F_{T_1}(t))\, I_{[0,x]}(t)\,. \tag{2.2.42}$$

For $T_1 \le t$, we want to show the following result:

$$P(B(t) \le x\,, T_1 \le t) = \int_0^t P(B(t-y) \le x)\, dF_{T_1}(y)\,. \tag{2.2.43}$$

This means that, on the event $\{T_1 \le t\}$, the process B "starts from scratch" at T_1. We make this precise by exploiting a "typical renewal argument". First observe that

$$\begin{aligned} P(B(t) \le x\,, T_1 \le t) &= P(t - T_{N(t)} \le x\,, N(t) \ge 1) \\ &= \sum_{n=1}^{\infty} P(t - T_{N(t)} \le x\,, N(t) = n) \\ &= \sum_{n=1}^{\infty} P(t - T_n \le x\,, T_n \le t < T_{n+1})\,. \end{aligned}$$

We study the summands individually by conditioning on $\{T_1 = y\}$ for $y \le t$:

$$
\begin{aligned}
&P(t-T_n \le x\,, T_n \le t < T_{n+1} \mid T_1 = y) \\
&= P\left(t - \left[y + \sum_{i=2}^{n} W_i\right] \le x\,, y + \sum_{i=2}^{n} W_i \le t < y + \sum_{i=2}^{n+1} W_i\right) \\
&= P\left(t - y - T_{n-1} \le x\,, T_{n-1} \le t - y \le T_n\right) \\
&= P\left(t - y - T_{N(t-y)} \le x\,, N(t-y) = n-1\right)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&P(B(t) \le x\,, T_1 \le t) \\
&= \sum_{n=0}^{\infty} \int_0^t P\left(t - y - T_{N(t-y)} \le x\,, N(t-y) = n\right)\, dF_{T_1}(y) \\
&= \int_0^t P(B(t-y) \le x)\, dF_{T_1}(y)\,,
\end{aligned}
$$

which is the desired relation (2.2.43). Combining (2.2.41)-(2.2.43), we arrive at

$$
P(B(t) \le x) = (1 - F_{T_1}(t))\, I_{[0,x]}(t) + \int_0^t P(B(t-y) \le x)\, dF_{T_1}(y)\,. \tag{2.2.44}
$$

This is a renewal equation of the form (2.2.39) with $u(t) = (1 - F_{T_1}(t))\, I_{[0,x]}(t)$, and $U(t) = P(B(t) \le x)$ is the unknown function.

A similar renewal equation can be given for $P(F(t) > x)$:

$$
P(F(t) > x) = \int_0^t P(F(t-y) > x)\, dF_{T_1}(y) + (1 - F_{T_1}(t+x))\,. \tag{2.2.45}
$$

We mentioned before, see (2.2.40), that the unique solution to the renewal equation (2.2.44) is given by

$$
U(t) = P(B(t) \le x) = \int_0^t (1 - F_{T_1}(t-y))\, I_{[0,x]}(t-y)\, dm(y)\,. \tag{2.2.46}
$$

Now consider a homogeneous Poisson process with intensity λ. In this case, $m(t) = EN(t) + 1 = \lambda t + 1$, $1 - F_{T_1}(x) = \exp\{-\lambda x\}$. From (2.2.46) for $x < t$ and since $B(t) \le t$ a.s. we obtain

$$
P(B(t) \le x) = P(t - T_{N(t)} \le x) = \begin{cases} 1 - \mathrm{e}^{-\lambda x} & \text{if } x < t\,, \\ 1 & \text{if } x \ge t\,. \end{cases}
$$

A similar argument yields for $F(t)$,

$$P(F(t) \leq x) = P(T_{N(t)+1} - t \leq x) = 1 - \mathrm{e}^{-\lambda x}, \quad x > 0.$$

