8

Plastic Limit Analyses of Oblique, Rhombic, and Rectangular Plates

8.1 Introduction

Plate structures are widely used in aerospace, shipping, civil, and mechanical engineering. Plastic limit analyses of flat plates with different geometries can approximately estimate the load-bearing capacities of the plates. A lot of analytical solutions for flat plates have been reported by Wood (1961), Sawczuk and Jaeger (1963), Save and Massonnet (1972), Golley (1997), Mishra et al. (1996), Moen et al. (1998). Their solutions are mainly based on the Tresca yield criterion, the Huber-von Mises yield criterion, or the Mohr-Coulomb strength criterion. The maximum principal stress criterion has also been applied for simplicity.

The Tresca-Mohr-Coulomb strength theory is a single-shear strength theory. It ignores the effect of the intermediate principal stress. The Tresca yield criterion and the Huber-von Mises yield criterion can be effectively applied for the analyses of the non-SD materials. The maximum principal stress criterion considers only one of the three principal stresses, which may be deficient in yielding valid analytical results.

The unified strength theory (UST) has attracted more and more attention in engineering applications. In this chapter the load-bearing capacity for simply supported plates of different geometries will be given. Amongst them, the unified solution to the load-bearing capacity for a simply supported oblique plate was presented by Li and Yu (2000).

In terms of the principal stresses, the mathematical expression of the UST is

$$\begin{cases} f = \sigma_1 - \dfrac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t, \text{when } \sigma_2 \leqslant \dfrac{\sigma_1 + \alpha\sigma_3}{1+\alpha}, \\ f' = \dfrac{\alpha}{1+b}(\sigma_1 + b\sigma_2) - \sigma_3 = \sigma_t, \text{when } \sigma_2 \geqslant \dfrac{\sigma_1 + \alpha\sigma_3}{1+\alpha}, \end{cases} \tag{8.1}$$

where f and f' are yield functions; σ_1, σ_2, and σ_3 are the maximum principal stress, the intermediate principal stress, and the minimum principal stress, respectively; σ_t and σ_c are the tensile and compressive strengths; α the tensile to compressive strength ratio, i.e., $\alpha = \sigma_t/\sigma_c$; b is a coefficient which reflects the relative effect of the intermediate principal stress and the intermediate principal shear stress. It is the parameter specifying the failure criterion in the unified strength theory. The unified strength theory parameter b can be obtained via the tensile strength σ_t, the compressive strength σ_c and the shear strength τ_0,

$$b = \frac{1 + \alpha - \sigma_t/\tau_0}{\sigma_t/\tau_0 - 1}.$$

The twin-shear strength theory (Yu et al., 1985) and the single-shear strength theory (Mohr-Coulomb, 1900) can be derived from Eq.(8.1) with $b = 1$ and $b = 0$, respectively. For the plane stress problem ($\sigma_2 = 0$) the UST can be simplified as

$$\begin{cases} f = \sigma_1 - \dfrac{\alpha}{1+b}\sigma_3 = \sigma_t, & \text{where} \quad 0 \leqslant \frac{1}{2}(\sigma_1 + \alpha\sigma_3), \\ f' = \dfrac{1}{1+b}\sigma_1 - \alpha\sigma_3 = \sigma_t, & \text{where} \quad 0 \geqslant \dfrac{1}{2}(\sigma_1 + \alpha\sigma_3). \end{cases} \tag{8.2}$$

The limit loci of the UST in the plane stress state and in the deviatoric plane are shown in Figs.8.1 and 8.2 respectively. The twelve mathematical expressions of the unified yield criterion in plane stress state are

$$\sigma_1 - \frac{\alpha b}{1+b}\sigma_2 = \sigma_t, \qquad \sigma_2 - \frac{\alpha b}{1+b}\sigma_1 = \sigma_t, \tag{8.3a}$$

$$\frac{1}{1+b}\sigma_1 + \frac{b}{1+b}\sigma_2 = \sigma_t, \qquad \frac{1}{1+b}\sigma_2 + \frac{b}{1+b}\sigma_1 = \sigma_t, \tag{8.3b}$$

$$\sigma_1 - \frac{\alpha}{1+b}\sigma_2 = -\sigma_t, \qquad \sigma_2 - \frac{\alpha}{1+b}\sigma_1 = -\sigma_t, \tag{8.3c}$$

$$\frac{1}{1+b}\sigma_1 - \alpha\sigma_2 = -\sigma_t, \qquad \frac{1}{1+b}\sigma_2 - \alpha\sigma_1 = -\sigma_t, \tag{8.3d}$$

$$-\frac{\alpha}{1+b}(b\sigma_1 + \sigma_2) = \sigma_t, \qquad -\frac{\alpha}{1+b}(b\sigma_2 + \sigma_1) = \sigma_t, \tag{8.3e}$$

$$\frac{b}{1+b}\sigma_1 - \alpha\sigma_2 = \sigma_t, \qquad \frac{b}{1+b}\sigma_2 - \alpha\sigma_1 = \sigma_t, \tag{8.3f}$$

