5

Plastic Limit Analysis for Simply Supported Circular Plates

5.1 Introduction

The circular plate has been used widely as an important structural element in many branches of engineering. Reliable prediction of the load-bearing capacity of circular plates is crucial for optimum structural design. The load-bearing capacity of circular plates by using the Tresca yield criterion and Huber-von Mises criterion has been given by Hopkins and Wang (1954), Hopkins and Prager (1954), and Ghorashi (1994), et al. The design of circular plates based on the plastic limit load was discussed by Hu (1960). Nine cases including a simply supported circular plate, clamped circular plate, annular plate, a built-in at inner edge and simply supported along the outer edge plate, shearing force along the outer edge and built-in at the inner edge, etc. were studied (Hu, 1960). A systematical summary was given by Mroz and Sawczuk (1960), Hodge (1959; 1963), Save and Massonnet (1972), Zyczkowski (1981), Save (1985) and Save et al. (1997).

Huang et al. (1989) applied the twin-shear stress criterion to derive the plastic limit transverse pressure for the simply supported circular plate. Previous studies showed that the limit analysis method is effective for the analysis of circular plates in the plastic limit state. However, the Tresca criterion, Huber-von Mises criterion, and twin-shear stress criterion are only applicable for certain materials. For instance, the Tresca criterion requires the shear strength and tensile strength of the material to satisfy the relation $\tau_s = 0.5\sigma_s$; the Huber-von Mises criterion is suitable for materials with $\tau_s = 0.577\sigma_s$, and the twin-shear stress criterion is valid for the materials with $\tau_s = 0.677\sigma_s$. All one of the solutions mentioned above is single solution adopted for only one kind of material.

A new unified solution to the plastic limit of a simply supported circular plate by using of the unified yield criterion was presented by Ma and He (1994), Ma et al. (1993; 1994; 1995a; 1995b; 1999). Unified plastic limit analysis of metal circular plates subjected to border uniformly distributed loading was derived by Wang et al. (2002). The unified solution can be adapted for more kinds of non-SD materials. The unified solution for simply supported circular plates using SD materials was derived by Wang and Yu (2002; 2003). The unified plastic limit of the plate for non-SD materials is a special case of the unified solution of the plate for SD materials, such as rock and concrete materials (Chen, 1975; 1981; 1988).

In this chapter, plastic limit analyses of simply supported circular plates with non-SD materials and SD materials under various transverse loading using the unified yield criterion and the unified strength theory are presented. Exact and unified solutions of the plastic limit load, moment field and velocity field in the plastic limit state are derived. The moment field and velocity field with respect to the Tresca criterion, the Huber-von Mises criterion (closed-form solution), and the twin-shear stress criterion are compared. This chapter presents an effective analytical method to compute the exact plastic limit load for circular plates using a piecewise linear yield criterion.

5.2 Basic Equations of Circular Plate

When considering a circular plate of radius a and thickness h subjected to axisymmetric transverse loading $P(r)$, a stress element of the circular plate is considered (Fig.5.1). Because of the axisymmetry of the structure and the loading, the non-zero stresses are the radial stress σ_r, the circumferential stress σ_θ, and the shear stress $\tau_{rz} = \tau_{rz}$. In the plastic limit state, the generalized stresses can be expressed as (Hodge, 1963; Chakrabarty, 1987)

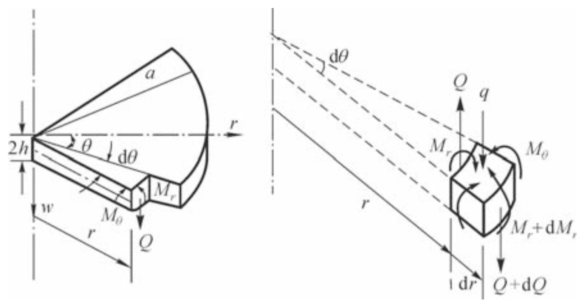


Fig. 5.1. Internal forces in a circular plate element

$$M_r = \int_{-h/2}^{h/2} \sigma_r z \mathrm{d}z, \quad M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z \mathrm{d}z, \tag{5.1a}$$

$$Q_{rz} = \int_{-h/2}^{h/2} \tau_{rz} \mathrm{d}z, \quad M_0 = \int_{-h/2}^{h/2} \sigma_y z \mathrm{d}z = \sigma_y h^2/4, \tag{5.1b}$$

where M_r, M_θ and M_0 are the radial, circumferential, and ultimate (fully plastic) bending moments, respectively; Q_{rz} is the transverse shear force which is generally not assumed to influence the plastic yielding. Defining dimensionless variables of $r = R/a$, $m_r = M_r/M_0$, $m_\theta = M_\theta/M_0$, $p(r) = P(r)a^2/M_0$, the equilibrium equation of a circular plate subjected to a constant uniform load can be written with reference to the axisymmetric condition as

$$\mathrm{d}(rm_r)/\mathrm{d}r - m_\theta = -\int_0^r p(r)r\mathrm{d}r, \tag{5.2}$$

where $p(r)$ is the transversely distributed loading per unit area.

The equilibrium equation for a uniformly-loaded circular plate can be simplified as

$$\mathrm{d}(rm_r)/\mathrm{d}r - m_\theta = -\frac{p}{2}r^2. \tag{5.3}$$

The relations between the curvature rate and the rate of deflection are

$$\dot{k}_r = -\mathrm{d}^2\dot{w}/\mathrm{d}r^2 \quad \text{and} \quad \dot{k}_\theta = -\mathrm{d}\dot{w}/(r\mathrm{d}r), \tag{5.4}$$

where $\dot{w}$, $\dot{k}_r$ and $\dot{k}_\theta$ are non-dimensional deflection rate, non-dimensional curvature rates in radial, and circumferential directions, respectively. The dimensionless deflection is defined as $w = W/a$, where W is the actual deflection, and a is the radius of circular plate. According to the associated flow rule,

$$\dot{k}_r = \dot{\lambda}\partial F/\partial m_r,\ \dot{k}_\theta = \dot{\lambda}\partial F/\partial m_\theta, \tag{5.5}$$

where $\dot{\lambda}$ is a plastic flow factor, F is plastic potential which is the same as the yield function according to the associated flow rule.

5.3 Unified Solutions of Simply Supported Circular Plate for Non-SD Materials

The plate is assumed to be made of a rigid-perfectly-plastic material, which satisfies the unified yield criterion. Fig.5.2 shows the generalized unified yield criterion in terms of m_r and m_θ. Fig.5.3 illustrates the flow vector of the curvature velocity at the corners when the unified yield criterion parameter b=0.5. The unified yield criterion is a piecewise linear function, and it has the form of

$$m_\theta = a_i m_r + b_i, \quad (i = 1, ..., 12). \tag{5.6}$$

Table 5.1 lists the respective constants a_i and b_i for the five lines L_i $(i = 1, ..., 5)$ of AB, BC, CD, DE and EF in Fig.5.2.

Substituting the yield criterion into Eq.(5.3) and then integrating Eq.(5.3), the radial moment m_r located on the segments L_i is obtained as

$$m_r = \frac{b_i}{1 - a_i} - \frac{pr^2}{2(3 - a_i)} + c_i r^{-1+a_i}, \quad (i = 1, ..., 5), \tag{5.7}$$

where c_i (i=1, ..., 5) are integral constants to be determined by boundary and continuity conditions. The field of velocity corresponding to each side L_i is obtained by equating Eq.(5.4) and Eq.(5.5). Considering the yield condition Eq.(5.6), the velocity field is integrated as

$$\dot{w} = \dot{w}_0 (c_{1i} r^{1-a_i} + c_{2i}), \quad (i = 1, ..., 5), \tag{5.8}$$

where c_{1i}, c_{2i} (i=1, ..., 5) are the integral constants, $\dot{w}_0$ is the velocity at the plate center.

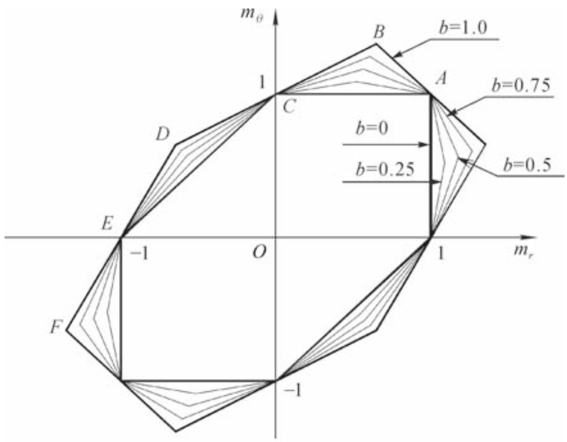


Fig. 5.2. Internal forces in a circular plate element

Table 5.1. Constants a_i and b_i in the unified yield criterion

	AB $(i = 1)$	BC $(i = 2)$	CD $(i = 3)$	DE $(i = 4)$	EF $(i = 5)$
a_i	$-b$	$b/(1+b)$	$1/(1+b)$	$1+b$	$(1+b)/b$
b_i	$1+b$	1	1	$1+b$	$(1+b)/b$

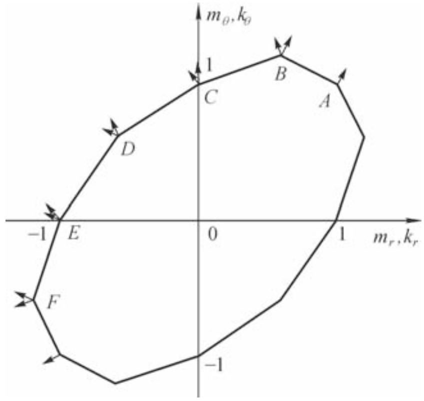


Fig. 5.3. Internal forces in a circular plate element

5.3.1 Uniformly Distributed Load

For a simply supported circular plate under uniformly distributed pressure (Fig.5.4), in the plastic limit state, moments at the center (r=0) of the simply supported circular plate satisfy $m_r = m_\theta$=1 (point A at the yield curve in Fig.5.2). According to the boundary condition of the plate and the requirement of stable flow of the plastic strains (Hodge, 1963), moments at the simply supported edge ($r = 1$) satisfy m_r =0 and m_θ =1 (point C at the yield curves in Fig.5.2). Bending moments at each point in the plate are located on the sides AB and BC in view of the normality requirement of plasticity.