The latter result is counterintuitive in a sense since, on the one hand, the inter-arrival times W_i are $\mathrm{Exp}(\lambda)$ distributed and, on the other hand, the time $T_{N(t)+1} - t$ until the next renewal has the same distribution. This reflects the *forgetfulness property* of the exponential distribution of the inter-arrival times. We refer to Example 2.1.7 for further discussions and a derivation of the distributions of $B(t)$ and $F(t)$ for the homogeneous Poisson process by elementary means. □

Comments

Renewal theory constitutes an important part of applied probability theory. Resnick [123] gives an entertaining introduction with various applications, among others, to problems of insurance mathematics. The advanced text on stochastic processes in insurance mathematics by Rolski et al. [127] makes extensive use of renewal techniques. Gut's book [65] is a collection of various useful limit results related to renewal theory and stopped random walks.

The notion of direct Riemann integrability has been discussed in various books; see Alsmeyer [2], p. 69, Asmussen [6], Feller [51], pp. 361-362, or Resnick [123], Section 3.10.1.

Smith's key renewal theorem will also be key to the asymptotic results on the ruin probability in the Cramér-Lundberg model in Section 4.2.2.

Exercises

(1) Let (T_i) be a renewal sequence with $T_0 = 0$, $T_n = W_1 + \cdots + W_n$, where (W_i) is an iid sequence of non-negative random variables.
   (a) Which assumption is needed to ensure that the renewal process $N(t) = \#\{i \geq 1 : T_i \leq t\}$ has no jump sizes greater than 1 with positive probability?
   (b) Can it happen that (T_i) has a limit point with positive probability? This would mean that $N(t) = \infty$ at some finite time t.

(2) Let N be a homogeneous Poisson process on $[0, \infty)$ with intensity $\lambda > 0$.
   (a) Show that $N(t)$ satisfies the central limit theorem as $t \to \infty$ i.e.,

$$\widehat{N}(t) = \frac{N(t) - \lambda t}{\sqrt{\lambda t}} \xrightarrow{d} Y \sim \mathrm{N}(0,1),$$

   (i) by using characteristic functions,
   (ii) by employing the known central limit theorem for the sequence $((N(n) - \lambda n)/\sqrt{\lambda n})_{n=1,2,\ldots}$, and then by proving that

$$\max_{t \in (n, n+1]} (N(t) - N(n))/\sqrt{n} \xrightarrow{P} 0.$$

(b) Show that N satisfies the multivariate central limit theorem for any $0 < s_1 < \cdots < s_n$ as $t \to \infty$:

$$(\sqrt{\lambda t})^{-1} \, (N(s_1 t) - s_1 \lambda t, \ldots, N(s_n t) - s_n \lambda t) \stackrel{d}{\to} \mathbf{Y} \sim \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}),$$

where the right-hand distribution is multivariate normal with mean vector zero and covariance matrix $\mathbf{\Sigma}$ whose entries satisfy $\sigma_{i,j} = \min(s_i, s_j)$, $i, j = 1, \ldots, n$.

(3) Let $F(t) = T_{N(t)+1} - t$ be the forward recurrence time from Example 2.2.14.
 (a) Show that the probability $P(F(t) > x)$, considered as a function of t, for $x > 0$ fixed satisfies the renewal equation (2.2.45).
 (b) Solve (2.2.45) in the case of iid $\mathrm{Exp}(\lambda)$ inter-arrival times.

2.3 The Mixed Poisson Process

In Section 2.1.3 we learned that an inhomogeneous Poisson process N with mean value function μ can be derived from a standard homogeneous Poisson process $\widetilde{N}$ by a deterministic time change. Indeed, the process

$$\widetilde{N}(\mu(t)), \quad t \geq 0,$$

has the same finite-dimensional distributions as N and is càdlàg, hence it is a possible representation of the process N. In what follows, we will use a similar construction by *randomizing the mean value function.*

Definition 2.3.1 (Mixed Poisson process)
Let $\widetilde{N}$ be a standard homogeneous Poisson process and μ be the mean value function of a Poisson process on $[0, \infty)$. Let $\theta > 0$ a.s. be a (non-degenerate) random variable independent of $\widetilde{N}$. Then the process

$$N(t) = \widetilde{N}(\theta \, \mu(t)), \quad t \geq 0,$$

is said to be a mixed Poisson process *with* mixing variable θ.