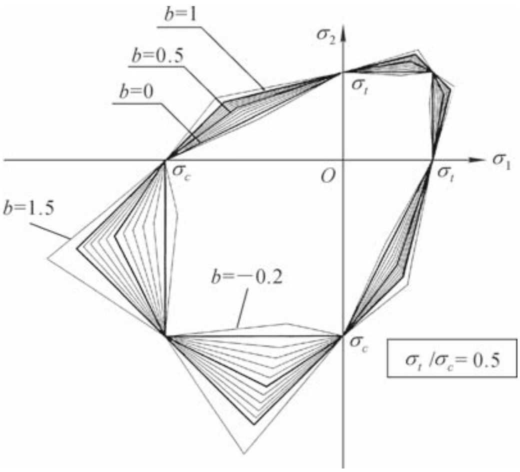

Fig. 8.1. Yield loci of the UST in the plane stress

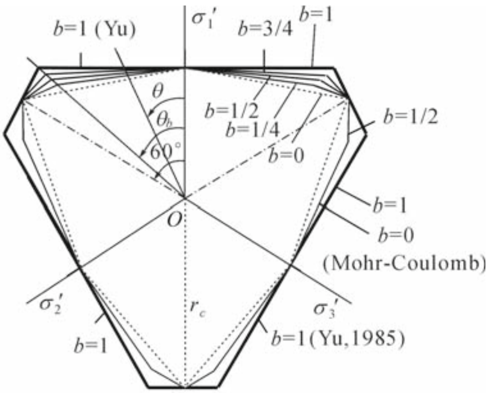

Fig. 8.2. Yield loci of the UST in the deviatoric plane

8.2 Equations for Oblique Plates

8.2.1 The Equilibrium Equation in Ordinary Coordinate System

For the oblique plate in Fig.8.3 with distribution of internal forces in Fig.8.4, u and v denote the ordinary coordinate axes; θ is the angle between the ordinary coordinate axes; $M_{n,1}$, $M_{n,2}$, and M_t are two positive bending moments and a shear moment per unit length of the oblique plate respectively. The unit of the moments is in Nm/m; $2l_1$ and $2l_2$ are respectively the total length of the two sides of the oblique plate; q is a transverse load over the plate.

The equilibrium equation of the plates in the Cartesian coordinate system is

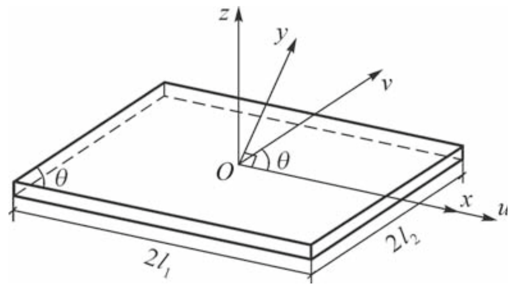

Fig. 8.3. Coordinate of the oblique plate

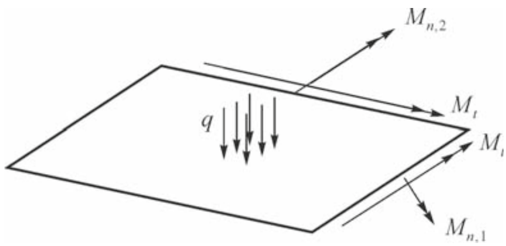

Fig. 8.4. Distribution of internal forces

$$\frac{\partial^2 M_{n,x}}{\partial x^2} + 2\frac{\partial^2 M_{n,xy}}{\partial x \partial y} + \frac{\partial^2 M_{n,y}}{\partial y^2} + q = 0, \tag{8.4}$$

where $M_{n,x}$, $M_{n,y}$, and $M_{n,xy}$ are the normal moments and shear moment per unit length in the rectangular Cartesian coordinate system.

The transformation between the rectangular Cartesian coordinate system and ordinary coordinate system can be expressed as

$$x = u + v\cos\theta, \quad y = v\sin\theta, \tag{8.5}$$

or

$$u = x - y\text{cot}\theta, \quad v = y/\sin\theta. \tag{8.6}$$

The equilibrium equation of plates in oblique coordinate system can be derived from Eq.(8.4),

$$\begin{aligned} &\frac{\partial^2 M_{n,x}}{\partial u^2} + 2\left(-\text{cot}\theta\frac{\partial^2 M_{n,xy}}{\partial u^2} + \frac{1}{\sin\theta}\frac{\partial^2 M_{n,xy}}{\partial u \partial v}\right) + \cot^2\theta\frac{\partial^2 M_{n,y}}{\partial u^2} \\ &- 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial^2 M_{n,y}}{\partial u \partial v} + \frac{1}{\sin^2\theta}\frac{\partial^2 M_{n,y}}{\partial v^2} + q = 0. \end{aligned} \tag{8.7}$$

8.2.2 Field of Internal Motion

Assuming that the oblique plate is simply supported around the four outer edges and subjected to a transverse load q, the functions of the internal forces for the oblique plates are

$$\begin{cases} M_{n,1} = c_1(l_1^2 - u^2), \\ M_{n,2} = c_2(l_2^2 - v^2), \\ M_t = c_3 uv, \end{cases}$$

where c_1, c_2, and c_3 are three coefficients to be determined.

According to the transformation of the internal moments between the ordinary coordinate system and the rectangular Cartesian coordinate system,

$$\begin{cases} M_{n,x} = -c_1 \dfrac{\cos 2\theta}{\sin^2\theta}(l_1^2 - u^2) + c_2 \cot^2\theta(l_2^2 - v^2) + c_3 \dfrac{\sin 2\theta}{\sin^2\theta} uv, \\ M_{n,y} = c_2(l_2^2 - v^2), \\ M_{n,xy} = -c_1 \cot\theta(l_1^2 - u^2) - c_2 \cot\theta(l_2^2 - v^2) - c_3 uv. \end{cases}$$

The uniform transverse load can then be derived from Eqs.(8.5) and (8.7),

$$q_l = [-2c_1(1 + 2\cos 2\theta) + 2c_2 + 2c_3 \sin\theta]/\sin^2\theta. \tag{8.8}$$

8.2.3 Moment Equation Based on the UST

The internal moments $M_{n,x}$, $M_{n,y}$, and $M_{n,xy}$ can be integrated form the stresses σ_x, σ_y and τ_{xy},

$$\begin{cases} M_{n,x} = \displaystyle\int_{-h}^{h} \sigma_x z \mathrm{d}z = \sigma_x h^2, \\ M_{n,y} = \displaystyle\int_{-h}^{h} \sigma_y z \mathrm{d}z = \sigma_y h^2, \\ M_{n,xy} = \displaystyle\int_{-h}^{h} \tau_{xy} z \mathrm{d}z = \tau_{xy} h^2, \end{cases} \tag{8.9}$$

where $2h$ is the thickness of the oblique plate. The UST can be rewritten for the plane stress state as

$$f = \frac{1}{1+b}\left[\frac{1+b-\alpha}{2}(\sigma_x+\sigma_y)+(1+b+\alpha)\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right]=\sigma_t,$$

$$\text{when } (1+\alpha)\frac{\sigma_x+\sigma_y}{2}+(1-\alpha)\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\geqslant 0, \tag{8.10a}$$

$$f' = \frac{1}{1+b}\left[\frac{1-\alpha b-\alpha}{2}(\sigma_x+\sigma_y)+(1+\alpha b+\alpha)\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right]=\sigma_t$$

$$\text{when } (1+\alpha)\frac{\sigma_x+\sigma_y}{2}+(1-\alpha)\sqrt{(\frac{\sigma_x-\sigma_y}{2})^2+\tau_{xy}^2}\leqslant 0. \tag{8.10b}$$

