9

Plastic Limit Analysis of Pressure Vessels

9.1 Introduction

Thin-walled vessels and thick-walled cylinders are applied widely in industry as pressure vessels, pipes, gun barrels, cylinders of rockets, etc. The limit analyses of thick-walled hollow spheres and cylinders under internal pressure were discussed in detail by Hill (1950), and Johnson and Mellor (1962). Further studies on this subject were reported by Derrington and Johnson (1958), Johnson and Mellor (1962), Tuba (1965), and Zyczkowski (1981). The Tresca yield criterion or the Huber-von Mises yield criterion is usually applied for the design of thin-walled pressure vessels. The result using the Huber-von Mises yield criterion for a spherical vessel is similar to that using the Tresca yield criterion. These solutions are applicable only for non-SD materials. It can be seen in the textbook of plasticity.

For SD materials two-parameter failure criteria have to be used (Drucker, 1973; Richmond et al. 1980). The limit pressure of a thick-walled hollow cylinder with material, following the Mohr-Coulomb strength theory, was discussed by Xu and Liu (1995). The limit pressures of the thick-walled hollow sphere and cylinder with material following the twin-shear strength theory were reported by Ni et al. (1998) and Zhuang (1998). Application of the twin-shear strength theory in the strength-calculation of gun barrels was given by Liu et al. (1998) and Li et al. (2007).

The elastic limit and plastic limit of the thin-walled vessel and thick-walled cylinder were studied with respect to the unified strength theory by Wang and Fan (1998), a series of unified solutions of limit loads for pressure vessels were given (Wang and Fan, 1998). Zhao et al. (1999), Feng et al. (2004a; 2004b) and recently Li et al. (2007) also give some results of unified solutions for pressure vessels. The unified strength theory is also applied to the unified limit load solution for fiber-reinforced concrete cylinder taking into consideration the strain softening of material by Chen et al. (2006). The effects of failure criterion on the elastic limit and plastic limit loads of the thin-walled pressure vessel and thick-walled pressure vessel were summarized by Yu (2004).

In most of the applications the thickness of the cylinder is constant and the cylinder is subjected to a uniform internal pressure p. The deformations of the cylinder are symmetrical with respect to the symmetric axis of the cylinder. The deformations at a cross section sufficiently far from the junction of the cylinder and its end caps are independent of the axial coordinate z. In particular, if the cylinder is open-ended (no end caps) and unconstrained, it undergoes axisymmetric deformations due to pressure p which is independent of z. If the deformation of cylinder is constrained by end caps, the displacements and stresses at cylinder cross sections near the end cap junctions differ from those at sections far away from the end cap junctions, if axially symmetrical loads and constraints are considered. Thus the solution is axisymmetrical; the solutions are functions of the radial coordinate r only. In the case of a thin-walled cylinder, the difference in stresses at the inner wall and outer wall is small if the thickness t is much less than the vessel diameter. The internal stresses can then be assumed to be independent of the radial coordinate r. Relationships between the internal pressure p, the dimensions of the thin-walled vessel, circumferential and axial stresses in a pressure vessel, can be found in textbooks of *Mechanics of Materials* or *Strength of Materials*.

The systematic results of elastic and plastic limit loads for thin-walled and thick-walled pressure vessels will be described in this chapter.

9.2 Unified Solution of Limit Pressure of Thin-walled Pressure Vessel

Considering the stresses in a thin-walled pressure vessel subjected to an internal pressure as shown in Fig.9.1, the pressure incurs a circumferential stress (or hoop stress) σ_1 and a longitudinal stress σ_m or σ_2 that can be expressed as

$$\sigma_1 = \frac{pD}{2t}, \ \sigma_2 = \sigma_m = \frac{pD}{4t}, \ \sigma_3 = 0. \tag{9.1}$$

Based on the unified strength theory,

$$F = \sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t, \quad \text{when} \quad \sigma_2 \leqslant \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha}, \tag{9.2a}$$

$$F' = \frac{1}{1+b}(\sigma_1 + b\sigma_2) - \alpha\sigma_3 = \sigma_t, \quad \text{when} \quad \sigma_2 \leqslant \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha}. \tag{9.2b}$$

The stresses of a thin-walled vessel satisfy the inequity condition $\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3) \leqslant \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha}$, therefore the first formula of the unified strength theory

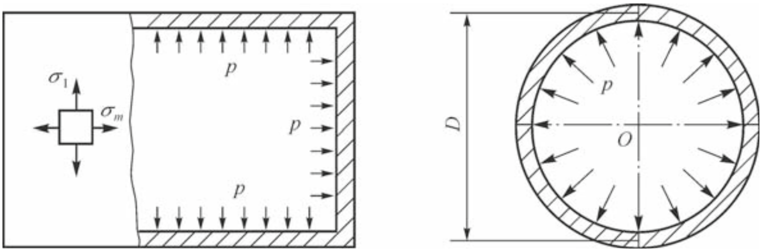

Fig. 9.1. Stresses in thin-walled pressure vessel

Eq.(9.2a) is valid for the yield condition of the vessel. Substituting Eq.(9.1) into Eq.(9.2a), the yield condition for a thin walled cylinder obeying the unified strength theory is obtained

$$F = \sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \frac{pD}{2t} - \frac{\alpha b}{1+b}\frac{pD}{4t} = \sigma_t. \tag{9.3}$$

The elastic limit pressure can be derived as

$$p_e = \frac{1+b}{2+2b-\alpha b}\frac{4t}{D}\sigma_t. \tag{9.4}$$

If the material has an allowable tensile stress of $[\sigma] = \sigma_t/n$, where n is the factor of safety, the allowable limit pressure is

$$[p] = \frac{1+b}{2+2b-\alpha b}\frac{4t}{D}[\sigma]. \tag{9.5}$$

If the internal pressure p and allowable stress $[\sigma]$ are given, the wall thickness should satisfy

$$t \geqslant \frac{2+2b+\alpha b}{1+b}\frac{pD}{4[\sigma]}. \tag{9.6}$$

The relationship between the limit pressure and wall thickness and the unified strength theory parameter b are illustrated in Fig.9.2 and Fig.9.3, respectively.

The unified solution with $b = 0$ and $\alpha \neq 1$ is with respect to the Mohr-Coulomb material and the solution for the Tresca material is a special case of the unified solution with $b = 0$ and $\alpha = 1$. The unified solution with $b = 1$ and $\alpha \neq 1$ corresponds to the generalized twin-shear criterion (Yu, 1983), and the unified solution with $b = \alpha = 1$ is the same as the solution of the twin-shear stress criterion (Yu, 1961) or the maximum deviatoric stress criterion (Haythorthwaite, 1960). Other solutions are new which can be applied to different materials. Therefore the unified solution can be adopted for analysis of structures made of various materials.

It is worth noting that:

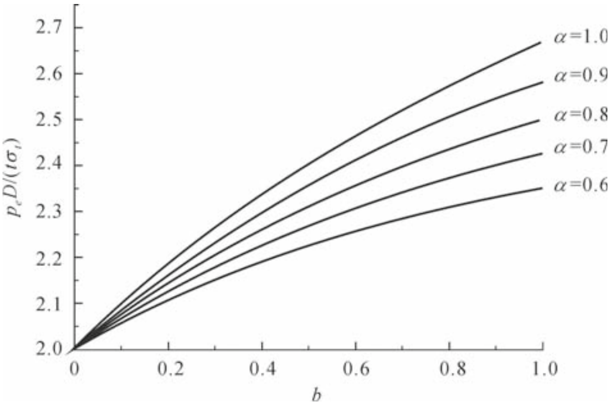

Fig. 9.2. Elastic limit pressure versus the unified strength theory parameter b

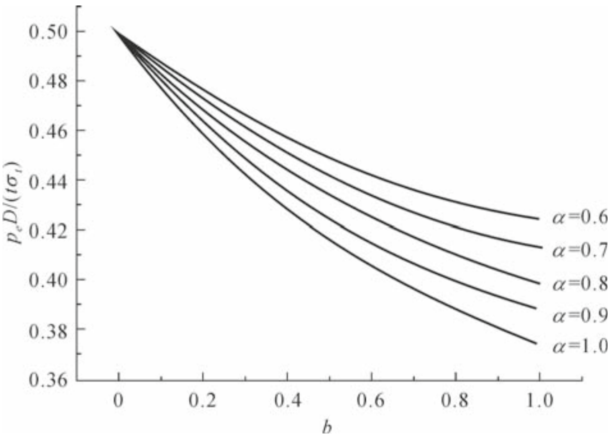

Fig. 9.3. Minimum wall thickness versus the unified strength theory parameter b

(1) The traditional solution is a single solution (with respect to $b = 0$ in the unified strength theory), which can be adopted only for one kind of material. The unified solution, however, gives a serial solution, which can be adopted for various materials and structures.