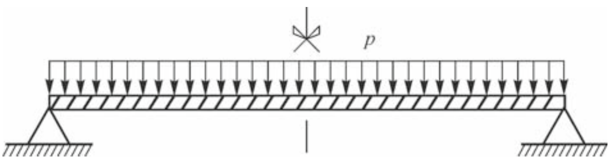


Fig. 5.4. Simply supported circular plate under uniformly distributed load

Assuming r_1 is a non-dimensional radius of a ring where the moments correspond to point B in Fig.5.2, the boundary conditions and continuity conditions can be put as: (1) $m_r(r = 0)$=1; (2) $m_r(r = r_1)$ is continuous and equals to (1+b)/(2+b); (3) $m_r(r = 1)$=1; (4)$\dot{w}(r = 0) = \dot{w}_0$; (5)$\dot{w}(r = r_1)$ and $\mathrm{d}\dot{w}/\mathrm{d}r(r = r_1)$ are continuous; (6)$\dot{w}(r = 1) = 0$. The integral coefficients c_1, c_2, c_{11}, c_{12}, and c_{22} in Eq.(5.7) and Eq.(5.8) are then derived as

$$\begin{cases} c_1 = 0,\ c_2 = -(1+b) + \dfrac{1+b}{2+b}\dfrac{3+b}{3+2b}r_1^{-2}, \\ c_{11} = -\dfrac{r_1^{-b(2+b)/(1+b)}}{(1+b)^2 - (2b+b^2)r_1^{1/(1+b)}}, \\ c_{21} = 1, \\ c_{12} = -c_{22} = -\dfrac{(1+b)^2}{(1+b)^2 - (2b+b^2)r_1^{1/(1+b)}}. \end{cases} \tag{5.9}$$

The plastic limit load p is derived as

$$p = \frac{6+2b}{2+b}\frac{1}{r_1^2}, \tag{5.10}$$

where r_1 satisfies the equation of

$$-(3+2b)(2+b) + (3+b)r_1^{-2} + 2b(2+b)r_1^{1/(1+b)} = 0. \tag{5.11}$$

Eq.(5.11) can be solved by half-interval search of r_1 in the interval of (0, 1) for a given value of b in the range of 0 to 1. The convergence with sufficient accuracy of Eq.(5.11) gives the approximation of r_1.

For a special case when b=0, the plastic solution becomes

$$\begin{cases} m_r = 1 - r^2\,,\ m_\theta = 1, \\ \dot{w} = \dot{w}_0(1-r), \\ p = 6\,,\ r_1 = 1\big/\sqrt{2}, \end{cases} \tag{5.12}$$

which is the same as those given by other researchers using the maximum principal stress criterion and the Tresca criterion (Hopkins and Prager, 1953; Hodge, 1963). Figs.5.5 and 5.6 show the moment fields and velocity fields of a simply supported circular plate with respect to three different criteria, namely, the Tresca criterion (b=0), the Huber-von Mises criterion (b=0.5), and the maximum deviatoric stress criterion or the twin-shear stress criterion (b=1). The plastic limit load p corresponding to the three criteria are 6.000, 6.489, and 6.839, respectively, when b equals 0, 0.5 or 1. The plastic limit load versus unified yield criterion parameter b of a simply supported circular plate is shown in Fig.5.6. The plastic limit load with respect to the Huber-von Mises criterion obtained by Hopkins and Wang (1954) is 6.51, which is very close to the present result with b=0.5. The upper bound limit load derived from the expanded Tresca hexagon which circumscribes the Huber-von Mises ellipse (Hopkins and Wang, 1954) is 6.83. The plastic limit loads obtained with b=0.5 and b=1 of the unified yield criterion differ considerably from that with b=0 by 8.15% and 14.0%, respectively.

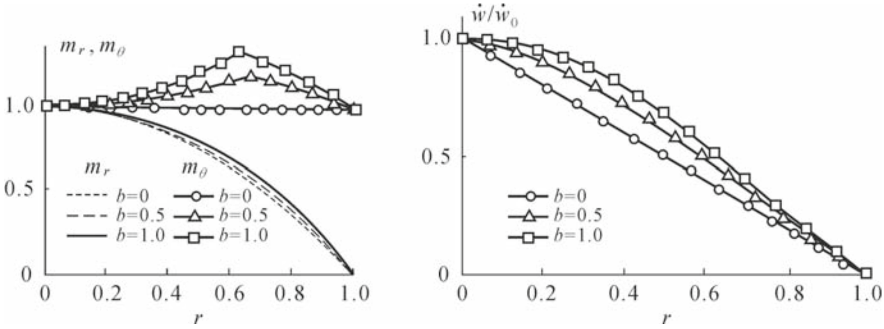


Fig. 5.5. Moment fields and velocity fields of simply supported circular plate

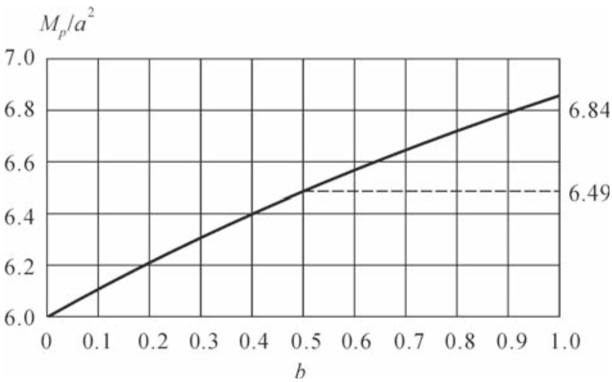


Fig. 5.6. Plastic limit load versus unified yield criterion parameter b of simply supported circular plate

It is seen that the choice of strength theory has a significant influence on the results of elasto-plastic analysis and the load-bearing capacities of a simply supported circular plate for non-SD materials. The unified yield criterion provides us with an effective approach to study these effects.

5.3.2 Arbitrary Axisymmetrical Load

This section presents the exact solution of a circular plate under an arbitrarily distributed axisymmetrical load. The plastic solution of a simply supported circular plate with a varying loading radius of the partial-uniform pressure in Fig.5.7(a) and the arbitrary loading variation in Fig.5.7(b) under different boundary conditions are discussed.

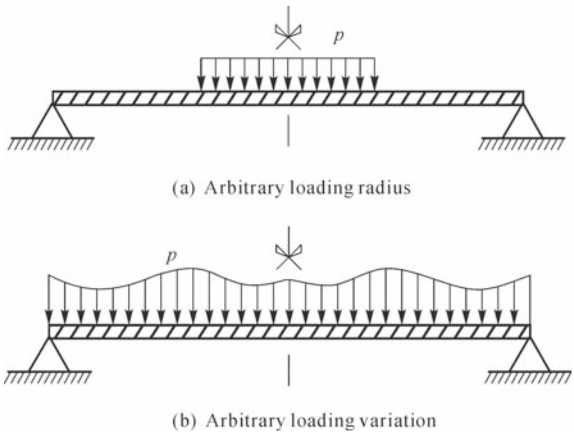


Fig. 5.7. Plastic limit load versus unified yield criterion parameter b of simply supported circular plate

5.3.2.1 Arbitrary Loading Radius

Defining dimensionless variables, $r = R/a, m_r = M_r/M_0, m = M/M_0$, and $q(r) = p(r)a^2/M_0$, the equilibrium equation of a circular plate subjected to a constant uniform load can be derived with reference to the axisymmetric condition as

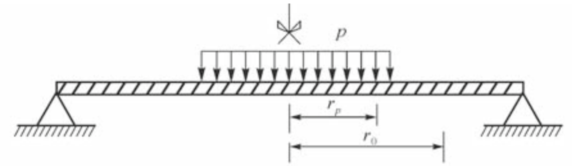
$$\mathrm{d}(rm_r)/\mathrm{d}r - m_\theta = -pr^2/2, \quad 0 \leqslant r \leqslant r_p, \tag{5.13}$$

$$\mathrm{d}(rm_r)/\mathrm{d}r - m_\theta = -pr_p^2/2, \quad r_p \leqslant r \leqslant 1, \tag{5.14}$$

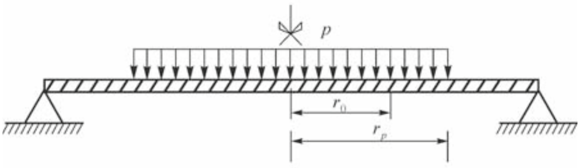
where $r_p = R_p/a$ is the normalized loading radius of the circular plate; R_p is the loading radius. $r_p = 1$ implies that the entire plate is uniformly loaded, whereas $r_p = 0$ indicates a point loaded at the center. When the unified yield criterion expressed by generalized stresses (Fig.5.2) is used, the expression of the limit condition is the same as that in Eq.(5.6).

For a circular plate under arbitrarily distributed load, the center point of the plate satisfies $m_r = m_\theta$ =1 (point A in Fig.5.2), and the simply supported boundary condition leads to $m_r = 0$ (point C in Fig.5.2). Stress states of all the points in the plate are still on the parts AB and BC. There are two possible cases, i.e., Case (1) $r_p \leqslant r_0$ and Case (2) $r_p > r_0$, where r_0 is the radius of the ring with the moments corresponding to the yield point B in Fig.5.2. These two cases are illustrated in Fig.5.8.

From the boundary conditions and continuity conditions, there are: (1) $m_r = 1$ at $r = 0$; (2) $m_r = 0$ at $r = 1$; (3) m_r is continuous at $r = r_p$; (4) $m_r = d_1 = (1 + b)/(2 + b)$ and is continuous at $r = r_0$. These conditions will be used to derive the integration coefficients in the moment equations.



(a) Case (1)



(b) Case (2)

Fig. 5.8. Plastic limit load versus unified yield criterion parameter b of simply supported circular plate

Case (1)
The moment fields of the plate corresponding to the first case can be integrated as

$$m_r = \frac{b_1}{1-a_1} - \frac{pr^2}{2(3-a_1)} + c_1 r^{-1+a_1}, \quad 0 \leqslant r \leqslant r_p, \tag{5.15a}$$

$$m_r = \frac{b_1}{1-a_1} - \frac{pr_p^2}{2(1-a_1)} + c_2 r^{-1+a_1}, \quad r_p \leqslant r \leqslant r_0, \tag{5.15b}$$

$$m_r = \frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 r^{-1+a_2}, \quad r_0 \leqslant r \leqslant 1, \tag{5.15c}$$

where c_1, c_2, and c_3 are integration coefficients. The integration coefficients, the plastic limit load p, and the demarcating radius r_0 are derived with application of the boundary and continuity conditions as

$$c_1 = 0, \tag{5.16a}$$

$$\frac{b_1}{1-a_1} - \frac{pr_p^2}{2(3-a_1)} = \frac{b_1}{1-a_1} - \frac{pr_p^2}{2(1-a_1)} + c_2 r_p^{-1+a_1}, \tag{5.16b}$$

$$\frac{b_1}{1-a_1} - \frac{pr_p^2}{2(1-a_1)} + c_2 r_0^{-1+a_1} = d_1, \tag{5.16c}$$

$$\frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 r_0^{-1+a_2} = d_1, \tag{5.16d}$$

$$\frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 = 0. \tag{5.16e}$$

When the loading radius r_p is specified, the unknowns c_2, c_3, p and r_0 in the above simultaneous equations can be derived as

$$c_3 = -\frac{d_1}{1-r_0^{-1+a_2}}, \tag{5.17}$$

$$p = \frac{2b_2}{r_p^2} - \frac{2(1-a_2)d_1}{(1-r_0^{-1+a_2})r_p^2}, \tag{5.18}$$

$$c_2 = \frac{1+a_1}{(1-a_1)(3-a_1)r_p^{-1+a_1}}\left[b_2 - \frac{(1-a_2)d_1}{1-r_0^{-1+a_2}}\right], \tag{5.19}$$

where the demarcating radius r_0 can be calculated from the equation of

$$\begin{aligned}&\frac{b_1}{1-a_1} - d_1 - \left[-\frac{1-a_2}{1-a_1}\frac{d_1}{1-r_0^{-1+a_2}} + \frac{b_2}{1-a_1}\right] \\ &\quad + \frac{1+a_1}{(1-a_1)(3-a_1)r_p^{-1+a_1}}\left[b_2 - \frac{(1-a_2)d_1}{1-r_0^{-1+a_2}}\right] r_0^{-1+a_1} = 0.\end{aligned} \tag{5.20}$$

When the derived values are substituted into Eqs.(5.15a)∼(5.15c), the moment fields of the plate are calculated.