Eqs.(8.10a) and (8.10b) can be expressed in terms of $M_{n,x}$, $M_{n,y}$, and $M_{n,xy}$ as

$$\begin{aligned}&(2+b)^2(M_{n,x}-M_{n,y})^2+4(2+b)^2M_{n,xy}^2-b^2(M_{n,x}+M_{n,y})^2\\&\quad=4(1+b)M_p[(1+b)M_p-b(M_{n,x}+M_{n,y})],\\&\quad\text{when } (1+\alpha)\frac{M_{n,x}+M_{n,y}}{2}+(1-\alpha)\sqrt{\left(\frac{M_{n,x}+M_{n,y}}{2}\right)^2+M_{n,xy}^2}\geqslant 0,\end{aligned} \tag{8.11a}$$

$$\begin{aligned}&(2+b)^2(M_{n,x}-M_{n,y})^2+4(2+b)^2M_{n,xy}^2-b^2(M_{n,x}+M_{n,y})^2\\&\quad=4(1+b)M_p[(1+b)M_p+b(M_{n,x}+M_{n,y})],\\&\quad\text{when } (1+\alpha)\frac{M_{n,x}+M_{n,y}}{2}+(1-\alpha)\sqrt{\left(\frac{M_{n,x}+M_{n,y}}{2}\right)^2+M_{n,xy}^2}\leqslant 0,\end{aligned} \tag{8.11b}$$

where M_p is the limit bending moment of the plate.

The limit loci of generalized stresses in terms of the unified strength theory for the plane plate are illustrated schematically in Fig.8.5.

8.3 Unified Solution of Limit Analysis of Simply Supported Oblique Plates

When the inequality $(1+\alpha)\frac{M_{n,x}+M_{n,y}}{2}+(1-\alpha)\sqrt{\left(\frac{M_{n,x}-M_{n,y}}{2}\right)^2+M_{n,xy}}\geqslant 0$ is satisfied, the unified yield function for u and v can be derived by substituting Eq.(8.7) into Eq.(8.11a). With calculus regarding u and v, the limit

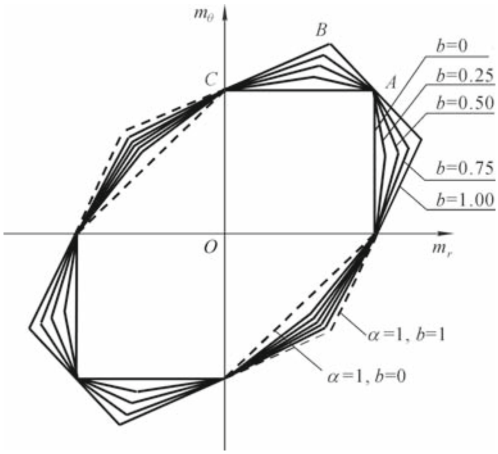

Fig. 8.5. Generalized unified yield criterion in plane stress state

loading at the points (0,0), ($\pm l_1$, 0), (0, $\pm l_2$), and ($\pm l_1$, $\pm l_2$) of the plate can then be derived. Defining

$$G = 4(1+b)^2 M_p^2, \tag{8.12a}$$

$$E_1 = \left[(1+b+\alpha)^2 \frac{\cos^2 2\theta}{\sin^4\theta} + 4(1+b+\alpha)^2 \cot^2\theta - (1+b-\alpha)^2 \frac{\cos^2 2\theta}{\sin^4\theta}\right] l_1^4, \tag{8.12b}$$

$$E_2 = \left[(1+b+\alpha)^2 \frac{\cos^2 2\theta}{\sin^4\theta} + 4(1+b+\alpha)^2 \cot^2\theta - (1+b-\alpha)^2 \csc^4\theta\right] l_2^4, \tag{8.12c}$$

$$E_3 = \left[(1+b+\alpha)^2 \frac{\sin^2 2\theta}{\sin^4\theta} + 4(1+b+\alpha)^2 - (1+b-\alpha)^2 \frac{\sin^2 2\theta}{\sin^4\theta}\right] l_1^2 l_2^2, \tag{8.12d}$$

$$F_1 = 4(1+b)(1+b-\alpha) M_p \frac{\cos 2\theta}{\sin^2\theta} l_1^2, \tag{8.12e}$$

$$F_2 = 4(1+b)(1+b-\alpha) M_p \csc^2\theta l_2^2, \tag{8.12f}$$

$$F_3 = 4(1+b)(1+b-\alpha) M_p \frac{\sin 2\theta}{\sin^2\theta} l_1 l_2. \tag{8.12g}$$

The limit load is derived from Eq.(8.8),

$$q_l = [-2c_1(1 + 2\cos 2\theta) + 2c_2 + 2c_3 \sin\theta]/\sin^2\theta, \tag{8.13}$$

where c_1, c_2, and c_3 are

$$\begin{cases} c_1 = \left(F_1 + \sqrt{F_1^2 + 4E_1G}\right)/(2E_1), \\ c_2 = \left(-F_2 + \sqrt{F_2^2 + 4E_2G}\right)/(2E_2), \\ c_3 = \left(-F_3 + \sqrt{F_3^2 + 4E_3G}\right)/(2E_3). \end{cases} \tag{8.14}$$