Example 2.3.2 (The negative binomial process as mixed Poisson process)
One of the important representatives of mixed Poisson processes is obtained by choosing $\mu(t) = t$ and θ gamma distributed. First recall that a $\Gamma(\gamma, \beta)$ distributed random variable θ has density

$$f_\theta(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} \, x^{\gamma-1} \, \mathrm{e}^{-\beta x}, \quad x > 0. \tag{2.3.47}$$

Also recall that an integer-valued random variable Z is said to be negative binomially distributed with parameter (p, v) if it has individual probabilities

$$P(Z = k) = \binom{v + k - 1}{k} p^v \, (1 - p)^k, \quad k \in \mathbb{N}_0, \quad p \in (0, 1), \quad v > 0.$$

Verify that $N(t)$ is negative binomial with parameter $(p, v) = (\beta/(t+\beta), \gamma)$. □

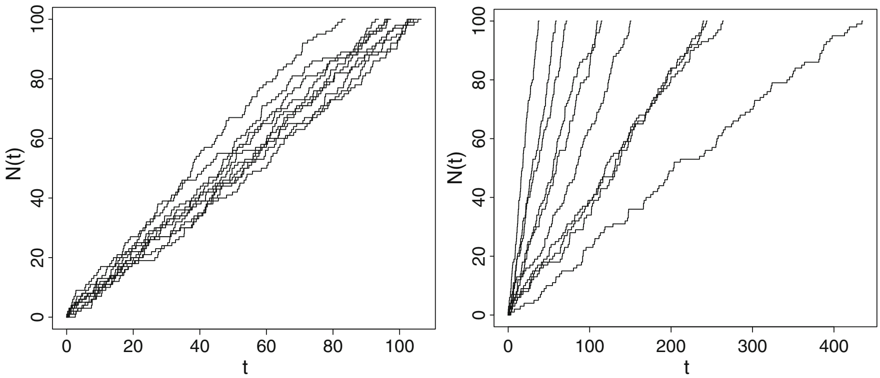

Figure 2.3.3 Left: *Ten sample paths of a standard homogeneous Poisson process.* Right: *Ten sample paths of a mixed homogeneous Poisson process with $\mu(t) = t$. The mixing variable θ is standard exponentially distributed. The processes in the left and right graphs have the same mean value function $EN(t) = t$.*

In an insurance context, a mixed Poisson process is introduced as a claim number process if one does not believe in one particular Poisson process as claim arrival generating process. As a matter of fact, if we observed only one sample path $\widetilde{N}(\theta(\omega)\mu(t), \omega)$ of a mixed Poisson process, we would not be able to distinguish between this kind of process and a Poisson process with mean value function $\theta(\omega)\mu$. However, if we had several such sample paths we should see differences in the variation of the paths; see Figure 2.3.3 for an illustration of this phenomenon.

A mixed Poisson process is a special *Cox process* where the mean value function μ is a general random process with non-decreasing sample paths, independent of the underlying homogeneous Poisson process $\widetilde{N}$. Such processes have proved useful, for example, in medical statistics where every sample path represents the medical history of a particular patient which has his/her "own" mean value function. We can think of such a function as "drawn" from a distribution of mean value functions. Similarly, we can think of θ representing different factors of influence on an insurance portfolio. For example, think of the claim number process of a portfolio of car insurance policies as a collection of individual sample paths corresponding to the different insured persons. The variable $\theta(\omega)$ then represents properties such as the driving skill, the age, the driving experience, the health state, etc., of the individual drivers.