Case (2)

The corresponding moment fields of Case (2) can be derived as

$$m_r = \frac{b_1}{1-a_1} - \frac{pr^2}{2(3-a_1)} + c_1 r^{-1+a_1}, 0 \leqslant r \leqslant r_0, \tag{5.21a}$$

$$m_r = \frac{b_2}{1-a_2} - \frac{pr^2}{2(3-a_2)} + c_2 r^{-1+a_2}, r_0 \leqslant r \leqslant r_p, \tag{5.21b}$$

$$m_r = \frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 r^{-1+a_2}, r_p \leqslant r \leqslant 1. \tag{5.21c}$$

With the same boundary and continuity conditions as for Case (1), the integral coefficients c_1, c_2 and c_3, the plastic limit load p, and the demarcating radius r_0 satisfy

$$c_1 = 0, \tag{5.22a}$$

$$\frac{b_1}{1-a_1} - \frac{pr_0^2}{2(3-a_1)} = d_1, \tag{5.22b}$$

$$\frac{b_2}{1-a_2} - \frac{pr_0^2}{2(3-a_2)} + c_2 r_0^{-1+a_2} = d_1, \tag{5.22c}$$

$$\frac{b_2}{1-a_2} - \frac{pr_p^2}{2(3-a_2)} + c_2 r_p^{-1+a_2} = \frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 r_p^{-1+a_2}, \tag{5.22d}$$

$$\frac{b_2}{1-a_2} - \frac{pr_p^2}{2(1-a_2)} + c_3 = 0. \tag{5.22e}$$

The integral coefficients, the plastic limit load and the demarcating radius can be derived as

$$p = \left(\frac{b_1}{1-a_1} - d_1\right)\frac{2(3-a_1)}{r_0^2}, \tag{5.23a}$$

$$c_2 = \left[\left(\frac{b_1}{1-a_1} - d_1\right)\frac{3-a_1}{3-a_2} - \left(\frac{b_2}{1-a_2} - d_1\right)\right] r_0^{1-a_2}, \tag{5.23b}$$

$$c_3 = -\frac{b_2}{1-a_2} + \left(\frac{b_1}{1-a_1} - d_1\right)\frac{3-a_1}{1-a_2}\frac{r_p^2}{r_0^2}, \tag{5.23c}$$

where r_0 can be numerically solved from Eq.(5.22d) by substituting Eqs.(5.23a) $\sim$(5.23c) into Eq.(5.22d).

Assuming that $r_p = r_0 = r_{p0}$, the two cases are identical. The value of r_{p0} in this special case can be derived as $r_{p0} = 1/2^{1+b}$ by solving Eqs.(5.18) and (5.22a) with application of $r_p = r_0 = r_{p0}$. When $r_p \leqslant r_{p0}$, the equations derived in Case (1) are adopted. On the other hand, when $r_p > r_{p0}$, the counterparts in Case (2) are adopted. Moment fields with six different values of loading radius r_p, i.e., 1.0, 0.75, 0.5, 0.25, 0.1 and 0.00001 are shown in Fig. 5.9. It can be seen that the moment field varies with the unified yield criterion parameter b. The plastic limit load increases with the increase of b. Table 5.2 gives the plastic limit load p with respect to different values of r_p and b.

Table 5.2. Plastic limit loads with different values of r_p

Criterion	$r_p = 1$	$r_p = 0.75$	$r_p = 0.5$	$r_p = 0.25$	$r_p = 0.1$	$r_p = 0.00001$
$b = 0.0$ (Tresca)	6.0000	7.1111	12.000	38.400	214.29	2×10^{10}
$b = 0.5$(Mises)	6.4887	7.6666	12.886	40.901	224.70	2×10^{10}
$b = 1.0$ (Twin-shear)	6.8392	8.0638	13.509	42.669	232.69	2×10^{10}

According to the hypothesis of the maximum principal stress condition or the Tresca condition, m_θ is equal to 1 in the whole circular plate regardless of the variation in the loading radius r_p. It is identical to the results of the unified yield criterion with $b = 0$, which is obviously unreasonable. When $0 < b \leqslant 1$, m_θ varies with the radius variable r and the loading radius r_p. Compared with that $b = 0$ or with the Tresca criterion, the varying tendency of m_θ seems more reasonable.

When r_p approaches zero, the problem is approximately the case of the circular plate under concentrated load at the center. The unified yield criterion reflects the moment singularity at the center of the circular plate under a concentrated load.

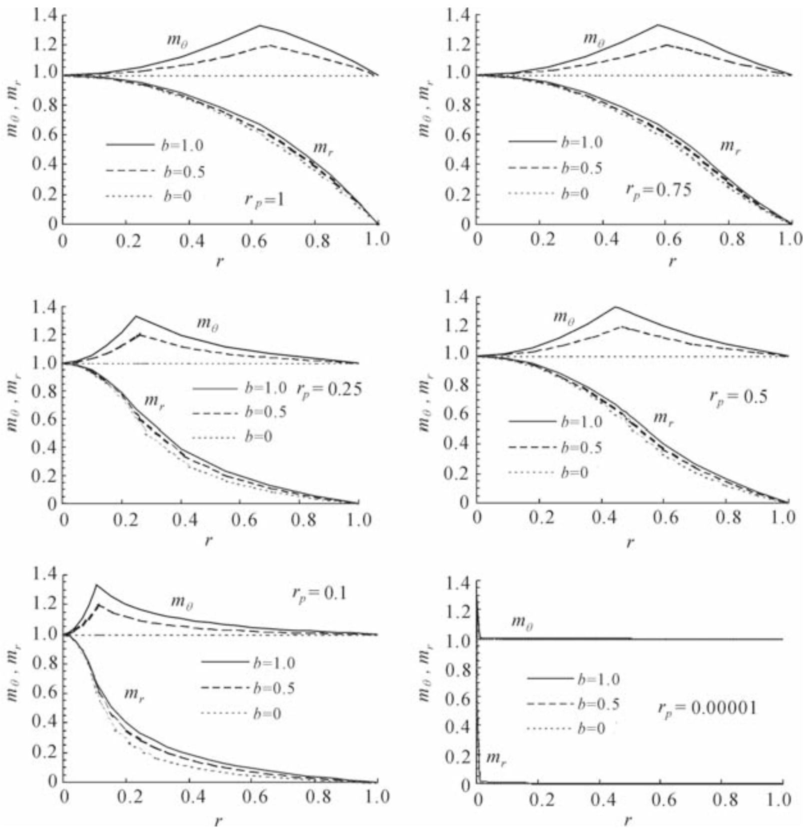


Fig. 5.9. Moment fields with different loading radii

The boundary conditions, continuity conditions, and the velocity field when the plate is subjected to a partial-uniform load are the same as those when the plate is under a uniformly distributed load, except that the demarcating radius r_0 is a function of the loading radius r_p. The plastic limit load derived in the present study satisfies the equilibrium conditions and the yield conditions. The velocity field of deflection which is compatible with the motion mechanism, is obtained. Therefore the solution of a plastic limit load given here is a complete solution. Velocity profiles corresponding to six different values of loading radius r_p, namely r_p=1, r_p=0.75, r_p=0.5, r_p=0.25, r_p=0.1, and r_p=0.00001 are plotted in Fig.5.10.

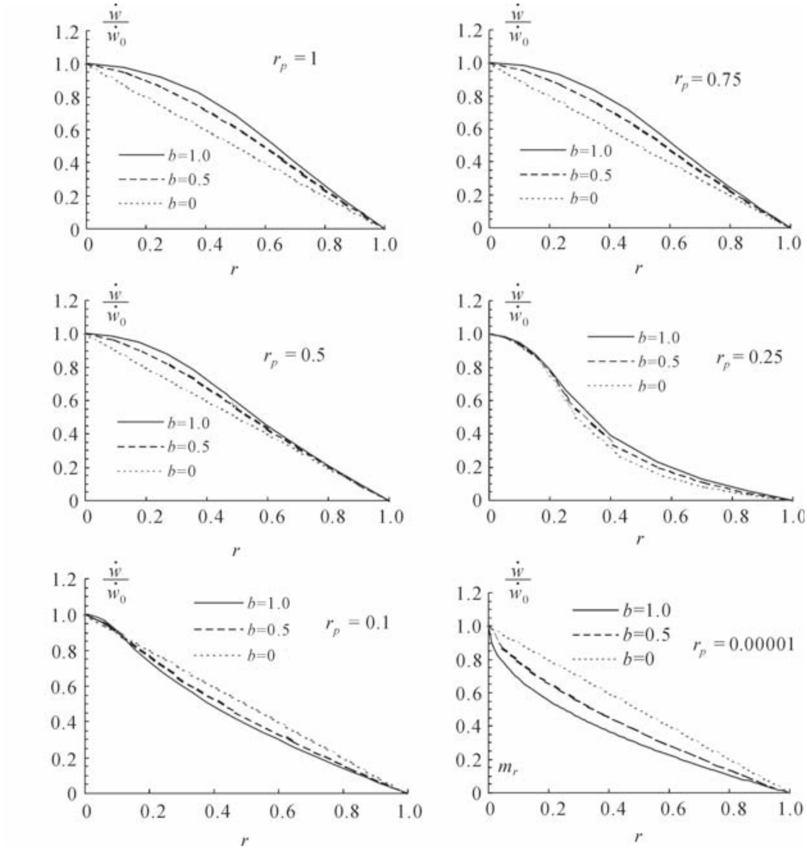


Fig. 5.10. Moment fields with different loading radii

It is seen that the velocity field with b=0 is independent of the loading radius r_p which is always a straight line and is not smooth at the plate center. The velocity field derived with the yield conditions with $0 < b \leqslant 1$, however, are functions of r_p and are smoothly connected at the plate center. This indicates that the velocity field in the plastic limit state in terms of the unified yield criterion is again more reasonable than that based on the Tresca criterion.

To verify the current results, the plastic limit solutions with specific parameters in the equations are explicitly given and compared with reported results:

(1) When $b = 0$, the moment field, velocity field, and the plastic limit load are obtained as

$$m_r = 1 - \frac{r^2}{(3 - 2r_p)r_p^2}, \; m_\theta = 1, \quad \text{when} \quad 0 \leqslant r \leqslant r_p, \tag{5.24a}$$

$$m_r = 1 - \frac{3r - 2r_p}{3 - 2r_p}, \; m_\theta = 1, \quad \text{when} \quad r_p \leqslant r \leqslant 1, \tag{5.24b}$$

$$\dot{w} = \dot{w}_0(1 - r), \tag{5.24c}$$

$$p = \frac{6}{(3 - 2r_p)r_p^2}. \tag{5.24d}$$

This result is identical to the solution in terms of the Tresca criterion given by Hodge (1963).

(2) When $r_p = 1$, the whole plate is under a uniformly distributed load. The solution is exactly the same as that in Section 5.3.1.

(3) Denoting P_T as the total load on the plate, i.e., $P_T = \pi r_p^2 p$, when r_p approaches zero, it approximates to a concentrated loading case. From the solution of the first case, it can be derived that $\lim\limits_{r_p \to 0} P_T \equiv 2\pi$ which is independent of the variable b. It agrees with the results using the Tresca criterion and the Huber-von Mises criterion (Hodge, 1963).