When $(1+\alpha)\frac{M_{n,x}+M_{n,y}}{2} + (1-\alpha)\sqrt{(\frac{M_{n,x}-M_{n,y}}{2})^2 + M_{n,xy}} \leqslant 0$, the limit load q_l can be derived from Eq.(8.11b) with the same form of Eq.(8.13), while the coefficients c_1, c_2, and c_3 are given as

$$c_1 = \left(-F_1 + \sqrt{F_1^2 + 4E_1G}\right)/(2E_1), \tag{8.15a}$$

$$c_2 = \left(F_2 + \sqrt{F_2^2 + 4E_2G}\right)/(2E_2), \tag{8.15b}$$

$$c_3 = \left(F_3 + \sqrt{F_3^2 + 4E_3G}\right)/(2E_3). \tag{8.15c}$$

The limit load q_l for a parallelogram plate with $\theta = \pi/3$ can be derived from Eq.(8.13),

$$q_l = \left[\frac{2}{l_2^2} + \frac{4(1+b)}{9(1+b)+3\alpha}\cdot\frac{1}{l_1 l_2}\right] M_p. \tag{8.16}$$

When $\theta = \pi/4$, the limit load q_l becomes

$$\begin{aligned} q_l = &\left[\frac{-4(1+b)}{(1+b+\alpha)}\frac{1}{l_1^2} + \frac{2(1+b)}{\alpha}\frac{1}{l_2^2}\right. \\ &\left. + \frac{(4+2\sqrt{2})(1+b)^2 + (4-2\sqrt{2})\alpha(1+b)}{(1+b)^2 + \alpha^2 + 6\alpha(1+b)}\frac{1}{l_1 l_2}\right] M_p. \end{aligned} \tag{8.17}$$

The relations between the limit load q_l and the unified strength theory parameter b of the UST for various oblique plates ($\theta = \pi/3$, $l_1 = 1.5l_2$; $\theta = \pi/3$, $l_1 = 2l_2$; $\theta = \pi/4$, $l_1 = 1.5l_2$; $\theta = \pi/4$, $l_1{=}2l_2$) are given in Fig.8.6 to Fig.8.9.

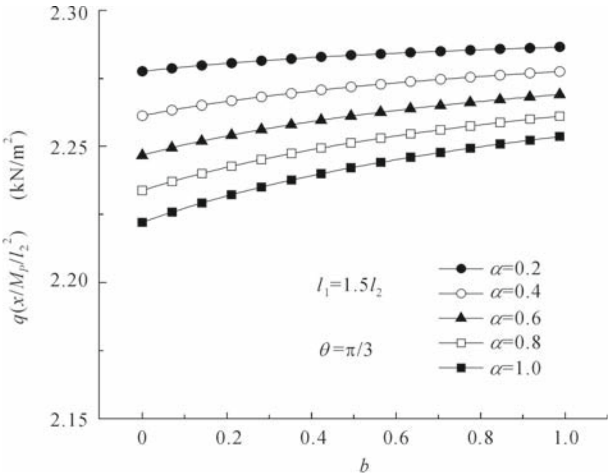

Fig. 8.6. Relations between q_l and unified strength theory parameter b ($\theta = \pi/3$, $l_1 = 1.5l_2$)

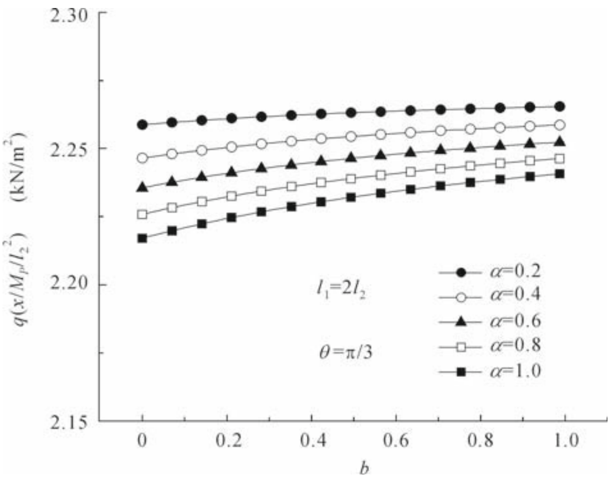

Fig. 8.7. Relations between q_l and unified strength theory parameter b ($\theta = \pi/3$, $l_1 = 2l_2$)

8.4 Limit Load of Rhombic Plates

For $l_1 = l_2$, the limit load can be obtained from Eqs.(8.16) and (8.17) for $\theta = \pi/3$,

$$q_l = \left[2 + \frac{4(1+b)}{9(1+b)+3\alpha}\right]\frac{M_p}{l_2^2} \quad \text{when} \quad \theta = \frac{\pi}{3}. \tag{8.18}$$

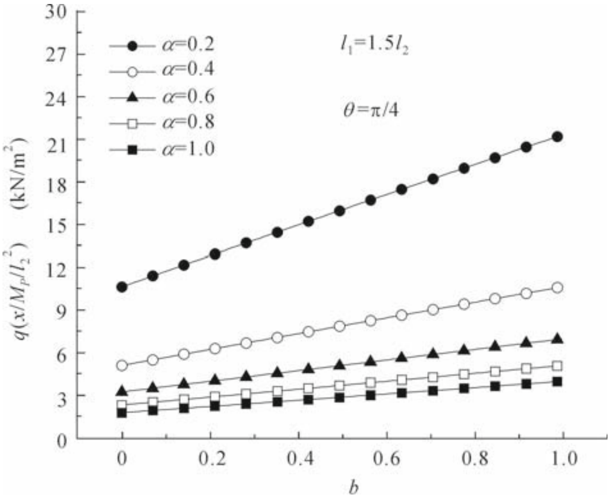

Fig. 8.8. Relations between q_l and unified strength theory b ($\theta = \pi/4$, $l_1 = 1.5l_2$)

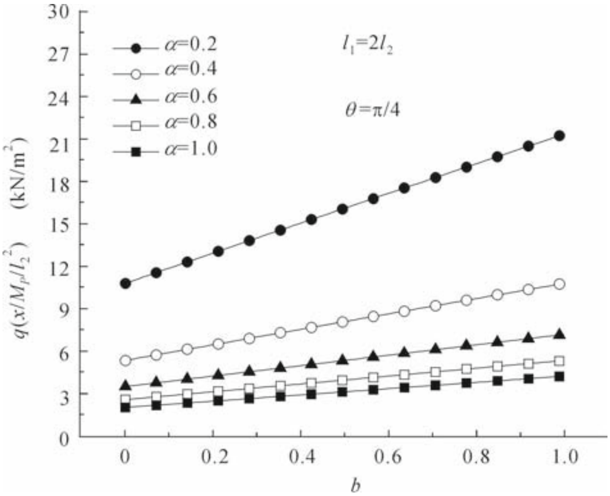

Fig. 8.9. Relations between q_l and unified strength theory b ($\theta = \pi/4$, $l_1 = 2l_2$)

The relations between limit load q_l and the unified strength theory parameter b are shown in Fig.8.10 and Fig.8.11 for $\theta = \pi/3$ and $\theta = \pi/4$ respectively.