In Figure 2.3.3 we see one striking difference between a mixed Poisson process and a homogeneous Poisson process: the shape and magnitude of the sample paths of the mixed Poisson process vary significantly. This property *cannot* be explained by the mean value function

$$EN(t) = E\widetilde{N}(\theta\,\mu(t)) = E\big(E[\widetilde{N}(\theta\,\mu(t)) \mid \theta]\big) = E[\theta\,\mu(t)] = E\theta\,\mu(t)\,, \quad t \ge 0\,.$$

Thus, if $E\theta = 1$, as in Figure 2.3.3, the mean values of the random variables $\widetilde{N}(\mu(t))$ and $N(t)$ are the same. The differences between a mixed Poisson and a Poisson process with the same mean value function can be seen in the variances. First observe that the Poisson property implies

$$E(N(t) \mid \theta) = \theta\,\mu(t) \quad \text{and} \quad \mathrm{var}(N(t) \mid \theta) = \theta\,\mu(t)\,. \tag{2.3.48}$$

Next we give an auxiliary result whose proof is left as an exercise.

Lemma 2.3.4 *Let A and B be random variables such that* $\mathrm{var}(A) < \infty$. *Then*

$$\mathrm{var}(A) = E[\mathrm{var}(A \mid B)] + \mathrm{var}(E[A \mid B])\,.$$

An application of this formula with $A = N(t) = \widetilde{N}(\theta\mu(t))$ and $B = \theta$ together with (2.3.48) yields

$$\begin{aligned}
\mathrm{var}(N(t)) &= E[\mathrm{var}(N(t) \mid \theta)] + \mathrm{var}(E[N(t) \mid \theta]) \\
&= E[\theta\,\mu(t)] + \mathrm{var}(\theta\,\mu(t)) \\
&= E\theta\,\mu(t) + \mathrm{var}(\theta)\,(\mu(t))^2 \\
&= EN(t)\left(1 + \frac{\mathrm{var}(\theta)}{E\theta}\,\mu(t)\right) \\
&> EN(t)\,,
\end{aligned}$$

where we assumed that $\mathrm{var}(\theta) < \infty$ and $\mu(t) > 0$. The property

$$\mathrm{var}(N(t)) > EN(t) \quad \text{for any } t > 0 \text{ with } \mu(t) > 0 \tag{2.3.49}$$

is called *over-dispersion*. It is one of the major differences between a mixed Poisson process and a Poisson process N, where $EN(t) = \mathrm{var}(N(t))$.

We conclude by summarizing some of the important properties of the mixed Poisson process; some of the proofs are left as exercises.

The mixed Poisson process *inherits* the following properties of the Poisson process:

- It has the *Markov property*; see Section 2.1.2 for some explanation.
- It has the *order statistics property*: if the function μ has a continuous a.e. positive intensity function λ and N has arrival times $0 < T_1 < T_2 < \cdots$, then for every $t > 0$,

  $$(T_1\,, \ldots, T_n \mid N(t) = n) \stackrel{d}{=} (X_{(1)}\,, \ldots, X_{(n)})\,,$$

  where the right-hand side is the ordered sample of the iid random variables $X_1, \ldots, X_n$ with common density $\lambda(x)/\mu(t)$, $0 \le x \le t$; cf. Theorem 2.1.11.

The order statistics property is remarkable insofar that it does not depend on the mixing variable θ. In particular, for a mixed homogeneous Poisson process the conditional distribution of $(T_1, \ldots, T_{N(t)})$ given $\{N(t) = n\}$ is the distribution of the ordered sample of iid $\mathrm{U}(0,t)$ distributed random variables.

The mixed Poisson process *loses* some of the properties of the Poisson process:

- It has *dependent increments.*
- In general, the distribution of $N(t)$ is *not Poisson.*
- It is *over-dispersed*; see (2.3.49).

Comments

For an extensive treatment of mixed Poisson processes and their properties we refer to the monograph by Grandell [61]. It can be shown that the mixed Poisson process and the Poisson process are the only *point processes* on $[0,\infty)$ which have the order statistics property; see Kallenberg [78]; cf. Grandell [61], Theorem 6.6.