(2) The solution for Tresca material ($b = 0$ and $\alpha = 1$) is identical to the solution for the Mohr-Coulomb material ($b = 0$ and $\alpha \neq 1$). It means that the SD effect of materials ($\alpha \neq 1$) cannot be considered by the Mohr-Coulomb strength theory in this case.

(3) All the solutions of the bearing capacity of structures with $b > 0$ are higher than the solution of the traditional Tresca or Mohr-Coulomb criterion. All the solutions of the required wall thickness of pressure vessels with $b > 0$ are lower than the solution of the traditional Tresca or Mohr-Coulomb criterion.

(4) The application of the unified strength theory and the unified solutions are more economical in terms of the effective use of materials and energy.

Example 9.1 Design of Space Shuttle

The satellite carrier rockets launched by the United States, Europe and China have very large diameters, for example the diameter of a satellite carrier rocket is 3.5 m with a pressurized body length of 15 m. Under an internal pressure p, the stresses of the rocket wall are

$$\sigma_1 = \frac{pD}{2t}, \ \sigma_2 = \frac{pD}{4t}, \ \sigma_3 = 0.$$

The stresses of a thin-walled vessel satisfy the inequity condition,

$$\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3) \leqslant \frac{\sigma_1 + \sigma_3}{2}.$$

Therefore any one of the two expressions of the unified yield criterion Eq.(9.2a) or Eq.(9.2b) is valid for the yield condition of the vessel. The calculated formulas of thickness of a missile body under inner pressure is

$$t \geqslant \frac{2 + 2b + \alpha b}{1 + b} \frac{pD}{4[\sigma]} \quad \text{(for SD materials)},$$

$$t \geqslant \frac{2 + 3b}{1 + b} \frac{pD}{4[\sigma]} \quad \text{(for non-SD materials)}.$$

From the above results, for a specific allowable stress $[\sigma]$, factor of safety n, internal pressure p and diameter D, the required wall thickness of the pressure vessels depends on the parameter b in the unified yield criterion. When $b = 0$ which corresponds to the single shear criterion, the required thickness is the largest; when $b = 1$ with respect to the twin shear stress criterion, the thickness is the smallest. The difference between the two required thicknesses is 33.3%.

A carrier rocket is a tool to launch a satellite and itself is not the target to be launched and positioned in space. Given the certain capacity of the launching system, a reduction in the rocket's selfweight is beneficial for increasing the satellite weight. Based on Zhang (1998) and Xia (1999), the cost of launching a satellite per ton mass is as high as tens of thousands of US dollars. The transportation cost per unit effective mass is about US$22,000 even if the satellite is positioned on a lower track (Xia, 1999; Zhang, 1998).

The application of new materials is an effective way of reducing the self-weight of the rocket. A new ultra-high-strength Al-Cu-Li 2195 alloy has been successfully used to fabricate large-scale tanks for the space shuttle, and the weight of the tanks decreases by 3405 kg.

On the other hand, using a more accurate analysis method may be an alternative way of making a cost-effective design of the carrier rocket. The current design of the rocket strength is based on the Tresca criterion which is a single shear criterion. If the unified yield criterion for $b = 1$ or $b = 1/2$

is adopted, the required wall thickness of the rocket can be reduced by more than 16.5%. A rough estimation shows that the body weight of the rocket may reduce by 440 kg if the wall thickness is 1 mm thinner. Thus the economic benefit is as high as US$9.2 million if the unified yield criterion with $b = 1$ is used.

9.3 Limit Pressure of Thick-walled Hollow Sphere

If a thick-walled sphere with inner radius r_a and outer radius r_b is subjected to an internal pressure p, as shown in Fig.9.4. The sphere will deform symmetrically about the center; the radial and any two orthogonal tangential directions will be the principal directions.

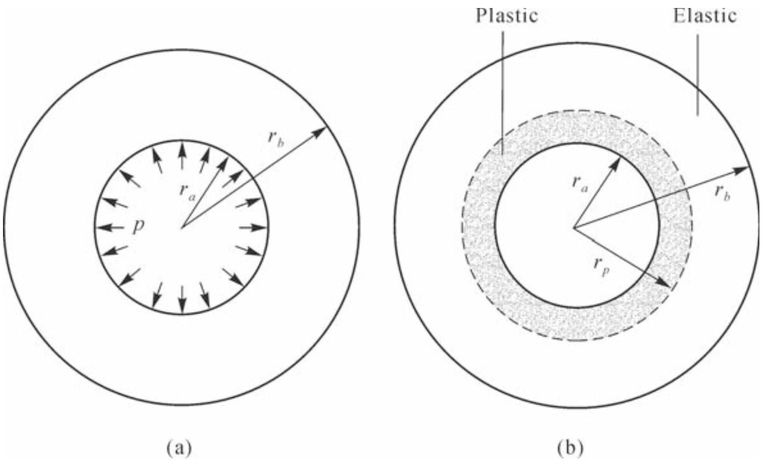

Fig. 9.4. Thick-walled sphere shell

The three corresponding principal strains are ε_r, ε_θ, ε_φ and $\varepsilon_\theta = \varepsilon_\varphi$. The equilibrium equation is

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 2\frac{\sigma_\theta - \sigma_r}{r}. \tag{9.7}$$

The elastic stress-strain relations are

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - 2\nu\sigma_\theta), \\ \\ \varepsilon_\theta = \dfrac{1}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r)]. \end{cases} \tag{9.8}$$

or

$$\begin{cases} \sigma_\theta = \sigma_\varphi = \dfrac{E}{(1+\nu)(1-2\nu)}(\varepsilon_\theta + \nu\varepsilon_r), \\ \\ \sigma_r = \dfrac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + 2\nu\varepsilon_\theta. \end{cases} \tag{9.9}$$

The compatibility equation has the form of

$$\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}r} + \frac{\varepsilon_\theta - \varepsilon_r}{r} = 0. \tag{9.10}$$

9.3.1 Elastic Limit Pressure of Thick-walled Sphere Shell

The Lame solutions of the elastic stress distribution had been given (Johnson and Mellor, 1962) as follows:

$$\sigma_r = \frac{pr_a^3}{r^3}\frac{(r_b^3 - r^3)}{(r_a^3 - r_b^3)}, \tag{9.11}$$

$$\sigma_\theta = \sigma_\varphi = \frac{pr_a^3}{2r^3}\frac{(2r^3 + r_b^3)}{(r_b^3 - r_a^3)}. \tag{9.12}$$

For convenience of formulation, the following dimensionless quantities are introduced:

$$K = \frac{r_b}{r_a}\ ,\ \rho = \frac{r}{r_a}\ ,\ \alpha = \frac{\sigma_t}{\sigma_c}, \tag{9.13}$$

where σ_t is the yield strength in uniaxial tension.

The stress expressions can then be written as

$$\sigma_r = \frac{\rho^3 - K^3}{\rho^3(K^3-1)}p\ ,\ \sigma_\theta = \sigma_\varphi = \frac{2\rho^3 + K^3}{2\rho^3(K^3-1)}p. \tag{9.14}$$

The magnitude of these stresses in elastic range is limited by the yield criterion. When the unified strength theory Eq.(9.2a) and Eq.(9.2b) are used, because $\sigma_\theta = \sigma_\varphi > \sigma_r$, i.e., $\sigma_1 = \sigma_\theta$ (or σ_φ), $\sigma_2 = \sigma_\varphi$ (or σ_θ), $\sigma_3 = \sigma_r$, $\tau_{12} = 0$, $\tau_{13} = \tau_{23}$, there is

$$\sigma_2 \geqslant \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha}.$$

Thus Eq.(9.2b) should be used as the yield condition when the unified strength theory is applied. The inner surface of the spherical shell yields first at the elastic limit pressure. Substituting the stress components at the inner surface into Eq.(9.2b), the elastic limit pressure p_e in terms of the unified strength theory is derived as

$$p_e = \frac{K^3 - 1}{(1-\alpha) + K^3(\frac{1}{2}+\alpha)}\sigma_t. \tag{9.15}$$

The relation of elastic limit pressure with the ratios of the outer radius to the inner radius $K = r_b/r_a$ is shown in Fig. 9.5. It is worth noting that, as K approaches infinity, the elastic limit pressure approaches a specific value. If $\alpha = 1$, this pressure is equal to 2/3 of the yield stress σ_y. When the ratio K is larger than 2, the increment of the limit pressure is the minimum in spite of different α.