5.3.2.2 Arbitrary Loading Distribution

Defining dimensionless variable $p(r) = P(r)a^2/M_0$ for a circular plate of radius a and thickness h subjected to an arbitrarily distributed axisymmetrical transverse pressure $\mu P(r)$, where μ is a plastic limit load factor, and $P(r)$ is a load distribution function, the equilibrium equation of the circular plate can be written with application of the axisymmetric condition as

$$\mathrm{d}(rm_r)/\mathrm{d}r - m_\theta = -\int \mu p(r) r \mathrm{d}r. \tag{5.25}$$

Substituting the yield criterion into Eq.(5.25) and then integrating Eq. (5.25), m_r located on the segments L_i are obtained as follows:

$$m_r = \frac{b_i}{1 - a_i} - r^{-1+a_i} \int r^{-a_i} [\int \mu p(r) \mathrm{d}r] \mathrm{d}r + c_i r^{-1+a_i}, \; (i = 1, ..., 5), \tag{5.26}$$

where c_i $(i = 1, ..., 5)$ are integration constants and can be determined from the continuity and boundary conditions.

Assuming the load function $p(r) = \sum\limits_{j=1}^{\infty} p_j r^{j-1}$, Eq.(5.26) becomes

$$m_r = \frac{b_i}{1-a_i} - \mu \sum_{j=1}^{\infty} p_j \frac{r^{j+1}}{(j+1)(j+2-a_i)} + c_i r^{-1+a_i}. \quad (i = 1, ..., 5) \quad (5.27)$$

The field of velocity corresponding to the five sides L_i is obtained as

$$\dot{w} = \dot{w}_0(c_{1i} r^{1-a_i} + c_{2i}) \quad (i = 1, ..., 5), \quad (5.28)$$

where c_{1i} and c_{2i} $(i = 1, ..., 5)$ are integration constants, and $\dot{w}_0$ is the velocity at the plate center.

The plastic limit load of a plate is always taken to be the total limit load on the plate. The dimensionless total limit load of the plate is obtained as

$$P_T = 2\pi \int_0^1 \mu p(r) r \mathrm{d}r \quad \text{or} \quad P_T = 2\pi\mu \sum_{j=1}^{\infty} \frac{p_j}{j+1}. \quad (5.29)$$

In the plastic limit state, moments at the center $(r = 0)$ of a simply supported circular plate satisfy $m_r = m_\theta = 1$ (point A on the yield curves in Fig. 5.2), and moments at the simply supported edge $(r = 1)$ satisfy $m_r = 0$ and $m_\theta = 1$ (point C on the yield curves in Fig. 5.2). Bending moments of other points in the plate are on the lines AB and BC according to the normality requirement of plasticity. Thus, index "i" in Eqs.(5.27) and (5.28) for a simply supported circular plate takes values of 1 or 2 only corresponding to the line AB or the line BC in Fig.5.2, respectively. Assuming r_1 is a non-dimensional radius of a ring where the moments exactly correspond to point B in Fig.5.2, the boundary and continuity conditions can then be described as: (1) $m_r(r = 0) = 1$; (2) $m_r(r = r_1)$ is continuous and equal to d_1; (3) $m_r(r = 1) = 1$; (4) $\dot{w}(r = 0) = \dot{w}_0$; (5) $\dot{w}(r = r_1)$ and $\mathrm{d}\dot{w}/\mathrm{d}r(r = r_1)$ are continuous, (6) $\dot{w}(r = 1) = 0$. Accordingly, the integral coefficients c_1, c_2, c_{11}, c_{12}, c_{21} and c_{22} in Eqs.(5.27) and (5.28) can be determined as

$$c_1 = 0, \; c_2 = -\frac{b_2}{1-a_2} + \mu \sum_{j=1}^{\infty} \frac{p_j}{(j+1)(j+2-a_2)}, \quad (5.30)$$

$$c_{11} = -\frac{r_1^{-b(2+b)/(1+b)}}{(1+b)^2 - (2b+b^2) r_1^{1/(1+b)}}, \; c_{21} = 1, \quad (5.31)$$

$$c_{12} = -c_{22} = -\frac{(1+b)^2}{(1+b)^2 - (2b+b^2) r_1^{1/(1+b)}}. \quad (5.32)$$

The loading factor μ is derived as

$$\mu = \frac{-d_1 + \frac{b_1}{1-a_1}}{\sum\limits_{j=1}^{\infty} \frac{p_j r_1^{j+1}}{(j+1)(j+2-a_1)}}, \quad (5.33)$$

where r_1 satisfies the equation

$$d_1 = \frac{b_2}{1-a_2} - \mu \sum_{j=1}^{\infty} \frac{p_j r_1^{j+1}}{(j+1)(j+2-a_2)} + c_2 r_1^{-1+a_2}. \tag{5.34}$$

The above equation is solved by a half-interval search method for r_1 in the interval (0, 1) for a given value of b between 0 and 1. Substituting the value of r_1 into Eqs.(5.30)∼(5.33), the moments and velocity distributions in Eqs.(5.27) and (5.28) can then be derived.

For a special case of b=0, the plastic solution becomes

$$\begin{aligned} m_r &= 1 - \mu \sum_{j=1}^{\infty} \tfrac{p_j}{(j+1)(j+2)} r^{j+1},\ m_\theta = 1, \\ \dot{w} &= \dot{w}_0(1-r) \text{ and } \mu = \frac{1}{\sum_{j=1}^{\infty} \frac{p_j}{(j+1)(j+2)}}, \end{aligned} \tag{5.35}$$

which are the same as those given by Ghorashi (1994) using the maximum principal stress criterion and the Tresca criterion. If a uniformly distributed load is applied, it becomes

$$m_r = 1 - r^2,\ m_\theta = 1,\ \dot{w} = \dot{w}_0(1-r) \text{ and } \mu = 6, \tag{5.36}$$

which are identical to the results given by Hodge (1963).

Table 5.3 lists the plastic limit load factors and the total limit loads of the simply supported circular plate for five linearly distributed load functions in terms of the three particular yield criteria, namely, the Tresca criterion, the Huber-von Mises criterion (approximated by the unified yield criterion with b=0.5), and the twin shear stress criterion.

Table 5.3. Plastic limit loads for linearly distributed load

$p(r)$	r		$1+r$		1		$2-r$		$1-r$	
		P_T		P_T		P_T		P_T		P_T
$b=0$	12.00	25.13	4.00	20.94	6.00	18.85	4.00	16.76	12.01	12.57
$b=1/2$	12.78	26.78	4.31	22.56	6.49	20.38	4.34	18.17	13.02	13.64
$b=1$	13.36	27.98	4.53	23.73	6.84	21.49	4.58	19.17	13.75	14.40

From Table 5.3, different yield criteria make differences in the plastic limit load factor, the total limit load, and the load distribution function. The values in terms of the Huber-von Mises criterion are about 6.5% to 8.5% higher, while the values with the twin shear stress criterion are about 10.5% to 14.5% higher than those with the Tresca criterion. The increasing load distribution along the plate radius leads to minimal changes among different yield criteria. On the other hand, it makes significant differences to the total limit load, implying the increasing load distribution and improves the load-bearing capacity of a plate. For a uniformly distributed load, the load factor corresponding to the three criteria (unified yield criterion with b=0, b=0.5, and b=1) are 6.00, 6.49 and 6.84, respectively. The load factor with respect to the Huber-von Mises criterion reported by Hopkins and Wang (1954) is 6.51, which approximates closely the result using the unified yield criterion with b=0.5. Fig.5.11 and Fig.5.12 illustrate schematically the moment fields and velocity fields corresponding to these criteria for a simply supported circular plate subjected to the two types of linearly distributed load, i.e., $p = r$ and $p = 1 - r$. It is seen that the radial moment does not change much, while the circumferential moment varies significantly with respect to different yield criteria. The circumferential moment is not constant if the unified yield criterion with non-zero parameter b is applied. Locations of the maximum circumferential moment shift with the loading condition and the yield criterion. The larger r_1, the larger the total limit loads. The velocity fields with respect to the unified yield criterion (UYC) with non-zero parameter b ($0 < b \leqslant 1$) distribute nonlinearly along the plate, while the velocity field with respect to the Tresca criterion (or UYC with b=0) varies linearly and is singular at the plate center. The distribution of velocity also depends on the loading function as illustrated in Figs.5.11(b) and 5.12(b).

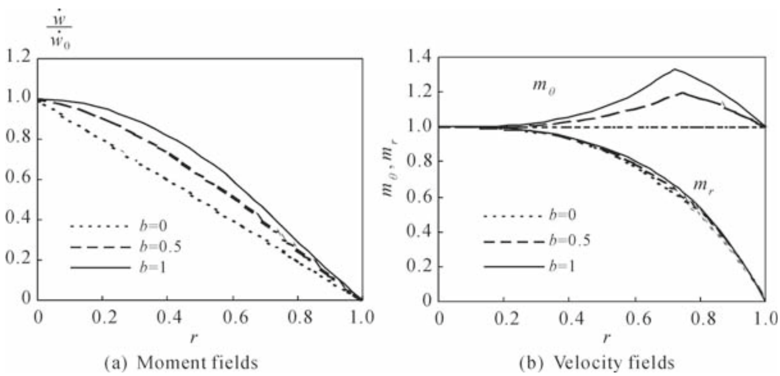


Fig. 5.11. Moment and velocity fields of simply supported circular plate ($p = r$)

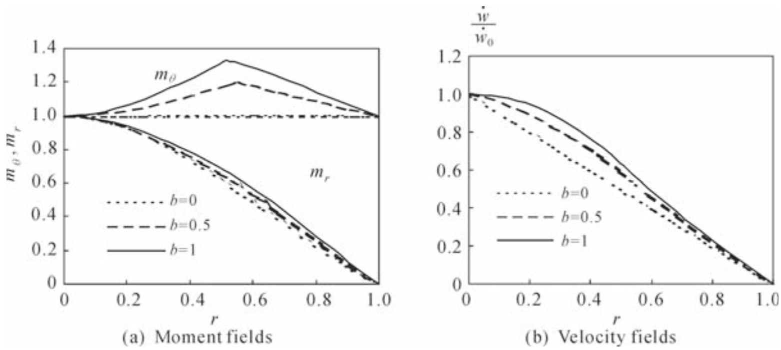


Fig. 5.12. Moments and velocity fields of simply supported circular plate ($p = 1-r$)

5.3.2.3 Edge Moment and Partial-uniform Load

For a plate loaded by edge moment and partial-uniform load as shown in Fig.5.13, there are two possible cases, i.e., Case (1) $d \leqslant r_0$ and Case (2) $d > r_0$, where d is the loading radius and r_0 is the dividing radius at which the moments m_r and m_θ correspond to the yield point B in Fig.5.2.