8.5 Limit Load of Rectangular Plates

When $\theta = \pi/2$, the limit load for rectangular plates can be derived from Eq.(8.13),

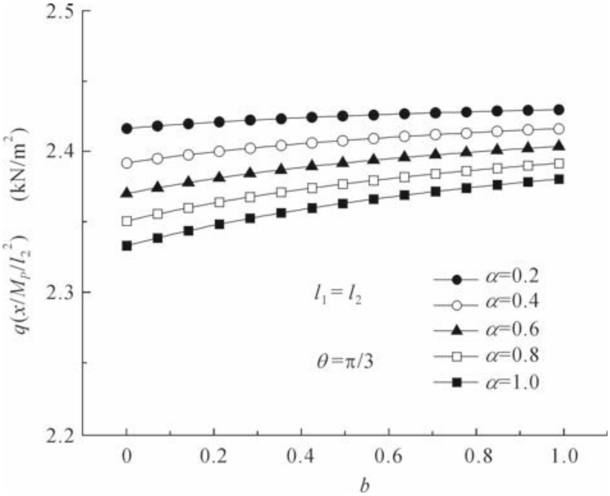

Fig. 8.10. Relations between limit load q_l and unified strength theory parameter b ($\theta = \pi/3$, $l_1 = l_2$)

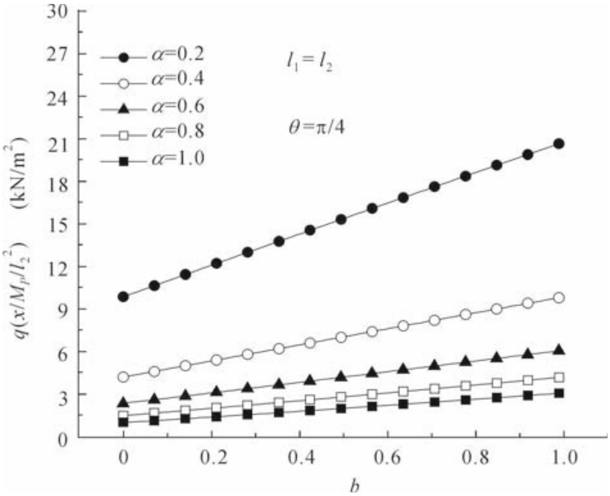

Fig. 8.11. Relations between limit load q_l and unified strength theory parameter b ($\theta = \pi/4$, $l_1 = l_2$)

$$q_l = \left[2\left(\frac{1}{l_1^2} + \frac{1}{l_2^2}\right) + \frac{2(1+b)}{(1+b+\alpha)l_1 l_2}\right] M_p. \tag{8.19}$$

If the plate consists of non-SD material ($\alpha = 1$), the limit load in Eq.(8.19) can be simplified as

$$q_l = \left[2\left(\frac{1}{l_1^2} + \frac{1}{l_2^2}\right) + \frac{2(1+b)}{(2+b)l_1 l_2}\right] M_p. \tag{8.20}$$

Figs.8.12 to 8.14 show the limit load q_l versus the unified strength theory parameter b for different rectangular plates.

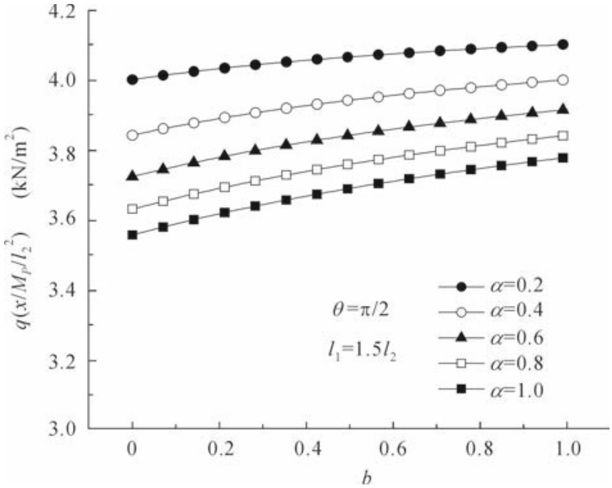

Fig. 8.12. Relations between q_l and unified strength theory parameter b ($\theta = \pi/2$, $l_1 = 1.5l_2$)

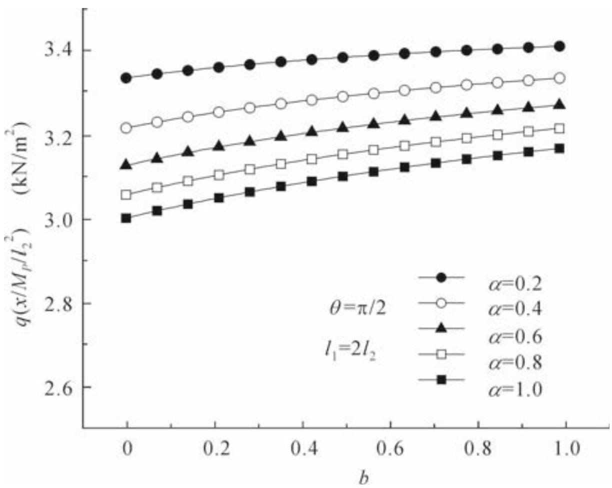

Fig. 8.13. Relations between and unified strength theory parameter b ($\theta = \pi/2$, $l_1 = 2l_2$)