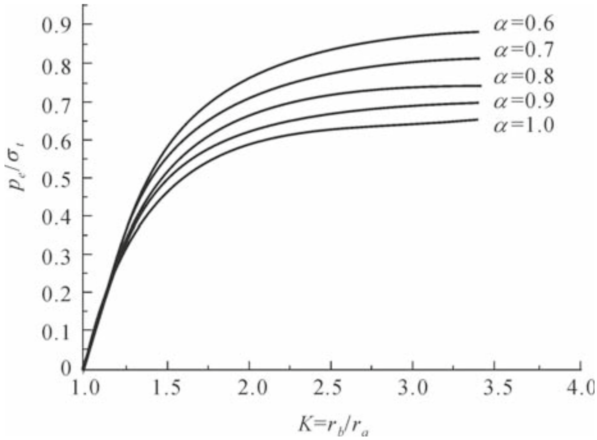

Fig. 9.5. Relation of elastic limit pressure of sphere shell to $K = r_b/r_a$

The elastic limit pressure for non-SD materials are a special case in Eq. (9.15) with α =1,

$$p_e = \frac{2}{3}\frac{K^3 - 1}{K^3}\sigma_y. \tag{9.16}$$

This result is identical with the previous result (Johnson and Mellor, 1962), which is a special case for results obtained based on the unified yield criterion.

The limit pressure of a thick-walled hollow sphere is independent of the strength parameter b. The reason is that the stress state of a spherical shell is spherically symmetrical about the center, and the three principal stresses satisfy $\sigma_1 = \sigma_2 > \sigma_3$. All the limit loci of the unified strength theory with different parameter b intersect each other for this stress state.

9.3.2 Plastic Limit Pressure of Thick-walled Sphere Shell

When the internal pressure p reaches the elastic limit pressure p_e, the inner surface of the hollow sphere shell yields. As the internal pressure increases, the plastic zone spreads towards the outer surface. Denoting the outer radius of the plastic zone as r_p, and assuming the material is perfectly plastic, the failure condition of the unified strength theory for the spherical shell can be simplified as

$$\sigma_\theta - \alpha\sigma_r = \sigma_t. \tag{9.17}$$

The stresses in the plastic region ($r_a \leqslant r \leqslant r_p$) can then be derived from the equilibrium equation (Eq.(9.9)) with application of the boundary condition of $r = r_a$, $\sigma_r = -p$,

$$\begin{cases} \sigma_r^p = \dfrac{\sigma_t}{1-\alpha}\left[1-\left(\dfrac{r_a}{r}\right)^{2(1-\alpha)}\right] - p\left(\dfrac{r_a}{r}\right)^{2(1-\alpha)}, \\ \sigma_\theta^p = \sigma_\varphi^p = \dfrac{\sigma_t}{1-\alpha}\left[1-\alpha\left(\dfrac{r_a}{r}\right)^{2(1-\alpha)}\right] - \alpha p\left(\dfrac{r_a}{r}\right)^{2(1-\alpha)}. \end{cases} \tag{9.18}$$

Since no stress-strain relation is required to derive the stresses, it is a statically determinate problem.

At the plastic zone boundary of $r = r_p$, the radial stress σ_r^p can be calculated by substituting the boundary condition into Eq.(9.18). The elastic part of the sphere is then considered as a new sphere with an inner radius of r_p and an outer radius of r_b with an internal pressure of σ_r^p at $r = r_p$. The stresses in the elastic region ($r_p \leqslant r \leqslant r_b$) can be written as

$$\begin{cases} \sigma_r^e = \dfrac{r_p^3(1-\frac{r_b^3}{r^3})}{r_p^3(1-\alpha)+r_b^3(\frac{1}{2}+\alpha)}\sigma_t, \\ \sigma_\theta^e = \sigma_\varphi^e = \dfrac{r_p^3(1+\frac{r_b^3}{2r^3})}{r_p^3(1-\alpha)+r_b^3(\frac{1}{2}+\alpha)}\sigma_t. \end{cases} \tag{9.19}$$

The pressure at the elastic-plastic boundary and the radius of the plastic zone can be derived from the stress continuous condition at the elasto-plastic boundary,

$$p_{ep} = \left\{-\frac{1}{1-\alpha}\left[\left(\frac{r_p}{r_a}\right)^{2(1-\alpha)} - 1\right] + \frac{(r_b^3 - r_p^3)\left(\frac{r_p}{r_a}\right)^{2(1-\alpha)}}{r_p^3(1-\alpha)+r_b^3(\frac{1}{2}+\alpha)}\right\}\sigma_t. \tag{9.20}$$

When r_p is equal to the outer radius of the sphere r_b, the sphere shell is completely plastic. The plastic limit of the internal pressure p_p can be derived as

$$p_p = \frac{\sigma_t}{1-\alpha}(K^{2(1-\alpha)} - 1). \tag{9.21}$$

Eq.(9.21) is the same as the result based on the twin-shear strength theory (Zhuang, 1998) and the result obtained by using the Mohr-Coulomb strength theory. The relationship of the plastic limit pressure with different ratios of the outer radius to the inner radius $K = r_b/r_a$ is shown in Fig.9.6. The plastic limit pressure increases with the increase of the ratio.

The plastic limit pressure of a thick-walled hollow sphere shell of non-SD materials can be calculated from Eq.(9.21) with α =1,

$$p_p = 2\ln K. \tag{9.22}$$

Eq.(9.22) is the same as the result based on the Tresca criterion (Johnson and Mellor, 1962).

The stresses in the plastic region ($r_p \leqslant r \leqslant r_b$) can be expressed as

$$\sigma_r^p = \frac{\sigma_t}{1-\alpha}\left[1 - \left(\frac{r_b}{r}\right)^{2(1-\alpha)}\right], \tag{9.23}$$

$$\sigma_\theta^p = \sigma_\varphi^p = \frac{\sigma_t}{1-\alpha}\left[1 - \alpha\left(\frac{r_b}{r}\right)^{2(1-\alpha)}\right]. \tag{9.24}$$

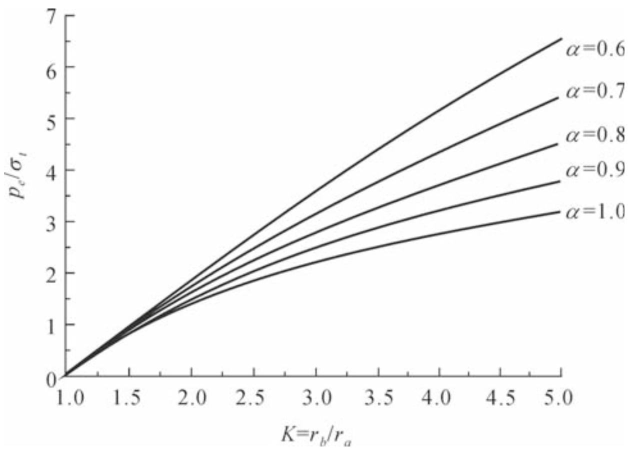

Fig. 9.6. Relation of plastic limit pressure of sphere shell to the ratio of $K = r_b/r_a$

9.4 Unified Solution of Elastic Limit Pressure of Thick-walled Cylinder

Considerable works on elasto-plastic analysis for a thick-walled cylinder under internal pressure have been reported by Turner (1909), Nadai (1931), Manning (1945), Allen and Sopwith (1951), and Crossland and Bones (1958). Discussions in depth on this subject can also be found in the books by Hill (1950), Johnson and Mellor (1962), Mendelson (1968), and Chakrabarty (1987).

The twin-shear yield criterion proposed by Yu (1961; 1983) was applied to study the limit pressure of a thick-walled cylinder by Li (1998), Huang and Zeng (1989). The generalized twin-shear strength theory (Yu et al., 1985) was also used to derive the limit pressure of a thick-walled cylinder and hollow sphere shell by Zhuang (1998), Zhao et al. (1999), and Ni et al. (1998). It was also applied to gun barrels by Liu et al. (1998). The elastic limit pressure, plastic limit pressure, and autofrettage pressure in an autofretted gun barrel were reported by Liu et al. (1998).