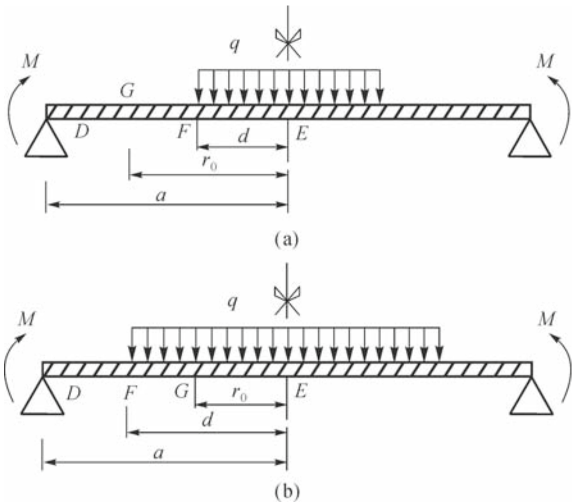


Fig. 5.13. Simply supported circular plate under partial-uniform load and edge moment

Case (1)

When point G lies on the line segment DF (Fig.5.13(a)), the equilibrium equations for EF, FG and GD are

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = (1+b)(m_0 - m_r) - \dfrac{q}{2}r^2 \\ m_\theta = (1+b)m_0 - bm_r \end{cases} \quad \text{for} \quad EF, \tag{5.37a}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = (1+b)(m_0 - m_r) - \dfrac{q}{2}d^2 \\ m_\theta = (1+b)m_0 - bm_r \end{cases} \quad \text{for} \quad FG, \tag{5.37b}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = m_0 - \dfrac{1}{1+b}m_r - \dfrac{q}{2}d^2 \\ m_\theta = m_0 - \dfrac{b}{1+b}m_r \end{cases} \quad \text{for} \quad GD, \tag{5.37c}$$

where q is the plastic limit load, and q satisfies

$$q = \frac{2(1+b)(3+b)m_0}{(2+b)\left[(3+b) - 2\left(\frac{d}{r_0}\right)^{1+b}\right]d^2}, \tag{5.38}$$

and r_0 satisfies

$$\begin{aligned} &2(1+b)\left(\frac{d}{r_0}\right)^{1+b} + (3+b)\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} - 2(2+b)\left(\frac{d}{r_0}\right)^{1+b}\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} \\ &- \frac{(2+b)m_b}{(1+b)}\left[(3+b-2\left(\frac{d}{r_0}\right)^{1+b}\right]\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} = 0, \end{aligned} \tag{5.39}$$

where $m_b = M_b/M_0$.

Case (2)
When point G is on line segment EF (Fig.5.13(b)), the equilibrium equations are

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = (1+b)(m_0 - m_r) - \dfrac{q}{2}r^2 \\ M_\theta = (1+b)m_0 - bm_r \end{cases} \quad \text{for} \quad EG, \tag{5.40}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = m_0 - \dfrac{1}{1+b}m_r - \dfrac{q}{2}r^2 \\ m_\theta = m_0 - \dfrac{b}{1+b}m_r \end{cases} \quad \text{for} \quad GF, \tag{5.41}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = m_0 - \dfrac{1}{1+b}m_r - \dfrac{q}{2}d^2 \\ m_\theta = m_0 - \dfrac{b}{1+b}m_r \end{cases} \quad \text{for} \quad GD, \tag{5.42}$$

where q satisfies

$$q = \frac{6+2b}{2+b}\frac{m_0}{r_0^2}, \tag{5.43}$$

and r_0 satisfies

$$\frac{3+b}{3+2b}\left(\frac{d}{r_0}\right)^{\frac{3+2b}{1+b}} 2(2+b)\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} + (3+b)\left(\frac{d}{r_0}\right)^2\left[\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} - \left(\frac{d}{r_0}\right)^{\frac{1}{1+b}}\right]$$
$$+\frac{(2+b)m_b}{(1+b)}\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} + \frac{2b(2+b)}{3+2b} = 0. \tag{5.44}$$

When $d = r_0$, point F and G overlap, the moment fields of Case (1) and Case (2) become the same, and r_0 satisfies

$$r_0 = \left(\frac{1}{2} + \frac{(2+b)m_b}{2(1+b)}\right)^{1+b} a. \tag{5.45}$$

5.3.2.4 Edge Moment and Partial-linear Load

For a circular plate subjected to partial-linear load and edge moment as shown in Fig.5.14, there also are two possible cases, i.e. Case (1) $d \leqslant r_0$ and Case (2) $d > r_0$.

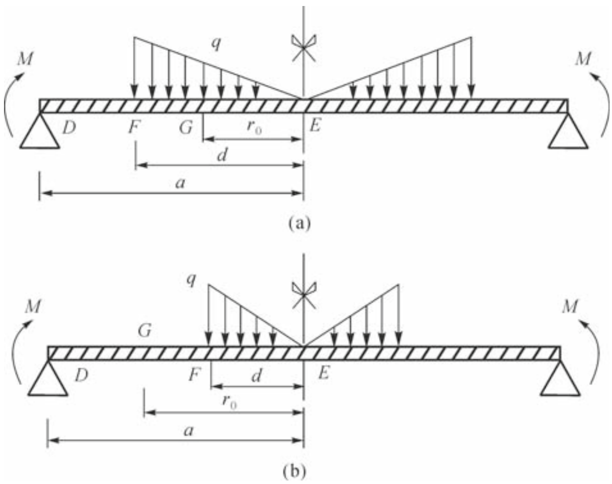


Fig. 5.14. Partial-linear load and edge moment

Case (1)

When point G lies on line segment EF (Fig.5.14(a)), the plastic limit loading is

$$q = \frac{3(4+b)}{2+b}\frac{dm_0}{r_0^2}, \tag{5.46}$$

and r_0 satisfies

$$\begin{aligned} &\left(\frac{(2+b)m_b}{(1+b)} + \frac{(4+b)d^3 - (2+b)r_0^3}{r_0^3}\right)\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} \\ &- \frac{3(1+b)(4+b)}{(4+3b)}\left(\frac{d}{r_0}\right)^3\left(\frac{d}{r_0}\right)^{\frac{1}{1+b}} + \frac{3b(2+b)}{4+3b} = 0. \end{aligned} \tag{5.47}$$

Case (2)

When point G is on line segment FD (Fig.5.14(b)), the plastic limit loading is

$$q = \frac{3(1+b)(4+b)m_0}{(2+b)\left[(4+b) - 3\left(\frac{d}{r_0}\right)^{1+b}\right]d^2}, \tag{5.48}$$

and r_0 satisfies

$$\begin{aligned} &3(1+b)\left(\frac{d}{r_0}\right)^{1+b} + (4+b)\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} - 3(2+b)\left(\frac{d}{r_0}\right)^{1+b}\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} \\ &- \frac{(2+b)m_b}{(1+b)}\left[(4+b) - 3\left(\frac{d}{r_0}\right)^{1+b}\right]\left(\frac{a}{r_0}\right)^{\frac{1}{1+b}} = 0 \end{aligned} \tag{5.49}$$

When $d = r_0$, i.e., points F and G overlap, and r_0 satisfies

$$r_0 = \left(\frac{2}{3} + \frac{(2+b)m_b}{3(1+b)}\right)^{1+b} a. \tag{5.50}$$

The relationship between the plastic limit load and parameter b is given in Table 5.4 and Fig.5.15 for the loading cases of (1) partial-uniform loading with edge moment, and (2) partial-linear load with edge moment, where $d = 0.6a$ and $M_b = 0.3M_0$. It is seen that the plastic limit load is different with respect to different yield criteria. When $b = 0$ (Tresca criterion), the plastic limit load is the minimum; when $b = 1$ (the twin-shear yield criterion), the plastic limit load is the maximum.

Table 5.4. Relationships of limit load q and parameter b

Loading type	b	0	0.1	0.2	0.3	0.4	0.5
1	r_0	0.7000	0.6797	0.6576	0.6418	0.6284	0.6171
	q	6.4815	6.6926	6.8850	7.0610	7.2226	7.3715
2	r_0	0.7412	0.7189	0.7002	0.6844	0.6709	0.6592
	q	10.6061	10.9061	11.1799	11.4303	11.6603	11.8721
Loading type	b	0.6	0.7	0.8	0.9	1.0	
1	r_0	0.6074	0.5990	0.5915	0.5847	0.5783	
	q	7.5093	7.6374	7.7568	7.8863	7.9729	
2	r_0	0.6491	0.6402	0.6324	0.6254	0.6192	
	q	12.0679	12.2495	12.4186	12.5765	12.7243	

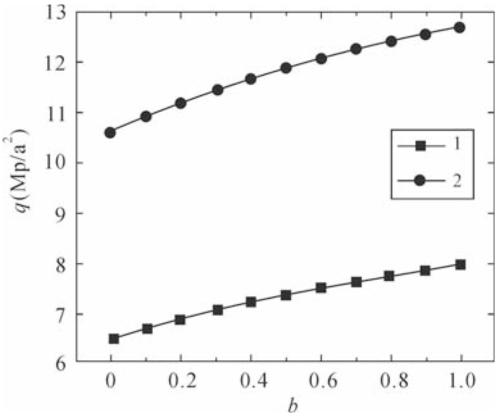


Fig. 5.15. Plastic limit load q versus unified yield criterion parameter b

5.4 Unified Solutions of Simply Supported Circular Plate for SD Materials

The unified solutions of a circular plate for non-SD materials are extended to SD materials. The unified plastic limit of a clamped circular plate with SD materials (strength differential effect in tension and compression) by using the unified strength theory was derived by Wei and Yu (2001). Unified plastic limit analyses of simply supported circular plates with different tensile and compressive strength under uniform annular load were derived by Wang and Yu (2002; 2003a). Plastic limit analysis of simply supported circular plates with different tensile and compressive strength under linear distributed load was given by Wang and Yu (2003). In this section, we will analyze the plastic limit load of simply supported circular plates that are made of SD materials. A parameter α is introduced, which is the ratio of the negative limit bending moment and positive limit bending moment, $\alpha = +m_p/-m_p$. The yield loci in terms of the generalized stresses m_r and m_θ is shown in Fig.5.16 with respect to different values of b.

The generalized yield criterion for a plate is similar to the yield criterion in the plane stress state. The generalized unified yield criterion for a plate is similar to the unified yield criterion in the plane stress state. However, two cases have to be considered, i.e., (a) $+m_p \neq -m_p$ and (b) $+m_p = -m_p$. They are shown in Fig.5.16.