Exercises

(1) Consider the mixed Poisson process $(N(t))_{t\geq 0} = (\widetilde{N}(\theta t))_{t\geq 0}$ with arrival times T_i, where $\widetilde{N}$ is a standard homogeneous Poisson process on $[0,\infty)$ and $\theta > 0$ is a non-degenerate mixing variable with $\mathrm{var}(\theta) < \infty$, independent of $\widetilde{N}$.

(a) Show that N does not have independent increments. (An easy way of doing this would be to calculate the covariance of $N(s,t]$ and $N(x,y]$ for disjoint intervals $(s,t]$ and $(x,y]$.)

(b) Show that N has the order statistics property, i.e., given $N(t) = n$, $(T_1, \ldots, T_n)$ has the same distribution as the ordered sample of the iid $\mathrm{U}(0,t)$ distributed random variables $U_1, \ldots, U_n$.

(c) Calculate $P(N(t) = n)$ for $n \in \mathbb{N}_0$. Show that $N(t)$ is not Poisson distributed.

(d) The negative binomial distribution on $\{0,1,2,\ldots\}$ has the individual probabilities

$$p_k = \binom{v+k-1}{k} p^v \, (1-p)^k \,, \quad k \in \mathbb{N}_0 \,, \quad p \in (0,1) \,, \quad v > 0 \,.$$

Consider the mixed Poisson process N with gamma distributed mixing variable, i.e., θ has $\Gamma(\gamma,\beta)$ density

$$f_\theta(x) = \frac{\beta^\gamma}{\Gamma(\gamma)} \, x^{\gamma-1} \, \mathrm{e}^{-\beta x} \,, \quad x > 0 \,.$$

Calculate the probabilities $P(N(t) = k)$ and give some reason why the process N is called *negative binomial process.*

(2) Give an algorithm for simulating the sample paths of an arbitrary mixed Poisson process.

(3) Prove Lemma 2.3.4.

(4) Let $N(t) = \widetilde{N}(\theta\, t)$, $t \ge 0$, be mixed Poisson, where $\widetilde{N}$ is a standard homogeneous Poisson process, independent of the mixing variable θ.

(a) Show that N satisfies the strong law of large numbers with random limit θ:

$$\frac{N(t)}{t} \to \theta \quad \text{a.s.}$$

(b) Show the following "central limit theorem":

$$\frac{N(t) - \theta\, t}{\sqrt{\theta\, t}} \stackrel{d}{\to} Y \sim \mathrm{N}(0,1)\,.$$

(c) Show that the "naive" central limit theorem does not hold by showing that

$$\frac{N(t) - EN(t)}{\sqrt{\mathrm{var}(N(t))}} \stackrel{\text{a.s.}}{\to} \frac{\theta - E\theta}{\sqrt{\mathrm{var}(\theta)}}\,.$$

Here we assume that $\mathrm{var}(\theta) < \infty$.

(5) Let $N(t) = \widetilde{N}(\theta\, t)$, $t \ge 0$, be mixed Poisson, where $\widetilde{N}$ is a standard homogeneous Poisson process, independent of the mixing variable $\theta > 0$. Write F_θ for the distribution function of θ and $\overline{F}_\theta = 1 - F_\theta$ for its right tail. Show that the following relations hold for integer $n \ge 1$,

$$P(N(t) > n) = t \int_0^\infty \frac{(t\,x)^n}{n!}\, \mathrm{e}^{-t\,x}\, \overline{F}_\theta(x)\, dx\,,$$

$$P(\theta \le x \mid N(t) = n) = \frac{\int_0^x y^n\, \mathrm{e}^{-y\,t}\, dF_\theta(y)}{\int_0^\infty y^n\, \mathrm{e}^{-y\,t}\, dF_\theta(y)}\,,$$

$$E(\theta \mid N(t) = n) = \frac{\int_0^\infty y^{n+1} \mathrm{e}^{-y\,t}\, dF_\theta(y)}{\int_0^\infty y^n\, \mathrm{e}^{-y\,t}\, dF_\theta(y)}\,.$$