Nowadays, gun barrels are made of high-strength steel having different strength in tension and compression. Therefore, the results with respect to the generalized twin-shear strength theory, which takes into account the SD effect of materials, should be more appropriate. The unified yield criterion (Yu and He, 1991) was used to drive the limit pressure for thick-walled tubes with different end conditions, e.g., the open-end condition, the closed-end condition, and the plane strain condition (Wang and Fan, 1998). For pressure-sensitive materials, a failure criterion should take into account the SD effect of material. The unified strength theory takes all the stress components into account and is suitable for both non-SD and SD materials. In this chapter the effects of yield criteria on elastic and plastic limit pressures for thick-walled tubes using the unified strength theory are summarized and discussed.

Considering a thick-walled cylinder with the inner and outer radii of the cylinder r_a and r_b under an internal pressure p and a longitudinal force P, the radius of the cylinder is assumed to be so large that the plane transverse sections remain on the plane during the expansion. It implies that the longitudinal strain ε_z is independent of the radius. Since the stresses and strains in a cross-section that is sufficiently far away from the ends do not vary along the length of the cylinder, the equation of equilibrium can be written as

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = \frac{\sigma_\theta - \sigma_r}{r}. \tag{9.25}$$

It should be mentioned that the z axis of the cylindrical coordinates (r, θ, z) is the longitudinal axis of the tube. Based on the generalized Hooke's law, the longitudinal stress in the elastic state can be written as

$$\sigma_z = E\varepsilon_z + \nu(\sigma_r + \sigma_\theta), \tag{9.26}$$

where E is Young's modulus and ν the Poisson's ratio. The radial strain ε_r and the circumferential strain ε_θ are

$$\varepsilon_r = -\nu\varepsilon_z + \frac{1+\nu}{E}\left[(1-\nu)\sigma_r - \nu\sigma_\theta\right], \tag{9.27a}$$

$$\varepsilon_\theta = -\nu\varepsilon_z + \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right]. \tag{9.27b}$$

The compatibility equation is $\frac{\mathrm{d}}{\mathrm{d}r}(\sigma_r+\sigma_\theta) = 0$, which indicates that $\sigma_r+\sigma_\theta$ have constant values at each stage of the elastic expansion. Integrating Eq. (9.25) and applying the boundary conditions of $\sigma_r = 0$ at $r = r_b$, and $\sigma_r = -p$ at $r = r_a$, the stresses are given as

$$\sigma_r = -p\left(\frac{\frac{r_b^2}{r^2}-1}{\frac{r_b^2}{r_a^2}-1}\right), \quad \sigma_\theta = p\left(\frac{\frac{r_b^2}{r^2}+1}{\frac{r_b^2}{r_a^2}-1}\right). \tag{9.28}$$

This is Lame's solution given by Lame (1852). If the resultant longitudinal load is denoted by P, the axial stress σ_z is $P/\left[\pi(r_b^2 - r_a^2)\right]$ since this stress is constant over the cross section. In particular, $p = 0$ represents an open-end condition and $P = \pi r_a^2 p$ represents a closed-end condition. For a plane strain condition ($\varepsilon_z = 0$), σ_z can be directly derived from Eqs.(9.26) and (9.28),

$$\sigma_z = \frac{p}{K^2-1}, \quad \text{closed end,} \tag{9.29a}$$

$$\sigma_z = 0, \quad \text{open end,} \tag{9.29b}$$

$$\sigma_z = \frac{2\nu p}{K^2-1}. \quad \text{plane strain.} \tag{9.29c}$$

The corresponding axial strains are

$$\varepsilon_z = \frac{(1-2\nu)p}{(K^2-1)E}, \quad \text{closed end,} \tag{9.30a}$$

$$\varepsilon_z = 0, \quad \text{open end,} \tag{9.30b}$$

$$\varepsilon_z = \frac{-2\nu p}{(K^2-1)E}, \quad \text{plane strain.} \tag{9.30c}$$

In all the three cases, σ_z is the intermediate principal stress. For the closed-end condition, σ_z is the average or the mean value of the other two principal stresses. If the material is assumed to be incompressible in both the elastic and plastic ranges, σ_z of the plane strain condition is identical to that of the closed-end condition. There are $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_z$, $\sigma_3 = \sigma_r$, and

$$\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3) \leqslant \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha}. \tag{9.31}$$

Thus the first equation of the unified strength theory should be applied as the yield conditions,

$$\sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t. \tag{9.32}$$

Substituting Eq.(9.31) into Eq.(9.32), the yield condition in the case of thick-walled cylinder with closed-end and plane strain condition can be expressed as

$$\frac{2+(2-\alpha)\alpha}{2(1+b)}\sigma_\theta - \frac{\alpha(2+b)}{2(1+b)}\sigma_r = \sigma_t. \tag{9.33}$$

For on open-end cylinder it is

$$\sigma_\theta - \frac{\alpha}{1+b}\sigma_r = \sigma_t. \quad \text{(open end)} \tag{9.34}$$

Substituting Eq.(9.28) into Eqs.(9.33) and (9.34), we get

$$[2+(2-\alpha)b]\frac{p}{K^2-1}\left(\frac{r_b^2}{r^2}+1\right) + \alpha(2+b)\frac{p}{K^2-1}\left(\frac{r_b^2}{r^2}-1\right) = 2(1+b)\sigma_t. \tag{9.35}$$

The elastic limit pressure in terms of the unified strength theory can be derived as

$$p_e = \frac{(1+b)(K^2-1)\sigma_t}{K^2(1+b+\alpha)+(1+b)(1-\alpha)}, \quad \text{closed end} \tag{9.36}$$

$$p_e = \frac{(1+b)(K^2-1)\sigma_t}{(1+b)(K^2+1)+\alpha(K^2-1)}, \quad \text{open end} \tag{9.37}$$

$$p_e = \frac{(1+b)(K^2-1)\sigma_t}{K^2(1+b+\alpha)+(1+b)(1-\alpha)}. \quad \text{plane strain} \tag{9.38}$$

The limit pressure for the closed-end cylinder based on the Mohr-Coulomb strength theory (single-shear theory) is

$$p_e = \frac{K^2-1}{(1+\alpha)K^2+(1-\alpha)}\sigma_t. \quad \text{(Mohr-Coulomb strength theory)} \tag{9.39}$$

The limit pressure of a thick-walled cylinder in terms of twin-shear strength theory was reported by Zhuang (1998) and Ni et al. (1998) as

$$p_e = \frac{2(K^2-1)}{(2+\alpha)K^2+2(1-\alpha)}\sigma_t. \quad \text{(twin shear strength theory)} \tag{9.40}$$

These limit pressures are specific cases of the solutions in terms of the unified solution with $b=0$ and $b=1$ respectively.

For non-SD materials, i.e., $\alpha=1$ or $\sigma_t=\sigma_c=\sigma_y$, Eqs.(9.36)~(9.38) are simplified as

$$p_e = \frac{(1+b)(K^2-1)}{K^2(2+b)}\sigma_y, \quad \text{closed end} \tag{9.41}$$

$$p_e = \frac{(1+b)(K^2-1)}{K^2(2+b)+b}\sigma_y, \quad \text{open end} \tag{9.42}$$

$$p_e = \frac{(1+b)(K^2-1)}{K^2(2+b)+b(1-2\nu)}\sigma_y. \quad \text{plane strain} \tag{9.43}$$

These results are identical to the solutions with Yu unified yield criterion (Wang and Fan, 1998).

The elastic limit pressure for the Tresca material at closed end, open end, and plane strain conditions can be obtained from Eqs.(9.41)∼(9.43) with $\alpha = 1$, $b = 0$. The solutions for different conditions are identical,

$$p_e = \frac{K^2-1}{2K^2}\sigma_y. \tag{9.44}$$

The elastic limit pressure for the Huber-von Mises material can be approximately derived from the unified solution with $\alpha = 1$, b=1/2,

$$p_e = \frac{3(K^2-1)}{5K^2}\sigma_y, \quad \text{closed end} \tag{9.45}$$

$$p_e = \frac{3(K^2-1)}{5K^2+(1-2\nu)}\sigma_y. \quad \text{plane strain} \tag{9.46}$$

The classical solutions for Huber-von Mises material are

$$p_e = \frac{K^2-1}{\sqrt{3K^2}}\sigma_y, \quad \text{closed end} \tag{9.47}$$

$$p_e = \frac{K^2-1}{\sqrt{3K^4+1}}\sigma_y, \quad \text{open end} \tag{9.48}$$

$$p_e = \frac{K^2-1}{\sqrt{3K^4+(1-2\nu)^2}}\sigma_y. \quad \text{plane strain} \tag{9.49}$$

The percentage difference between the approximated elastic limit pressure with regard to the unified solution with $\alpha = 1$, $b = 1/2$, and the exact solution based on the Huber-von Mises criterion is as low as 0.38%.