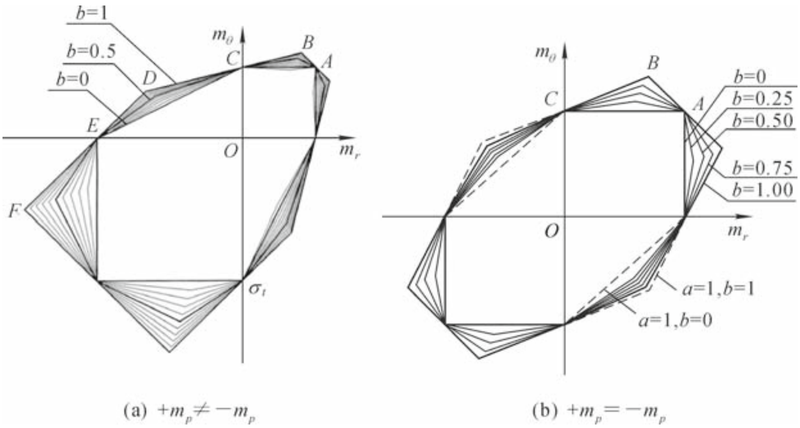


Fig. 5.16. Generalized unified yield criterion for plate

The yield criterion of the unified strength theory in terms of generalized stresses is

$$m_r - \frac{\alpha b}{1+b} m_\theta = m_p, \quad m_\theta - \frac{\alpha b}{1+b} m_r = m_p, \tag{5.51a}$$

$$m_r - \frac{\alpha}{1+b} m_\theta = m_p, \quad m_\theta - \frac{\alpha}{1+b} m_r = m_p, \tag{5.51b}$$

$$\frac{\alpha}{1+b}(bm_r + m_\theta) = -m_p, \quad \frac{\alpha}{1+b}(bm_\theta + m_r) = -m_p, \tag{5.51c}$$

$$\frac{1}{1+b}(m_r + bm_\theta) = m_p, \quad \frac{1}{1+b}(m_\theta + bm_r) = m_p, \tag{5.51d}$$

$$\frac{1}{1+b}(m_r + \alpha m_\theta) = m_p, \quad \frac{1}{1+b}(m_\theta + \alpha m_r) = m_p, \tag{5.51e}$$

$$\frac{b}{1+b}(m_r + \alpha m_\theta) = m_p, \quad \frac{b}{1+b}(m_\theta + \alpha m_r) = m_p. \tag{5.51f}$$

The center point of the simply supported plate satisfies $m_r(r = 0) = m_\theta(r = 0)$ (point A in Fig.5.16); and simply supported boundary satisfies $m_r(r = a) = 0$ (point C in Fig. 5.16), $m_\theta = (1+\alpha)\, m_r$ (point B in Fig.5.16). Stress states of all points in the plate are located on parts AB and BC. The yield conditions of parts AB, BC in Fig.5.16 can be expressed as

$$AB: \ \frac{b}{1+b}m_r + \frac{1}{1+b}m_\theta = m_p, \tag{5.52a}$$

$$BC: \ m_\theta - \frac{\alpha b}{1+b}m_r = m_p. \tag{5.52b}$$

5.4.1 Partial-uniform Load

There are two possible cases where the plate is subjected to a partial-uniform load as shown in Figs.5.17(a) and 5.17(b), i.e., Case (1) $d \leqslant r_0$ and Case (2) $d > r_0$, respectively, where d is the loading radius. The moments m_r and m_θ at point G with a radius of r_0 are located at point B in Fig.5.16.

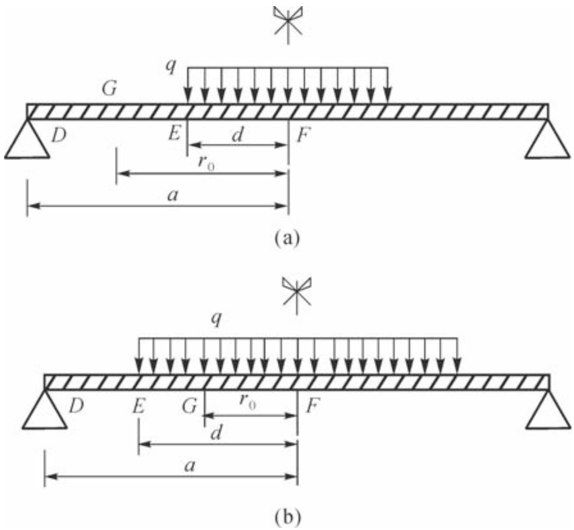


Fig. 5.17. Simply support circular plate subjected to partial-uniform load

Case (1)
When point G lies on segment DE, the equilibrium equations for different parts are given by

$$EF:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=(1+b)m_p-(1+b)m_r-\dfrac{pr^2}{2},\\ m_\theta=(1+b)m_p-bm_r, \end{cases} \tag{5.53a}$$

$$GE:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=(1+b)m_p-(1+b)m_r-\dfrac{pd^2}{2},\\ m_\theta=(1+b)m_p-bm_r, \end{cases} \tag{5.53b}$$

$$DG:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=m_p-\dfrac{1+b-\alpha b}{1+b}m_r-\dfrac{pd^2}{2},\\ m_\theta=m_p+\dfrac{\alpha b}{1+b}m_r. \end{cases} \tag{5.53c}$$

Case (2)
When point G is located on segment EF, the equilibrium equations are

$$GF:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=(1+b)m_p-(1+b)m_r-\dfrac{pr^2}{2},\\ m_\theta=(1+b)m_p-bm_r, \end{cases} \tag{5.54a}$$

$$EG:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=m_p-\dfrac{1+b-\alpha b}{1+b}m_r-\dfrac{pr^2}{2},\\ m_\theta=m_p+\dfrac{\alpha b}{1+b}m_r, \end{cases} \tag{5.54b}$$

$$DG:\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r}=m_p-\dfrac{1+b-\alpha b}{1+b}m_r-\dfrac{pd^2}{2},\\ m_\theta=m_p+\dfrac{\alpha b}{1+b}m_r. \end{cases} \tag{5.54c}$$

The boundary and continuity conditions are: (1) at point F, $m_r(r=0)$ is a finite value; (2) at point D, $m_r(r=0)=0$; (3) at point E, $m_r(r=d)$ is continuous; (4) at point G, $m_r(r=r_0)$ continuous and equal to $(1+b)m_p/(1+b+\alpha)$.
Case (1)
Integrate Eqs.(5.53a) and (5.53c),

$$m_r=m_p-\frac{pr^2}{6+2b}+c_1r^{-(1+b)}m_\theta=(1+b)m_p-bm_r\quad\text{for}\quad EF, \tag{5.55a}$$

$$m_r=m_p-\frac{pd^2}{2(1+b)}+c_2r^{-(1+b)}m_\theta=(1+b)m_p-bm_r\quad\text{for}\quad GE, \tag{5.55b}$$

$$m_r=\frac{1+b}{1+b-\alpha b}(m_p-\frac{pd^2}{2})+c_3r^{-\frac{1+b-\alpha b}{1+b}}m_\theta=m_p+\frac{\alpha b}{1+b}m_r\quad\text{for}\quad DG. \tag{5.55c}$$

With reference to the boundary and continuity conditions (1) to (4), we have

$$c_1 = 0, \tag{5.56a}$$

$$c_2 = \frac{pd^{3+b}}{(1+b)(3+b)}, \tag{5.56b}$$

$$c_3 = \left[-\frac{1+b}{1+b-\alpha b}\left(m_p - \frac{pd^2}{2}\right)\right] a^{\frac{1+b-\alpha b}{1+b}}. \tag{5.56c}$$

The plastic limit load is

$$p = \frac{2(1+b)(3+b)\alpha m_p}{(1+b+\alpha)d^2\left[(3+b) - 2\left(\frac{d}{r_0}\right)^{1+b}\right]}, \tag{5.57}$$

where r_0 satisfies

$$\begin{aligned} &2\alpha(1+b)\left(\frac{d}{r_0}\right)^{1+b} + (3+b)(1+b-\alpha b)\left(\frac{a}{r_0}\right)^{\frac{1+b-\alpha b}{1+b}} \\ &-2(1+b+\alpha)\left(\frac{d}{r_0}\right)^{1+b}\left(\frac{a}{r_0}\right)^{\frac{1+b-\alpha b}{1+b}} = 0. \end{aligned} \tag{5.58}$$

Substituting c_1, c_2 and c_3 into Eqs.(5.55a) and (5.55c), the moment fields for Case (1) can be derived.

Case (2)

When point G located between E and F, from Eq.(5.54),

$$\begin{cases} m_r = m_p - \dfrac{pr^2}{6+2b} + c_4 r^{-(1+b)} \\ m_\theta = (1+b)m_p - bm_r \end{cases} \quad \text{for} \quad GF, \tag{5.59a}$$

$$\begin{cases} m_r = \dfrac{1+b}{1+b-\alpha b}m_p - \dfrac{(1+b)pr^2}{2(3+3b-\alpha b)} + c_5 r^{-\frac{1+b-\alpha b}{1+b}} \\ m_\theta = m_p + \dfrac{\alpha b}{1+b}m_r \end{cases} \quad \text{for} \quad EG, \tag{5.59b}$$

$$\begin{cases} m_r = \dfrac{1+b}{1+b-\alpha b}m_p - \dfrac{1+b}{1+b-\alpha b}\dfrac{pd^2}{2} + c_6 r^{-\frac{1+b-\alpha b}{1+b}} \\ m_\theta = m_p + \dfrac{\alpha b}{1+b}m_r \end{cases} \quad \text{for} \quad DG. \tag{5.59c}$$

Applying the boundary and continuity conditions (1) to (4), we obtain

$$c_4 = 0, \tag{5.60a}$$

$$c_5 = -\frac{1+b}{1+b-\alpha b} m_p \alpha^{\frac{1+b-\alpha b}{1+b}} + \frac{(1+b)pd^2}{2(1+b-\alpha b)} \left(\alpha^{\frac{1+b-\alpha b}{1+b}} - d^{\frac{1+b-\alpha b}{1+b}} \right) + \frac{(1+b)p}{2(3+3b-\alpha b)} d^{\frac{3+3b-\alpha b}{1+b}}, \tag{5.60b}$$

$$c_6 = \left(-\frac{1+b}{1+b-\alpha b} m_p + \frac{1+b}{1+b-\alpha b} \frac{pd^2}{2} \right) \alpha^{\frac{1+b-\alpha b}{1+b}}. \tag{5.60c}$$

The plastic limit load is derived as

$$p = \frac{(6+2b)\alpha m_p}{(1+b-\alpha) r_0^2}, \tag{5.61}$$

where r_0 satisfies

$$\alpha(1+b) - \frac{\alpha(3+b)(1+b-\alpha b)}{3+3b-\alpha b} - (1+b+\alpha) \left(\frac{a}{r_0} \right)^{\frac{1+b-\alpha b}{1+b}} + \alpha(3+b) \left(\frac{d}{r_0} \right)^2 \left[\left(\frac{a}{r_0} \right)^{\frac{1+b-\alpha b}{1+b}} - \left(\frac{d}{r_0} \right)^{\frac{1+b-\alpha b}{1+b}} \right] + \frac{\alpha(3+b)(1+b-\alpha b)}{3+3b-\alpha b} \left(\frac{d}{r_0} \right)^{\frac{3+3b-\alpha b}{1+b}} = 0. \tag{5.62}$$

Special Case

When $r_0 = d$, i.e., point G overlaps point E, the moment fields of the two cases are the same as

$$\begin{cases} m_r = m_p - \dfrac{pr^2}{6+2b} + c_7 r^{-(1+b)} \\ m_\theta = (1+b)m_r - bm_r \end{cases} \quad \text{for } EF, \tag{5.63a}$$

$$\begin{cases} m_r = \dfrac{1+b}{1+b-\alpha b} \left(m_p - \dfrac{pd^2}{2} \right) + c_8 r^{-\frac{1+b-\alpha b}{1+b}} \\ m_\theta = m_p + \dfrac{\alpha b}{1+b} m_r \end{cases} \quad \text{for } DE. \tag{5.63b}$$

From the boundary and continuity conditions (1) to (4),

$$c_7 = 0, \tag{5.64a}$$

$$c_8 = \left[-\frac{1+b}{1+b-\alpha b} \left(m_p - \frac{pd^2}{2} \right) \right] \alpha^{\frac{1+b-\alpha b}{1+b}}, \tag{5.64b}$$

the plastic limit load is simplified as

$$p = \frac{\alpha(6+2b)m_p}{(1+b+\alpha)d^2}, \tag{5.65}$$

where $d = r_0$ and

$$\left(\frac{\alpha}{d}\right)^{\frac{1+b-\alpha b}{1+b}} = \frac{-2\alpha}{1+b-\alpha b-2\alpha}. \tag{5.66}$$

Denoting the critical loading radius as d_0, when $d \leqslant d_0$, i.e. Case (1), Eq.(5.58) gives a unique solution in the region of $d < r \leqslant a$, while Eq.(5.62) has no solution in $0 < r \leqslant d$. Thus the equations for Case (1) are adopted to solve the plastic limit load and moment fields. When, on the other hand, Eq.(5.58) has no solution in $d < r \leqslant a$, while Eq.(5.62) can be used to solve r_0 in the region of $0 < r \leqslant d$. In this case, point G is on the segment EF. The plastic limit load and moment fields can be obtained from Case (2).