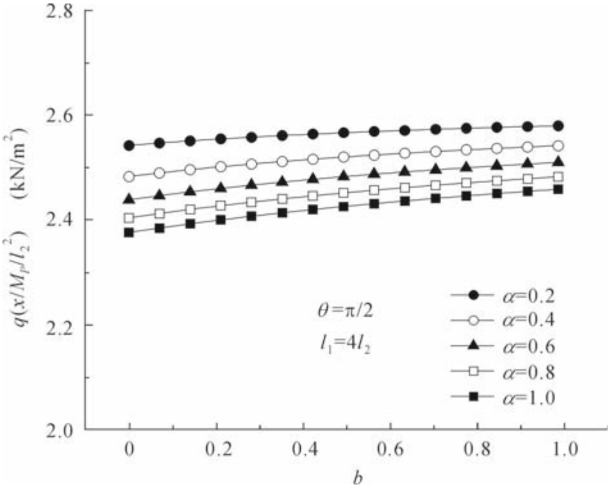

Fig. 8.14. Relations between q_l and unified strength theory parameter b ($\theta = \pi/2$, $l_1 = 4l_2$))

It is seen that both α and b have significant influences on the limit load. For a given value of α, the limit load increases with increasing parameter b. On the other hand, for a given value of b, the limit load decreases with the increase of α.

The limit load q_l versus the ratio is shown in Figs.8.15 to 8.17 for different parameter b.

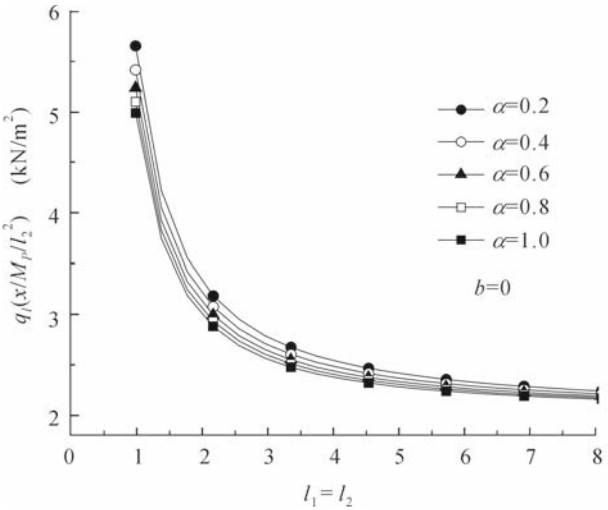

Fig. 8.15. Variation of q_l for different rectangular plates ($b = 0$)

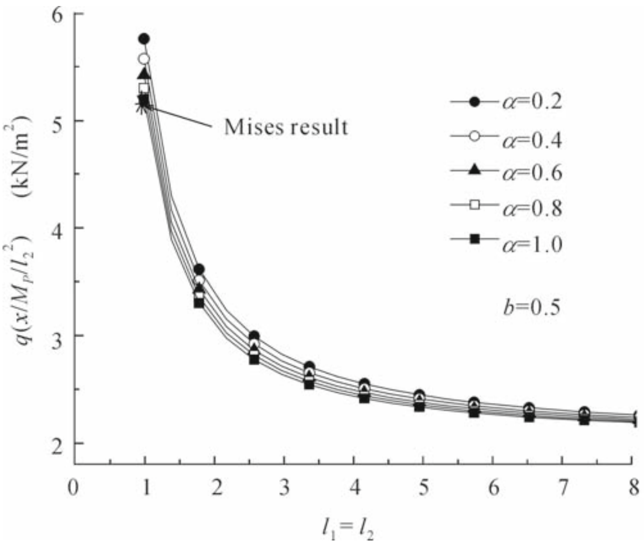

Fig. 8.16. Variation of q_l for different rectangular plates ($b = 0.5$)

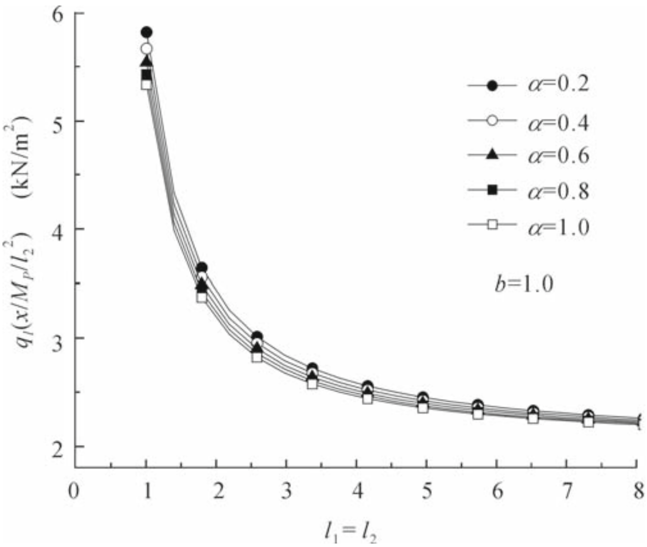

Fig. 8.17. Variation of q_l for different rectangular plates ($b = 1.0$)

It is seen that the limit load q_l decreases with the increase of the ratio l_1/l_2 and α for a given value of b. On the other hand, $q_l(M_p/l_2^2)$ approaches a constant of 2 (kN/m^2), and is independent of b and α.

8.6 Unified Limit Load of Square Plates

The limit load of square plates can be obtained by further simplifying the limit load solution in Eq.(8.10) with $l_1 = l_2$

$$q_l = 2\left[2 + \frac{(1+b)}{1+b+\alpha}\right]\frac{M_p}{l_2^2}. \tag{8.21}$$

When $\alpha = 1$, the limit load based on the twin shear yield criterion can be derived as

$$q_l = 2\left[2 + \frac{(1+b)}{2+b}\right]\frac{M_p}{l_2^2}, \tag{8.22}$$

and

$$q_l = 5.155\frac{M_p}{l_2^2}, \quad \text{when} \quad b = \frac{1}{1+\sqrt{3}}, \tag{8.23}$$

which is identical to the following solution based on the von Mises criterion (Wang, 1998)

$$q_l = 5.2\frac{M_p}{l_2^2}, \quad \text{when} \quad b = \frac{1}{2}. \tag{8.24}$$

Fig. 8.18 shows the limit load q_l with respect to the parameter b for square plates.