The elastic limit pressure in terms of the twin-shear yield criterion can be derived from the unified solution with $\alpha = 1$, $b = 1$,

$$p_e = \frac{2(K^2-1)}{3K^2}\sigma_y, \quad \text{closed end} \tag{9.50}$$

$$p_e = \frac{2(K^2-1)}{3K^2+1}\sigma_y, \quad \text{open end} \tag{9.51}$$

$$p_e = \frac{2(K^2 - 1)}{3K^2 + (1 - 2\nu)}\sigma_y. \quad \text{plane strain} \tag{9.52}$$

The percentage difference of the solutions between the Tresca material and the twin-shear material is as high as 33.4%.

It can be noted from the above derivation that all the solutions with regard to the prevailing yield criteria can be approximated or deduced from the unified solution in view of the unified strength theory with specific values of α and b. The variations of the unified solution regarding different values of α and b are illustrated in Figs.9.7 and 9.8.

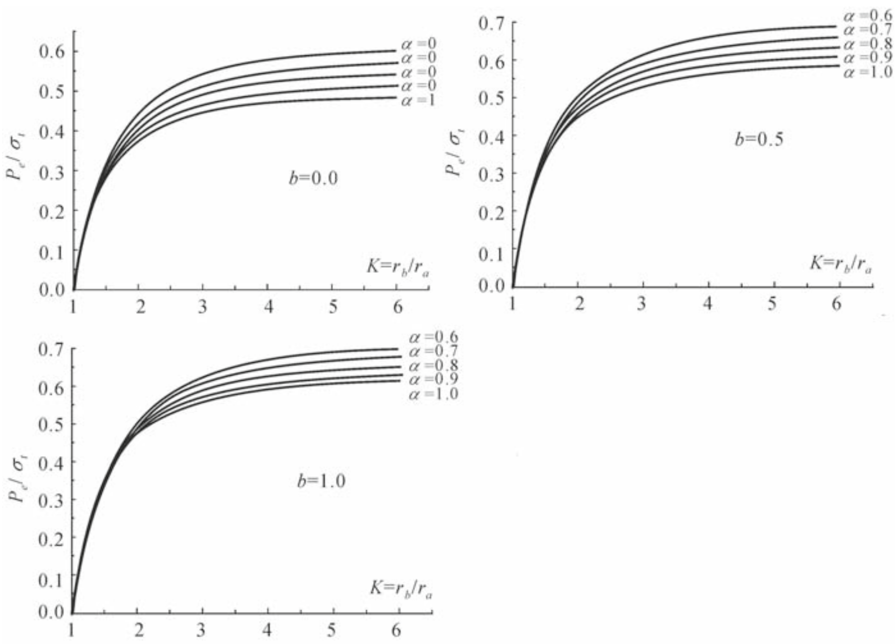

Fig. 9.7. Relation of elastic pressure with $K = r_b/r_a$

The results of the elastic limit pressures of a thick-walled cylinder for closed end and open end in view of different yield criteria are summarized in Table 9.1 and Table 9.2.

When a uniform pressure p is applied externally to a thick-walled cylinder with wall ratio r_b/r_a, the elastic stress distribution of σ_r and σ_θ can be derived from Eq.(9.28) by exchanging the positions of r_a and r_b in the formulation. In this case both the stresses are compressive, and the magnitude of σ_θ is higher than that of σ_r.

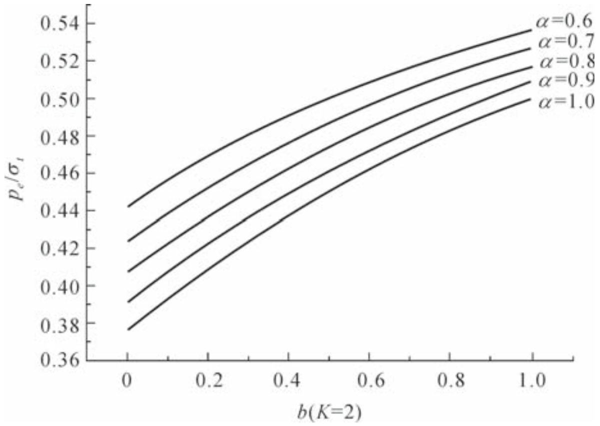

Fig. 9.8. Relation of elastic pressure with the unified strength theory parameter b

Table 9.1. Summary of elastic limit pressures for closed end

	Materials	Elastic limit pressures	Failure criterion used
1	SD material $\alpha \neq 1$	$p_e = \frac{(1+b)(K^2-1)\sigma_t}{K^2(1+b+\alpha)+(1+b)(1-\alpha)}$	Unified strength theory
2	SD material $\alpha \neq 1$	$p_e = \frac{K^2-1}{(1+\alpha)K^2+(1-\alpha)}\sigma_t$	Unified strength theory $b = 0$, Mohr-Coulomb
3	SD material $\alpha \neq 1$	$p_e = \frac{2(K^2-1)}{K^2(2+\alpha)+2(1-\alpha)}\sigma_t$	Unified strength theory $b = 1$, twin-shear theory
4	$\alpha = 1$ materials	$p_e = \frac{(1+b)(K^2-1)}{K^2(2+b)}\sigma_y$	Unified yield criterion
5	$\alpha = 1$ materials	$p_e = \frac{K^2-1}{2K^2}\sigma_y$	Unified yield criterion $b = 0$, Tresca criterion
6	$\alpha = 1$ materials	$p_e = \frac{K^2-1}{\sqrt{3}K^2}\sigma_y$	von Mises yield criterion
7	$\alpha = 1$ materials	$p_e = \frac{3(K^2-1)}{5K^2}\sigma_y$	Unified yield criterion $b = 1/2$
8	$\alpha = 1$ materials	$p_e = \frac{2(K^2-1)}{3K^2}\sigma_y$	Unified yield criterion $b = 1$, twin-shear criterion

9.5 Unified Solution of Plastic Limit Pressure of Thick-walled Cylinder

9.5.1 Stress Distribution

When the internal pressure exceeds p_e, a plastic zone starts at the inner surface and spreads towards the outer surface. If the outer radius of the

Table 9.2. Summary of elastic limit pressures for open end

	Materials	Elastic limit pressures	Failure criterion used
1	SD material $\alpha \neq 1$	$p_e = \frac{(1+b)(K^2-1)\sigma_t}{(1+b)(K^2+1)+\alpha(K^2-1)}$	Unified strength theory
2	SD material $\alpha \neq 1$	$p_e = \frac{K^2-1}{(1+\alpha)K^2+(1-\alpha)}\sigma_t$	Unified strength theory $b = 0$, Mohr-Coulomb
3	SD material $\alpha \neq 1$	$p_e = \frac{2(K^2-1)}{K^2(2+\alpha)+2(1-\alpha)}\sigma_t$	Unified strength theory $b = 1$, twin-shear theory
4	$\alpha = 1$ materials	$p_e = \frac{(1+b)(K^2-1)}{K^2(2+b)+b}\sigma_y$	Unified yield criterion
5	$\alpha = 1$ materials	$p_e = \frac{K^2-1}{2K^2}\sigma_y$	Unified yield criterion $b = 0$, Tresca criterion
6	$\alpha = 1$ materials	$p_e = \frac{K^2-1}{\sqrt{3K^4+1}}\sigma_y$	von Mises yield criterion
7	$\alpha = 1$ materials	$p_e = \frac{3(K^2-1)}{5K^2+1}\sigma_y$	Unified yield criterion $b = 1/2$
8	$\alpha = 1$ materials	$p_e = \frac{2(K^2-1)}{3K^2+1}\sigma_y$	Unified yield criterion $b = 1$, twin-shear criterion

elastic-plastic boundary is denoted as r_c, in the elastic region ($r_c \leqslant r \leqslant r_b$), the radial and circumferential stresses can be derived from Lame's equations with application of the boundary conditions of $\sigma_r = 0$ at $r = r_b$, and the stresses at $r = r_c$ satisfying the yield conditions. The pressure reaches its maximum value when the plastic zone reaches the outer surface of the thick-walled tube.