Figs.5.18 to 5.21 show the moment fields when $\alpha = 0.1$, $d = a$, $0.5a$, $0.1a$ and $0.00001a$, respectively. The plastic limit loads with respect to different values of unified yield criterion parameter b are plotted in Fig.5.22.

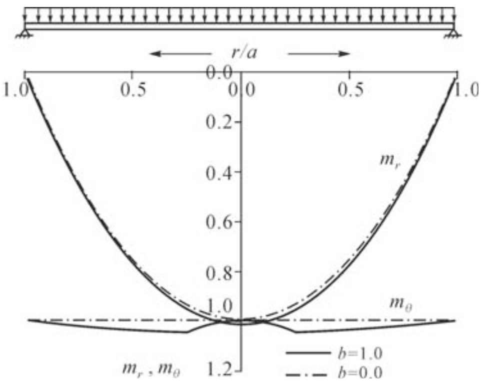


Fig. 5.18. Moment fields when $d = a$

From Fig.5.18 to Fig.5.21, the moment fields depend on the unified yield criterion parameter b. The higher the parameter b, the higher the corresponding moment and the plastic limit load. When $b = 0$, the moment m_θ is independent of the loading radius, and $m_\theta = m_p$. When $b \neq 0$, m_θ varies along the radial direction, and the variation of m_θ gives a more reasonable representation than that with $b = 0$. When d approaches zero, which corresponds to a concentrated loading case, the solution of m_θ has no singularity at $r = 0$ when the parameter b is equal to 0. When $b \neq 0$, the solutions of both m_θ and m_r have singularity at $r = 0$. It can be said that when using the unified strength theory, the singularity of the moment fields at the plate center for a concentrated loading case can be properly represented.

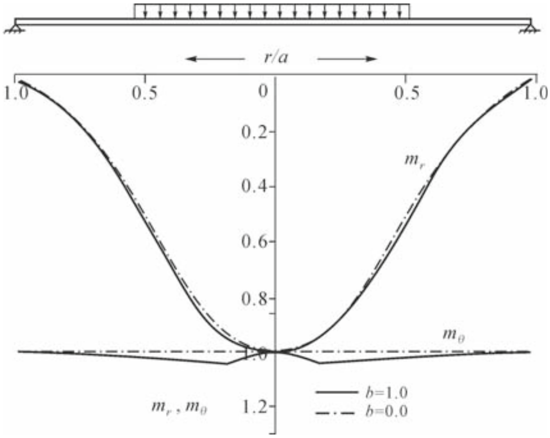


Fig. 5.19. Moment fields when $d = 0.5a$

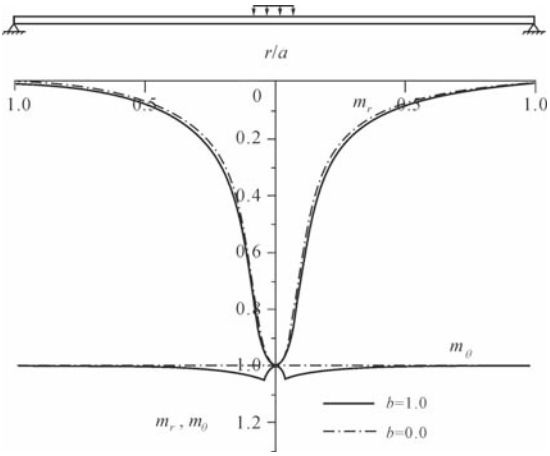


Fig. 5.20. Moment fields when $d = 0.1a$

5.4.2 Linearly Distributed Load

For a simply supported circular plate under linearly distributed load, two different loading cases are discussed in the following context.

Case (1)

For a linear pressure load as shown in Fig.5.23, when the moment of point F falls on point B in Fig.5.16, the moment fields satisfy the boundary and continuity conditions of (1) $m_r = m_\theta$ at $r = 0$; (2) m_r and m_θ are continuous at point F or $r = r_0$; (3) $m_r = 0$ at $r = a$ of the outer edge.

The equilibrium equations of lines EF and FD are

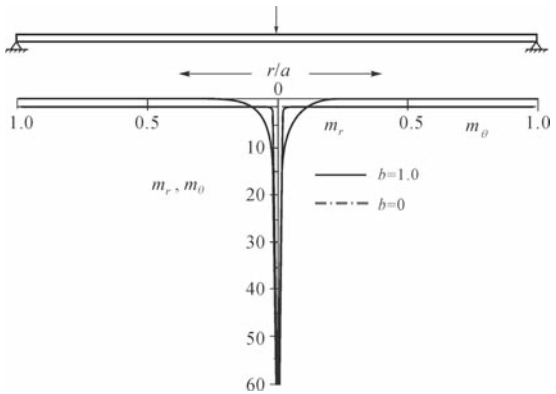


Fig. 5.21. Moment fields when $d = 0.00001a$

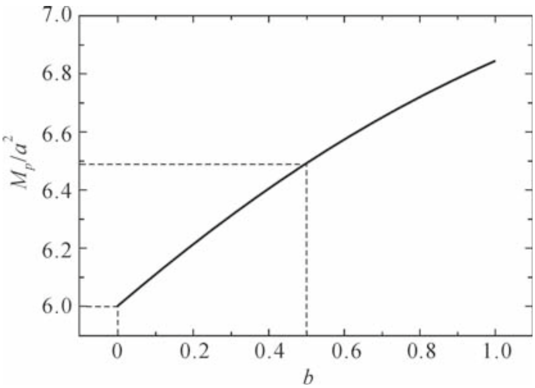


Fig. 5.22. Plastic limit loads with respect to different values of unified yield criterion parameter b

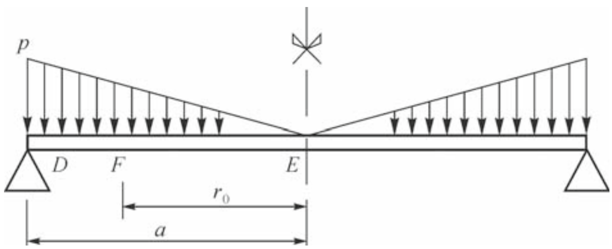


Fig. 5.23. Linearly distributed loading (Case (1))

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = (1+b)m_p - (1+b)m_r - \dfrac{p}{3a}r^3 \\ m_\theta = (1+b)m_p - bm_r \end{cases} \quad \text{for } EF, \tag{5.67a}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = m_p - \dfrac{1+(1-\alpha)b}{1+b}m_r - \dfrac{p}{3a}r^3 \\ m_\theta = (1+b)m_p - \dfrac{\alpha b}{1+b}m_r \end{cases} \quad \text{for } FD. \tag{5.67b}$$

Solving Eq.(5.67) with references to the boundary and continuity conditions yields

$$\begin{cases} m_r = c_1 r^{-(1+b)} + m_p - \dfrac{pr^3}{3a(4+b)} \\ m_\theta = -c_1 b r^{-(1+b)} + m_p + \dfrac{bpr^3}{3a(4+b)} \end{cases} \quad \text{for} \quad EF, \tag{5.68a}$$

$$\begin{cases} m_r = c_2 r^{-\frac{1+(1-\alpha)b}{1+b}} + \dfrac{1+b}{1+(1-\alpha)b} m_p - \dfrac{(1+b)pr^3}{3\alpha(4+4b-b\alpha)} \\ m_\theta = c_2 \dfrac{\alpha b}{1+b} r^{-\frac{1+(1-\alpha)b}{1+b}} + \dfrac{1+b}{1+(1-\alpha)b} m_p - \dfrac{\alpha bpr^3}{3\alpha(4+3b)} \end{cases} \quad \text{for} \quad FG. \tag{5.68b}$$

The two coefficients c_1 and c_2 are derived as

$$c_1 = 0, \tag{5.69a}$$

$$c_2 = \left[-\frac{(1+b)m_p}{1+(1-\alpha)b} + \frac{(1+b)pa^2}{3(4+4b-b\alpha)} \right] \alpha^{\frac{1+(1-\alpha)b}{1+b}}. \tag{5.69b}$$

The limit loading is

$$p = \frac{3\alpha(4+b)\alpha m_p}{(1+\alpha+b)r_0^3}, \tag{5.70}$$

and r_0 satisfies

$$\begin{aligned} &\alpha(4+b)[1+(1-\alpha)b] \left(\frac{a}{r_0}\right)^3 \left(\frac{a}{r_0}\right)^{\frac{1+(1-\alpha)b}{1+b}} - (4+4b-b\alpha)(1+\alpha \\ &+ b) \left(\frac{a}{r_0}\right)^{\frac{1+(1-\alpha)b}{1+b}} + 3\alpha b(1+\alpha+b) = 0 \end{aligned} \tag{5.71}$$

Case (2)

For a linear pressure load as shown in Fig.5.24, when the moment of point F is at point B in Fig.5.16, the equilibrium equations of lines EF and FD are

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = (1+b)m_p - (1+b)m_r - \dfrac{pr^2}{2} + \dfrac{p}{3a}r^3 \\ m_\theta = (1+b)m_p - bm_r \end{cases} \quad \text{for} \quad EF, \tag{5.72a}$$

$$\begin{cases} r\dfrac{\mathrm{d}m_r}{\mathrm{d}r} = m_p - \dfrac{1+(1-\alpha)b}{1+b}m_r - \dfrac{pr^2}{2} + \dfrac{p}{3a}r^3 \\ m_\theta = (1+b)m_p - \dfrac{\alpha b}{1+b}m_r \end{cases} \quad \text{for} \quad FD. \tag{5.72b}$$

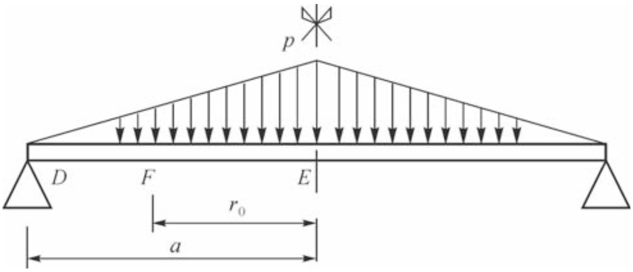


Fig. 5.24. Linearly distributed loading (Case (2))