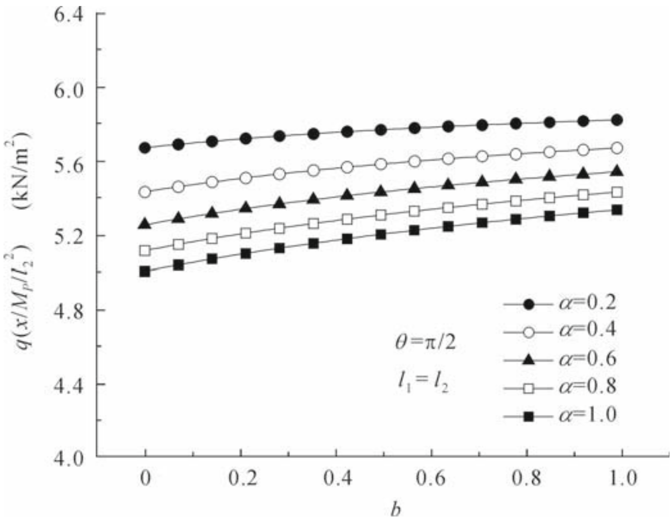

Fig. 8.18. Relations of q_l to unified strength theory parameter b ($\theta = \pi/2$, $l_1 = l_2$)

8.7 Tabulation of the Limit Load for Oblique, Rhombic and Square Plates

For convenient comparison and easier reference, the relations of the limit load $q_l(M_p/l_2^2)$ to the unified strength theory parameter b, the SD ratio α, the angle θ for the oblique plate, and the length ratio of l_1/l_2 are tabulated in Table 8.1 to Table 8.4.

Table 8.1. Relation of $q_l(M_p/l_2^2)$ to b and α with $\theta = 45°$

	α	b=0	b=0.25	b=0.5	b=0.75	b=1.00
	0.2	8.820	12.585	15.313	18.009	20.678
	0.4	4.193	5.603	7.015	8.422	8.820
$l_1/l_2 = 1.0$	0.6	2.352	3.262	4.193	5.132	6.074
	0.8	1.484	2.130	2.803	3.493	4.193
	1.0	1.000	1.484	1.998	2.531	3.078
	0.2	10.621	13.323	15.993	18.637	21.260
	0.4	5.097	6.492	7.879	8.255	10.621
$l_1/l_2 = 1.5$	0.6	3.235	4.164	5.097	6.028	6.955
	0.8	2.316	3.003	3.699	4.397	5.097
	1.0	1.778	2.316	2.865	3.420	3.978
	0.2	10.743	13.405	16.039	18.652	21.248
	0.4	5.311	6.684	8.047	8.400	10.743
$l_1/l_2 = 2.0$	0.6	3.468	4.390	5.311	6.227	7.140
	0.8	2.547	3.237	3.929	4.620	5.311
	1.0	2.000	2.547	3.099	3.652	4.206

The limit load q_l versus the ratio for different lengths of rectangular plates and the unified strength theory parameter b are listed in Table 8.4. The limit load for square plate and rectangular plates with $l_1/l_2 = 2$, $l_1/l_2 = 4$, $l_1/l_2 = 7$, $l_1/l_2 = 10$, and $l_1/l_2 = \infty$ are given. It is seen that the limit load q_l decreases with the increase of the ratio l_1/l_2 and α for a given value of b, and the limit load q_l increases with the increase in the unified strength theory parameter b in any case.

In Table 8.4, the result of $\alpha = 1$ and $b = 0$ is the same as the result for Tresca material; the result for $\alpha = b = 1$ is the same as the result for twin-shear yield criterion material; the result of $\alpha = 1$ and $b = 1/2$ is the

Table 8.2. Relation of $q_l(M_p/l_2^2)$ to b and α with $\theta = 60^{\rm o}$

	α	b=0	b=0.25	b=0.5	b=0.75	b=1.00
$l_1/l_2 = 1.0$	0.2	2.417	2.422	2.426	2.428	2.430
	0.4	2.392	2.402	2.408	2.413	2.417
	0.6	2.370	2.383	2.392	2.399	2.404
	0.8	2.351	2.366	2.377	2.386	2.392
	1.0	2.333	2.351	2.364	2.373	2.381
$l_1/l_2 = 1.5$	0.2	2.278	2.281	2.284	2.285	2.287
	0.4	2.261	2.268	2.272	2.275	2.278
	0.6	2.247	2.255	2.261	2.266	2.269
	0.8	2.234	2.244	2.252	2.257	2.261
	1.0	2.222	2.234	2.242	2.249	2.254
$l_1/l_2 = 2.0$	0.2	2.208	2.211	2.213	2.214	2.215
	0.4	2.196	2.201	2.204	2.206	2.208
	0.6	2.185	2.192	2.196	2.199	2.202
	0.8	2.175	2.183	2.189	2.193	2.196
	1.0	2.167	2.175	2.182	2.187	2.190

linear approximation of the result of the Huber-von Mises criterion. Because the Huber-von Mises criterion has a nonlinear mathematical expression, it is relatively complicated to derive analytical solutions in structural plasticity. For practical application, approximating the numerical solutions based on the Huber-von Mises criterion is adopted. UST with $\alpha = 1$ and $b = 1/2$ can be considered as a linear approximation of the Huber-von Mises yield criterion which is more suitable for the derivation of analytic solutions. The results of the Huber-von Mises criterion and the unified strength theory with $\alpha = 1$ and $1/(1+\sqrt{3})$ are very close with a percentage difference less that 3%, which is even as low as 0.87% for the plastic limit load of the square plate.

For a given value of α, the limit load q_l can be derived from Eq.(8.13) and Eq.(8.2) for different materials. As a result the material properties of the plates can be taken into account more appropriately if the UST is applied.