The elastic part of the elastic-plastic thick-walled tube can be considered as a new tube with the inner radius r_c, outer radius r_b and an internal pressure p_e. The stress distribution in the elastic region for incompressible material can be written as

$$\sigma_\theta = \frac{p_e r_c^2}{r_b^2 - r_c^2}\left(1 + \frac{r_b^2}{r^2}\right), \tag{9.53}$$

$$\sigma_r = \frac{p_e r_c^2}{r_b^2 - r_c^2}\left(1 - \frac{r_b^2}{r^2}\right), \tag{9.54}$$

$$\sigma_z = \frac{\nu}{2}(\sigma_\theta + \sigma_r), \tag{9.55}$$

where

$$p_e = \frac{2(1+b)(r_b^2 - r_c^2)}{(2+2b-\alpha b)(r_b^2 + r_c^2) + \alpha(2+b)(r_b^2 - r_c^2)}\sigma_t. \tag{9.56}$$

9.5.2 Plastic Zone in the Elasto-plastic Range

In the plastic zone, for elastic-perfectly-plastic material, the stress state satisfies Eq.(9.2a) or Eq.(9.2b) when the unified strength theory is adopted. According to the stress state condition of Eq. (9.3a), the first equation of the unified strength theory, i.e., Eq. (9.2a), should be applied as the yield condition,

$$\frac{2+(2-\alpha)b}{2(1+b)}\sigma_\theta - \frac{\alpha(2+b)}{2(1+b)}\sigma_r = \sigma_t. \tag{9.57}$$

Substituting Eq.(9.57) into the equilibrium equation in Eq.(9.25), we get

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}\frac{\sigma_r}{r} - \frac{2(1+b)}{2+(2-\alpha)b}\frac{\sigma_t}{r} = 0. \tag{9.58}$$

The general solution to this differential equation is

$$\sigma_r = \frac{c}{r^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}}} + \frac{\sigma_t}{1-\alpha}. \tag{9.59}$$

The integration constant can be determined by the boundary condition of $r = r_a$, $\sigma_r = -p$ as $-p = \frac{c}{r_a^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}}} + \frac{\sigma_t}{1-\alpha}$, which gives

$$c = (-p - \frac{\sigma_t}{1-\alpha})A^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}}. \tag{9.60}$$

Therefore, the stress distribution in the plastic region ($r_a \leqslant r \leqslant r_c$) is

$$\sigma_r = -\left(p + \frac{\sigma_t}{1-\alpha}\right)\left(\frac{r_a}{r}\right)^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}} + \frac{\sigma_t}{1-\alpha}, \tag{9.61}$$

$$\sigma_\theta = \frac{2(1+b)\sigma_t}{2+(2-\alpha)b} - \frac{\alpha(2+b)}{2+(2-\alpha)b}\left[\left(p + \frac{\sigma_t}{1-\alpha}\right)\left(\frac{r_a}{r}\right)^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}} + \frac{\sigma_t}{1-\alpha}\right], \tag{9.62}$$

$$\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta). \tag{9.63}$$

Eqs.(9.61)∼(9.63) give the stresses of a thick-walled cylinder at the plastic region. Since no stress-strain relation is required to derive the stresses, the problem is considered statically determinate.

9.5.3 Plastic Zone Radius in the Elasto-plastic Range

The pressure on the elastic and plastic zone boundary satisfies Eq.(9.57) of the elastic zone solution. Assuming that the radius of the plastic zone is r_c, for a given internal pressure p, the plastic zone radius r_c can be determined from Eq.(9.57). When pressure increases, the plastic zone radius r_c increases gradually from r_a to r_b.

The stress continuity of radial stress σ_r across $r = r_c$ gives

$$\sigma_{r=r_c}(\text{elastic zone}) = \sigma_{r=r_c}(\text{plastic zone}).$$

Substituting the radial stress in Eq.(9.55) and the radial stress in Eq.(9.60) into the stress continuity condition, the relation of pressure p to plastic zone radius is derived,

$$p = \left(\frac{r_c}{r_a}\right)^{\frac{2(1+b)(1-\alpha)}{2+(2-\alpha)b}} \left[\frac{2(1+b)(r_b^2 - r_c^2)}{(2+2b-\alpha b)(r_b^2 + r_c^2) + \alpha(2+b)(r_b^2 - r_c^2)} + \frac{1}{1-\alpha}\right]\sigma_t - \frac{\sigma_t}{1-\alpha}. \tag{9.64}$$

As an example, the relation of pressure versus the plastic zone radius is illustrated schematically in Fig.9.9 for the ratio of the external radius r_b to the internal radius r_a, $K = r_b/r_a = 2$.

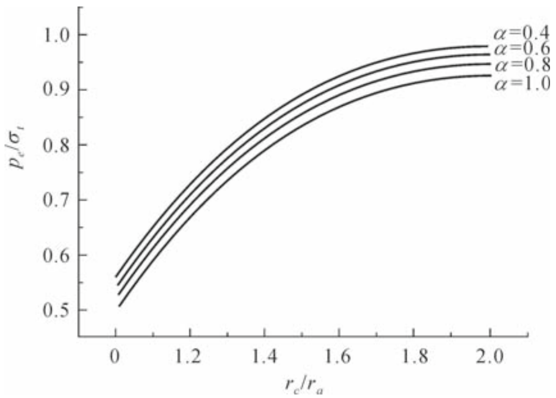

Fig. 9.9. Plastic zone radius versus internal pressure for different α ($K = 2, b = 1.0$)

9.5.4 Plastic Limit Pressure

9.5.4.1 Plastic Limit Pressure for SD Materials

When r_c is equal to r_b, the thick-walled tube is completely plastic. The plastic limit pressure for the thick-walled cylinder is

$$p_p = \frac{\sigma_t}{1-\alpha}\left(K^{\frac{2(1+b)(1-\alpha)}{2+2b-\alpha b}} - 1\right). \tag{9.65}$$

The solution in Eq.(9.65) is for a thick-walled cylinder with closed end or plane strain conditions. It can be referred to as the unified solution of plastic limit pressure for thick-walled cylinder.

When $b = 0$, the plastic limit pressure in terms of the Mohr-Coulomb theory is deduced from the unified solution,

$$p_p = \frac{\sigma_t}{1-\alpha}(K^{(1-\alpha)} - 1). \tag{9.66}$$

When $b = 1$, the unified solution becomes the plastic limit pressure in terms of the twin-shear strength theory,

$$p_p = \frac{\sigma_t}{1-\alpha}(K^{\frac{4(1-\alpha)}{4-\alpha}} - 1). \tag{9.67}$$

9.5.4.2 Plastic Limit Pressure for Non-SD Materials

The unified solution for non-SD materials can be derived from the unified solution with $\alpha = 1$. The plastic limit pressure of a thick-walled cylinder based on the unified yield criterion can be expressed as

$$p_p = \frac{2(1+b)\sigma_t}{2+b}\ln K. \tag{9.68}$$

The limit pressure in terms of the Tresca yield criterion can be derived from the unified solution with $b = 0$,

$$p_p = \sigma_t \ln K, \tag{9.69}$$

which is identical to the classical solution based on the Tresca yield criterion.

The plastic limit pressure in terms of the linear Huber-von Mises yield criterion can be approximately obtained with the unified solution with $b = 1/2$,

$$p_p = \frac{6}{5}\sigma_t \ln K. \tag{9.70}$$

The plastic limit pressure in terms of the twin-shear yield criterion can be obtained from the unified solution with $b = 1$,

$$p_p = \frac{4}{3}\sigma_t \ln K. \tag{9.71}$$

Eqs.(9.70) and (9.71) are identical to the plastic limit pressure based on the twin-shear strength theory (Zhuang, 1998).

The relation of the plastic limit pressure with respect to the different parameter b and different thickness of cylinder ($K = r_b/r_a = 1.8$, $K = 2.0$, $K = 2.5$, $K = 3.0$) are shown in Fig.9.10 to Fig.9.13. From these figures the effect of failure criteria is prominent.