Eq.(5.72) can be solved with application of the boundary and continuity conditions as

$$\begin{cases} m_r = c_3 r^{-(1+b)} + m_p - \dfrac{pr^2}{2(3+b)} + \dfrac{pr^3}{3a(4+b)} \\ m_\theta = -c_1 b r^{-(1+b)} + m_p + \dfrac{bpr^2}{2(3+b)} - \dfrac{bpr^3}{3a(4+b)} \end{cases} \quad \text{for } EF, \tag{5.73a}$$

$$\begin{cases} m_r = c_4 r^{-\frac{1+(1-\alpha)b}{1+b}} + \dfrac{1+b}{1+(1-\alpha)b} m_p \\ \qquad - \dfrac{(1+b)pr^2}{2(3+3b-b\alpha)} + \dfrac{(1+b)pr^3}{3\alpha(4+4b-b\alpha)} \\ m_\theta = c_4 \dfrac{\alpha b}{1+b} r^{-\frac{1+(1-\alpha)b}{1+b}} + \dfrac{1+b}{1+(1-\alpha)b} m_p \\ \qquad + \dfrac{\alpha bpr^2}{2(3+3b-b\alpha)} - \dfrac{\alpha bpr^3}{3\alpha(4+3b)} \end{cases} \quad \text{for } FD. \tag{5.73b}$$

The two coefficients c_3 and c_4 are derived as

$$c_3 = 0, \tag{5.74a}$$

$$c_4 = \left[\frac{(1+b)(6+6b-b\alpha)pa^2}{6(3+3b-b\alpha)(4+4b-b\alpha)} - \frac{(1+b)}{1+(1-\alpha)b} m_p \right] \alpha^{\frac{1}{1+b}}. \tag{5.74b}$$

The limit loading is obtained as

$$p = \frac{6\alpha a(3+b)(4+b)m_p}{(1+\alpha+b)[3a(4+b)r_0^2 - 2(3+b)r_0^3]}, \tag{5.75}$$

and r_0 satisfies

$$
\begin{aligned}
&6\alpha ab(1+\alpha+b)(4+b)(4+b-b\alpha)r_0^2-(1+\alpha+b)(3+3b \\
&\quad -b\alpha)(4+4b-b\alpha)(3a^2(4+b)r_0^2-2(3+b)r_0^3)\left(\frac{a}{r_0}\right)^{\frac{(1+(1-\alpha)b)}{1+b}} \\
&\quad -6\alpha b(1+\alpha+b)(3+3b-b\alpha)(3+b)r_0^3=0.
\end{aligned}
\tag{5.76}
$$

From Eqs.(5.70) and (5.75), the relationship of the plastic limit load and the unified strength theory parameters b and α can be determined as shown in Figs.5.25 and 5.26. It is seen that the plastic limit load is significantly affected by the unified yield criterion parameters b and α. When α is given, the plastic limit load increases with the increase in b. When $b = 0$, which corresponds to the Mohr-Coulomb criterion, the plastic limit load is the minimum, and when $b = 1$ corresponding to the twin-shear strength criterion, the plastic limit load gives the maximum value. For any specific value of parameter b, the plastic limit load increases with the increase in α. When $\alpha = 1$, it gives the same results as those based on the unified yield criterion.

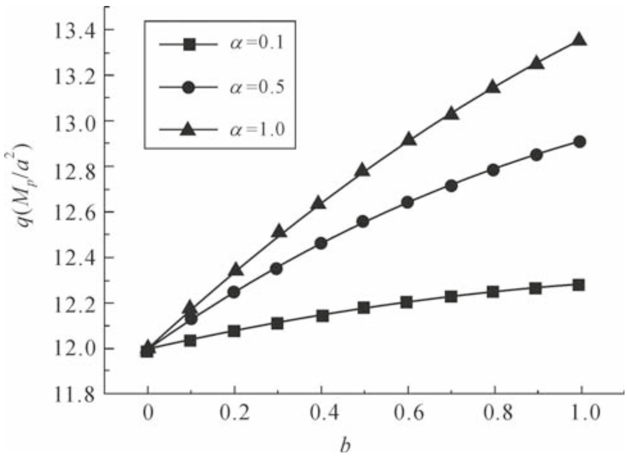


Fig. 5.25. Relation curves of limit loading and unified strength theory parameters b, α (Case (1))

It is seen from Fig.5.25 and Fig.5.26 that the unified strength theory parameters b and the ratio of material strength in tension and in compression α have a significant influence on the limit bearing capacity of a simply supported circular plate. The unified strength theory provides us with an effective approach for studying these effects and for raising the bearing capacities of engineering structures more than the Tresca criterion and the Mohr-Coulomb criterion ($b = 0$). So, there is a considerable economical benefit in using the

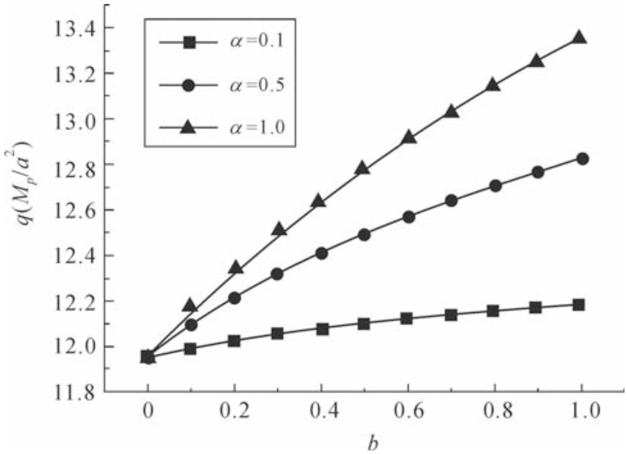


Fig. 5.26. Relation curves of limit loading and unified strength theory parameters b, α (Case (2))

new results if the strength of material is adapted to the new yield criterion ($b > 0$). This brings a tremendous energy saving and reduction in pollution.

5.5 Summary

The unified solution to the plastic limit load for simply supported circular plates made of either non-SD materials or SD materials under various loading conditions are derived. They are obtained by applying the unified strength theory in the plane stress state. The unified solution gives a series of new results, and establishes a relationship between various results and encompasses solutions using the Tresca criterion, the maximum stress criterion, the Huber-von Mises criterion, the Mohr-Coulomb criterion and the twin-shear criterion as special cases. The plastic limit load of a simply supported circular plate under a uniformly distributed load is $p = (6 + 2b)/(2 + b)r_0^2$ for non-SD materials, $p = \alpha(6 + 2b)/((1 + b + \alpha)r_0^2)$ for SD materials, where p is the normalized plastic limit load, and $p = Pa^2/M_p$. A series of solutions for various materials can be deduced from the unified solution. For an easier understanding of the current results, some specific solutions are given below:

Non-SD materials:

$p = 6.0$, it follows the Tresca criterion (or the unified yield criterion with b=0);

$p = 6.51$, it follows the Huber-von Mises criterion (Hopkins-Wang 1954, numerical integrated method);

$p = 6.46$, it follows the Huber-von Mises criterion (Sokolovsky's solution, 1955);

$p = 6.49$, it follows the unified yield criterion with $b = 1/2$ (Ma and He, 1994; Ma et al., 1995);

$p = 6.84$, it satisfies the twin-shear yield criterion (Li, 1988; Huang and Zeng, 1989) or the unified yield criterion with $b = 1$ (Ma and He, 1994; Ma et al, 1995).

SD materials:

$p = 6\alpha/((1+\alpha)r_0^2)$ for SD materials satisfying the Mohr-Coulomb criterion ($b = 0$);

$p = 14\alpha/((3+2\alpha)r_0^2)$ for SD materials satisfying a new criterion ($b = 1/2$);

$p = 9\alpha/((2+\alpha)r_0^2)$ for SD materials satisfying the twin-shear strength criterion ($b = 1$).

A series of research exercises were carried out to show the effects of strength theory on the results of elasto-plastic analysis and the load-bearing capacities of a simply supported circular plate for non-SD materials and SD materials. The choice of strength theory has a significant influence on these results. The unified yield criterion and unified strength theory provide us with an effective approach for studying these effects. The unified plastic limit of a clamped circular plate for non-SD materials and SD materials will be described in Chapter 6.

5.6 Problems

Problem 5.1 Determine the limit bearing capacity of a simply supported circular plate by using of the Tresca criterion.

Problem 5.2 Determine the limit bearing capacity of a simply supported circular plate by using of the unified yield criterion ($b = 0$).

Problem 5.3 Determine the limit bearing capacity of a simply supported circular plate by using of the unified yield criterion ($b = 0.5$).

Problem 5.4 Determine the limit bearing capacity of a simply supported circular plate by using of the unified yield criterion ($b = 0.8$).

Problem 5.5 Determine the limit bearing capacity of a simply supported circular plate by using of the unified yield criterion ($b = 1.0$).

Problem 5.6 Determine the limit bearing capacity of a simply supported circular plate by using of the unified strength theory ($b = 0$).

Problem 5.7 Determine the limit bearing capacity of a simply supported circular plate by using of the unified strength theory ($b = 0.5$).

Problem 5.8 Determine the limit bearing capacity of a simply supported circular plate by using of the unified strength theory ($b = 0.8$).

Problem 5.9 Determine the limit bearing capacity of a simply supported circular plate by using of the unified strength theory ($b = 1.0$).

Problem 5.10 A simply supported circular plate under uniform annular load is shown in Fig.5.27. The relationship of limit load q and b for a special case is shown in Fig.5.28. Please derive the unified solution for the plate. The referenced figure similar to Fig.5.8 is shown in Fig.5.29.

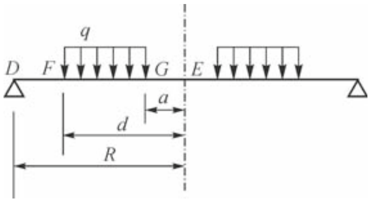


Fig. 5.27.

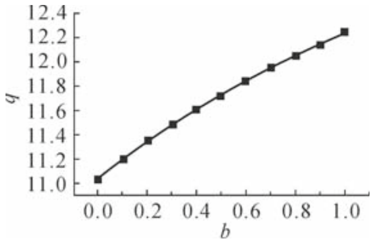


Fig. 5.28.

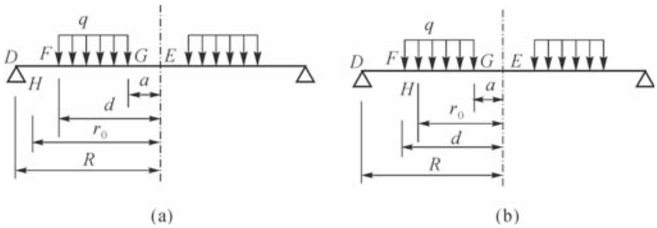


Fig. 5.29.

Problem 5.11 A simply supported circular plate is under linear and uniform load, as shown in Fig.5.30. Please derive the unified solution for the plate.

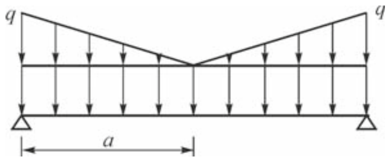


Fig. 5.30.

Problem 5.12 A simply supported circular plate is under linear and uniform load as shown in Fig.5.31. Please derive the unified solution for the plate.

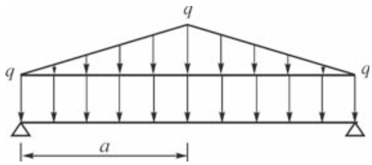


Fig. 5.31.

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