8.8 Summary

Based on the unified strength theory, the plastic limit analyses for oblique plates are carried out and the unified limit load is derived. It gives a se-

Table 8.3. Relation of $q_l(M_p/l_2^2)$ to b and α with $\theta = 90^o$

	α	b=0	b=0.25	b=0.5	b=0.75	b=1.00
	0.2	5.667	5.724	5.765	5.795	5.818
	0.4	5.429	5.515	5.579	5.628	5.667
$l_1/l_2 = 1.0$	0.6	5.250	5.351	5.429	5.489	5.538
	0.8	5.111	5.220	5.304	5.373	5.429
	1.0	5.000	5.111	5.200	5.273	5.333
	0.2	4.556	4.718	4.845	4.948	5.032
	0.4	4.317	4.485	4.620	4.731	4.824
$l_1/l_2 = 1.5$	0.6	4.139	4.304	4.441	4.556	4.654
	0.8	4.000	4.160	4.295	4.411	4.511
	1.0	3.889	4.043	4.175	4.289	4.389
	0.2	4.167	4.387	4.569	4.722	4.853
	0.4	3.929	4.139	4.318	4.472	4.605
$l_1/l_2 = 2.0$	0.6	3.750	3.949	4.122	4.272	4.405
	0.8	3.611	3.799	3.963	4.109	4.239
	1.0	3.500	3.676	3.833	3.974	4.100

ries of solutions covering those from the single-shear theory (Mohr-Coulomb strength theory) to the twin-shear strength theory (Yu, 1985). The unified solution of the limit load for the oblique, rhombic, rectangular, and square plates encompasses the solutions as special cases as reported by other researchers as well as a series of new solutions.

The parameter b has a significant influence on the load-bearing capacities of oblique plates and the influences vary for different conditions. The influences vary with the state of stress. When the intermediate principal stress σ_2 is close to the minimum principal stress $\sigma_3$the difference in the limit load based on different strength criteria is minimal. However, when the intermediate principal stress σ_2 is close to $\sigma_2 = (\sigma_1 + \sigma_3)/2$, the influence on the limit load is significant.

The limit load q_l for different materials and structures can be obtained when α and b vary and when the different l_1/l_2 and θ are adopted.

Table 8.4. Relations of limit load $q_l(M_p/l_2^2)$ for different rectangular plates

b	α	$l_1/l_2 = 1.0$ (square)	$l_1/l_2 = 2$	$l_1/l_2 = 4$	$l_1/l_2 = 7$	$l_1/l_2 = 10$	$l_1/l_2 = \infty$
	0.2	5.667	3.333	2.542	2.279	2.187	2.0
	0.4	5.429	3.214	2.482	2.245	2.163	2.0
0	0.6	5.250	3.125	2.438	2.219	2.145	2.0
	0.8	5.111	3.056	2.403	2.200	2.131	2.0
	1.0	5.000	3.000	2.375	2.184	2.120	2.0
	0.2	5.765	3.382	2.566	2.293	2.196	2.0
	0.4	5.579	3.289	2.520	2.266	2.178	2.0
0.5	0.6	5.429	3.214	2.482	2.245	2.163	2.0
	0.8	5.304	3.152	2.451	2.227	2.150	2.0
	1.0	5.200	3.100	2.425	2.212	2.140	2.0
$\frac{1}{1+\sqrt{3}}$	1.0	5.155(Mises)	3.077	2.414	2.206	2.135	2.0
1.0	0.2	5.818	3.409	2.580	2.301	2.202	2.0
(twin-	0.4	5.667	3.333	2.542	2.279	2.187	2.0
shear)	0.6	5.538	3.269	2.510	2.261	2.174	2.0
	0.8	5.429	3.214	2.482	2.245	2.163	2.0
	1.0	5.333	3.167	2.458	2.231	2.153	2.0

8.9 Problems

Problem 8.1 Try your hand at an application of the unified yield criterion for limit analysis of a square plate.

Problem 8.2 Try your hand at an application of the unified yield criterion for limit analysis of a rectangular plate with different l_1, l_2 and θ.

Problem 8.3 Try your hand at an application of the unified yield criterion for limit analysis of a rhombic plate with different θ.

Problem 8.4 Try your hand at an application of the unified yield criterion for limit analysis of an oblique plate with different l_1, l_2 and θ.

Problem 8.5 Why does the solution obtained by using the unified yield criterion contain all the solutions of the Tresca yield criterion, the von Mises yield criterion, the twin-shear yield criterion and other possible yield criteria adopted for those materials with the same yield stress in tension and in compression?

Problem 8.6 Write a paper regarding the plastic analysis of an oblique plate with different l_1, l_2 and θ using the unified yield criterion.

Problem 8.7 Can you introduce a unified plastic solution for an oblique plate with different l_1, l_2 and θ using the unified strength theory? The ratio of tensile strength σ_t to compressive strength σ_c is $\alpha = \sigma_t/\sigma_c = 0.8$.

Problem 8.8 A high-strength alloy has the strength ratio in tension and compression $\alpha = 0.9$. Find the unified solution for a square plate made of this alloy.

Problem 8.9 A high-strength alloy has the strength ratio in tension and compression $\alpha = 0.9$. Find the unified solution for a rectangular plate made of this alloy.

Problem 8.10 A high-strength alloy has the strength ratio in compression and tension $\alpha = 0.9$. Find the unified solution for a rhombic plate made of this alloy.

Problem 8.11 A high-strength alloy has the strength ratio in compression and tension $\alpha = 0.9$. Find the unified solution for an oblique plate made of this alloy.

Problem 8.12 Compare the plastic solutions of a square plate using the unified yield criterion and the unified strength theory with $\alpha = 0.8$.

Problem 8.13 Compare the plastic solutions of a rectangular plate using the unified yield criterion and the unified strength theory with $\alpha = 0.8$.

Problem 8.14 Compare the plastic solutions of a rhombic plate using the unified yield criterion and the unified strength theory with $\alpha = 0.8$.

Problem 8.15 Compare the plastic solutions of an oblique plate using the unified yield criterion and the unified strength theory with $\alpha = 0.8$.

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