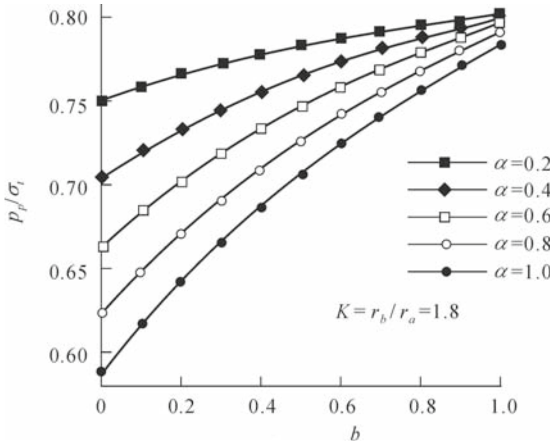

Fig. 9.10. Relation of plastic limit pressure to the unified strength theory parameter b when $K = 1.8$

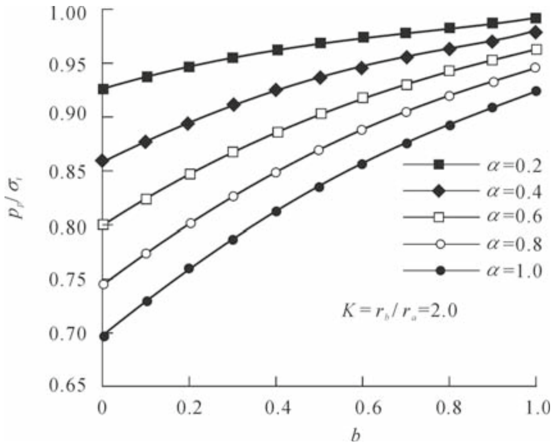

Fig. 9.11. Relation of plastic limit pressure to the unified strength theory parameter when $K = 2.0$

From Figs.9.13, 9.14 and Table 9.3, the elastic limit pressure in terms of the unified strength theory increases monotonically with increasing b for all the three end conditions.

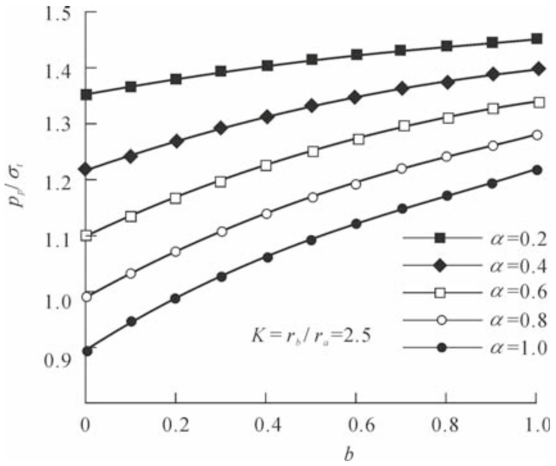

Fig. 9.12. Relation of plastic limit pressure to the unified strength theory parameter when $K = 2.5$

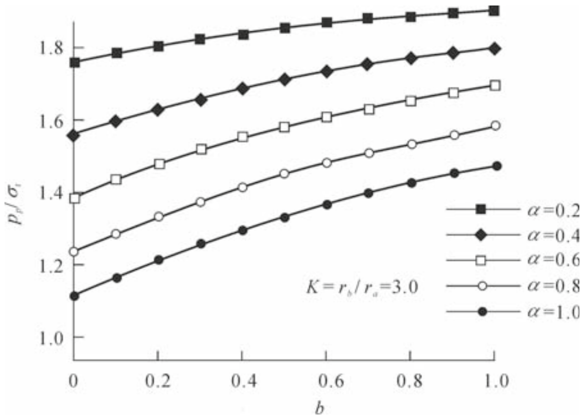

Fig. 9.13. Relation of plastic limit pressure to the unified strength theory parameter when $K = 3.0$

The elastic limit pressure in terms of the Tresca criterion is equal to that of the unified strength theory with $b = 0$ and $\alpha = 1$. The elastic limit pressure in terms of the von Mises criterion is equal to that of the unified strength theory with $b \simeq 4$. Therefore it can be concluded that the Huber-von Mises and the Tresca criteria are encompassed in the unified strength theory with regard to the elastic limit pressure. The maximum elastic limit pressure in terms of the twin-shear yield criterion is obtained with $b = 1$. It is 33.4% and 15.5% higher than those obtained from the Tresca criterion and the Huber-von Mises criterion respectively. It was also found that the higher values obtained from the unified strength theory were insensitive to the variations

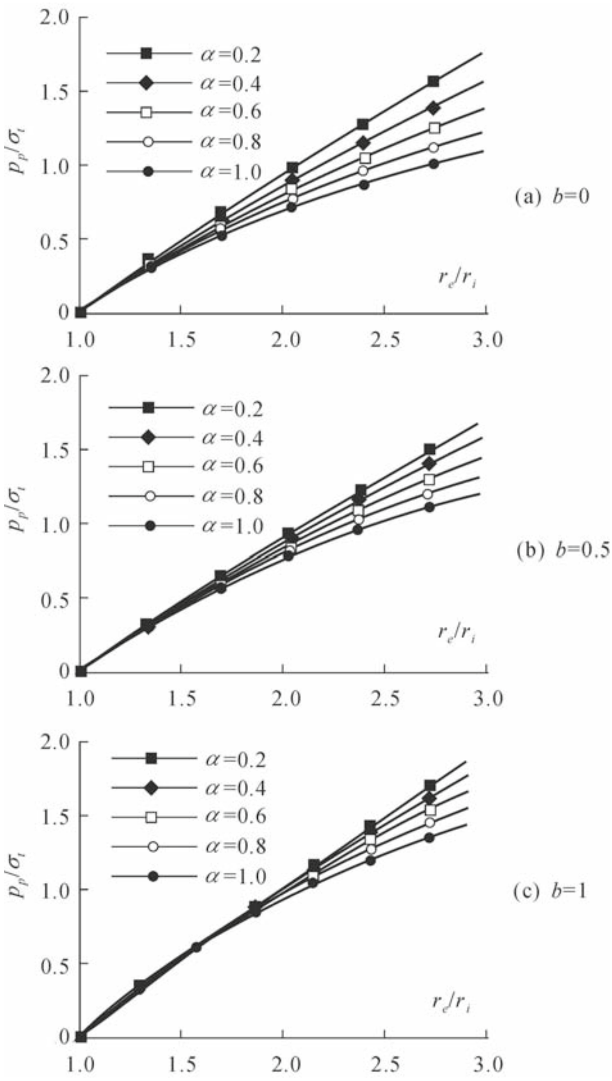

Fig. 9.14. Relation of p_p/σ_t and r_e/r_i with different b and α

of the inner-to-outer-radius ratio regardless of the different end conditions. For the plastic limit pressure, similar statements can be made.

The results of plastic limit pressures of a thick-walled cylinder under the closed end condition in terms of different yield criteria are summarized in Table 9.3.

The elastic limit pressure and plastic limit pressure, two important parameters in the design of a cylinder, have been derived using the unified strength theory. It was found that the percentage difference of the elastic-plastic limit

Table 9.3. Summary of elastic limit pressures for open end

	Materials	Elastic limit pressures	Failure criterion used
1	SD material $\alpha \neq 1$	$p_p = \frac{\sigma_t}{1-\alpha}(K^{\frac{2(1+b)(1-\alpha)}{2+2b-\alpha b}} - 1)$	Unified strength theory
2	SD material $\alpha \neq 1$	$p_p = \frac{\sigma_t}{1-\alpha}(K^{(1-\alpha)} - 1)$	Unified strength theory $b = 0$, Mohr-Coulomb
3	SD material $\alpha \neq 1$	$p_p = \frac{\sigma_t}{1-\alpha}(K^{\frac{4(1-\alpha)}{4-\alpha}} - 1)$	Unified strength theory $b = 1$, twin-shear theory
4	$\alpha = 1$ materials	$p_p = \frac{2(1+b)\sigma_t}{2+b} \ln K$	Unified yield criterion $\alpha = 1$
5	$\alpha = 1$ materials	$p_p = \sigma_t \ln K$	Tresca yield criterion $\alpha = 1$, $b = 0$
6	$\alpha = 1$ materials	$p_p = \frac{6}{5}\sigma_t \ln K$	Unified yield criterion b=1/2
7	$\alpha = 1$ materials	$p_p = \frac{4}{3}\sigma_t \ln K$	Twin-shear yield criterion $\alpha = 1$, $b = 1$

pressures derived from different criteria could differ from one from another as much as 33.4%. If the unified strength criterion is used in the design instead of the Tresca or the Huber-von Mises criterion, it could lead to a substantial saving of material.

9.6 Summary

The limit analysis of the thick-walled hollow sphere and cylinder under pressure was discussed in detail in literature. The Tresca yield criterion or the Huber-von Mises yield criterion has been applied for analysis and design purposes. The solution is adopted only for one kind of material.

In the last decade the elastic limit and plastic limit of thin-walled vessels and thick-walled cylinders were studied by researchers with respect to the unified strength theory. Unified limit solutions for a thick-wall cylinder subject to external and internal pressure are given. The unified solution is a series of results which can be adopted for more materials. They are described in this chapter.

The application of the unified strength theory is also extended from ideal elasto-plastic materials to hardening material. The unified limit analysis of a

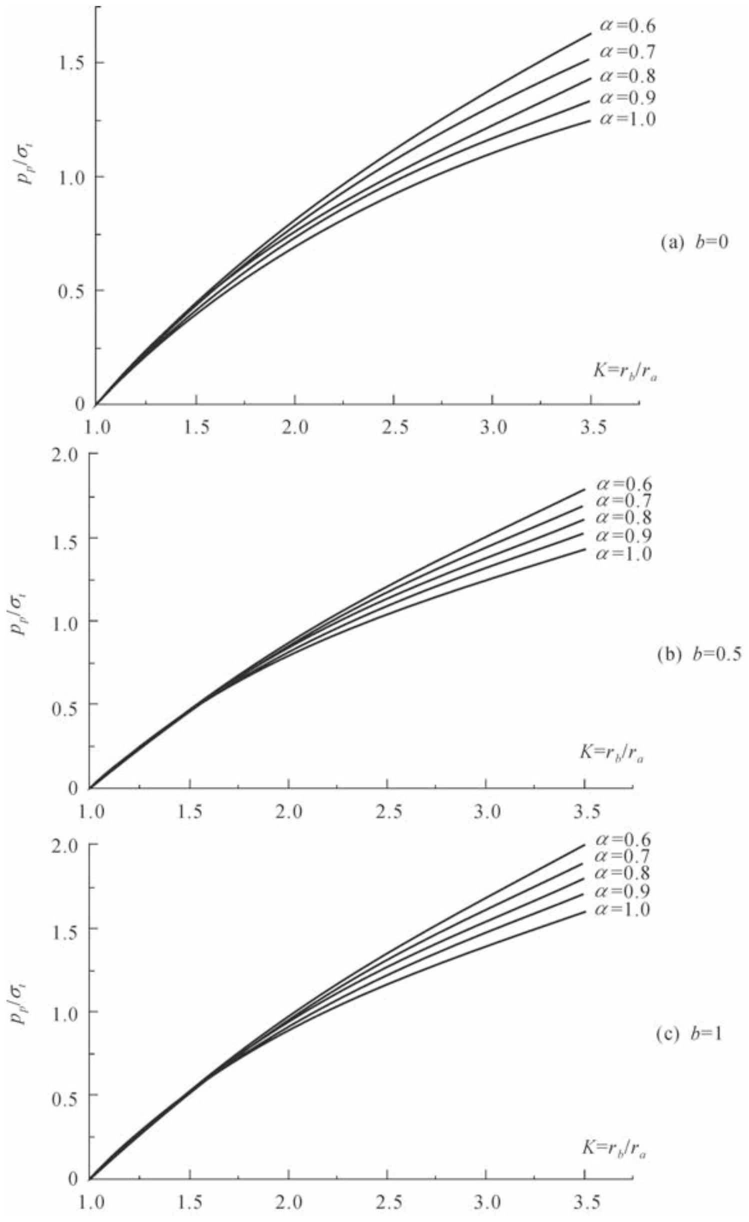

Fig. 9.15. Relation of plastic limit pressure with the thickness of cylinder

thick-wall cylinder of linearly strengthened material is derived by Ma (2004). It was also extended recently to brittle materials such as concrete or rock. Based on Yu unified strength theory with a material's strain softening property considered, a unified strength criterion for strain softening materials has also been proposed to find out the bearing capacity of a thick-walled cylinder subject to external pressure.

A unified solution for a cylinder is generalized to take into account the strain softening material, elasto-brittle-plastic materials and fiber-reinforced concrete. The bearing capacity analysis for a thick walled cylinder, take into account elasto-brittle-plastic and strain softening, is presented by Xu and Yu (2004), Chen et al. (2006a; 2006b).

9.7 Problems

Problem 9.1 Derive the elastic limit pressure equation for a spherical shell under internal pressure in terms of the Mohr-Coulomb theory.

Problem 9.2 Derive the elastic limit pressure equation for a spherical shell under internal pressure in terms of the twin-shear strength theory.

Problem 9.3 Explain why we would expect the Mohr-Coulomb strength theory, the twin-shear strength theory and the unified strength theory to coincide in the case of a spherical shell with spherical symmetry.

Problem 9.4 Derive the elastic limit pressure equation for a thick-walled cylinder under internal pressure using the Mohr-Coulomb theory

$$p_e = \frac{K^2 - 1}{K^2(1+\alpha)(1-\alpha)}\sigma_t, \ K = r_b/r_a.$$

Problem 9.5 Derive the elastic limit pressure equation for a thick-walled cylinder under internal pressure by using the twin-shear strength theory

$$p_e = \frac{K^2 - 1}{K^2(1+\alpha/2)(1-\alpha)}\sigma_t, \ K = \frac{r_b}{r_a}.$$

Problem 9.6 Compare the results of Problem 9.4 with those of Problem 9.5.

Problem 9.7 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the elastic limit external pressure equation for a thick-walled cylinder under external pressure using the Mohr-Coulomb strength theory.

Problem 9.8 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the elastic limit external pressure equation for a thick-walled cylinder under external pressure using the twin-shear strength theory.

Problem 9.9 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the plastic limit external pressure equation for a thick-walled cylinder under external pressure using the unified strength theory.

Problem 9.10 Compare the results obtained for Problems 9.7, 9.8 and 9.9.

Problem 9.11 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the elastic limit external pressure equation for a thick-walled cylinder under external pressure using the Mohr-Coulomb strength theory.

Problem 9.12 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the plastic limit external pressure equation for a thick-walled cylinder under external pressure using the twin-shear strength theory.

Problem 9.13 A uniform pressure p is applied externally to a thick-walled cylinder of wall ratio r_b/r_a. In this case both the stresses are negative, where σ_θ is more compressive than σ_r. Introduce the plastic limit external pressure equation for a thick-walled cylinder under external pressure using the unified strength theory.

Problem 9.14 Compare the results obtained in Problems 9.11, 9.12 and 9.13.

Problem 9.15 Explain why we have to determine the stress state condition $\sigma_2 \leqslant \frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$ or $\sigma_2 \geqslant \frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$ using the unified strength theory.

Problem 9.16 How do you choose between the two equations in the unified strength theory?

Problem 9.17 What is the result if you use the second equation of the unified strength theory for the stress state of $\sigma_2 \leqslant \frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$?

Problem 9.18 What is the result if you use the first equation of the unified strength theory for the stress state of $\sigma_2 \geqslant \frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$?

Problem 9.19 Complete discussions of the effects of pressure and temperature on yielding of thick-walled spherical shells given by Johnson and Mellor (1962), Mendelson (1968), and Chakrabarty (1987). The Tresca yield criterion was used in these studies. Can you obtain a more complete study on this subject using the unified yield criterion ($\alpha = 1$)?

Problem 9.20 A complete discussion of the effects of pressure and temperature on yielding of thick-walled spherical shells was given by Johnson and Mellor (1962), Mendelson (1968), and Chakrabarty (1987). The Tresca yield criterion was used in these studies. Can you obtain a more complete study on this subject using the unified strength theory ($\alpha \neq 1$)?

Problem 9.21 Complete discussions of the effects of pressure and temperature on yielding of thick-walled cylinder given by Johnson and Mellor (1962), Mendelson (1968), and Chakrabarty (1987). The Tresca yield criterion was used in these studies. Can you obtain a more complete study on this subject using the unified yield criterion ($\alpha = 1$)?

Problem 9.22 Complete discussions of the effects of pressure and temperature on yielding of thick-walled cylinder given by Johnson and Mellor (1962), Mendelson (1968), and Chakrabarty (1987). The Tresca yield criterion was used in these studies. Can you obtain a more complete study on this subject using the unified strength theory ($\alpha \neq 1$)?

Problem 9.23 The unified yield criterion can be used in many fields. Write an article regarding the application of the unified yield criterion.

Problem 9.24 The unified strength theory can be used in many fields. Write an article regarding the application of the unified strength theory.

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