# 4

# Theorems of Limit Analysis

## 4.1 Introduction

To understand plastic limit analysis it is helpful to review the behavior of an elastic-plastic solid or structure subjected to mechanical loading. An inelastic solid will yield at a specific magnitude of the applied load. The corresponding load is called the elastic limit of the structure. If the external load exceeds the elastic limit, a plastic region starts to spread through the structure. With further expansion of the yield area, the displacement of the structure progressively increases. At another critical load, the plastic region becomes so large as not to resist the unconstrained plastic flow in the solid. The load cannot be increased beyond this point. The collapse load is called the plastic limit of the structure. Plastic limit analysis involves an associated flow rule of the adopted yield criterion. The plastic limit load is also registered as the load-bearing capacity of the structure.

Limit analysis and design of steel structures have been well explored (Symonds and Neal, 1951; Neal, 1956; Hodge, 1959; 1963; Baker and Heyman, 1969; Heyman, 1971; Save and Massonnet, 1972; Horne, 1979; Zyczkowski, 1981; Mrazik et al., 1987; Save et al., 1997). Exploitation of the strength reserve of the load-bearing capacity yields a design of structures with increased admissible loads or decreased cross-sections, which results in a reduction in the amounts of materials and costs.

To save material, one of the choices is to transfer part of the load from the most highly stressed cross-sections to those that are understressed in the elastic state. The number of fully exploited cross-sections can be increased by the redistribution of the internal forces. The load-bearing capacity of structures may be more accurately estimated by choosing an appropriate strength theory or yield criterion.

In plastic limit analysis, direct integration of the equilibrium equations governed by certain yield condition leads to the load-bearing capacity for specific boundary conditions. The associated flow rule is often used to determine the velocity field. Only in some special cases is it possible to derive closed form solutions. But in general it is always feasible to derive approximations through numerical integration of the basic equations.

Determination of the load-bearing capacity of a structure is the simplest when the yield curve is polygonal in shape, as is the case for the Tresca yield criterion and the twin-shear yield criterion. The reason is that only linear equations need to be solved when these kinds of criteria are applied. For other criteria nonlinear equations are involved. Numerical techniques are more appropriate. Thus, the replacement of the Tresca yield conditions by the Huber-von Mises criterion usually renders the analytical solution impractical. However, the unified yield criterion and the unified strength criterion have the advantages of the piecewise linear form, and uniform solutions of the load bearing capacity with respect to different yield conditions can be derived for some simple structures, such as the axial-symmetrical plates, cylinders, tubes and thick-wall vessels.

Ma and He (1994), Ma et al. (1993; 1994; 1995a; 1995b; 1995c) gave a unified plastic limit solution to circular plates under uniform loads and partially uniform loads. Ma and Hao (1998) derived a unified solution to simply supported and clamped circular plates with the Yu's unified yield criterion. Further applications of the unified yield criterion to plastic limit analysis of circular plates under arbitrary loads were reported by Ma et al. (1999). The unified solutions of the limit speed of the rotating disc and cylinder using the unified yield criterion were given by Ma et al. (2001).

The unified plastic limit solution to circular plates under uniform loads and partially uniform loads using the Yu unified strength theory for SD materials was presented by Wei and Yu (2001; 2002), Wang and Yu (2002).

A general formulation of limit design theorems for perfectly plastic materials was given by Gvozdev (1938; 1960). However his work was not known in the Western world until the late 1950s, and before that a very similar theory had been developed by Prager at Brown University (Drucker et al., 1952; Prager, 1947).

One of the most important developments in plastic theory is the upper and lower bound theorems. The contents of these theorems were known by intuition long before Gvozdev's and Prager's school works. However, a complete and precise formulation was given by Gvozdev, Drucker and Greenberg. And Prager's formulation has been proved very valuable. These important principles were also stated by Prager (1947), Hill (1950), Mendelson (1968), Kachanov (1971), Save and Masonnet (1972), Martin (1975), Chen (1975), Zyczkowski (1981), and Nielsen (1999).

The early applications of plasticity to structural concrete were mainly for those reinforced concrete structures whose strength was governed by reinforcement. For such structures a plastic limit design has been standardized. Examples are the yield hinge method for beams and frames (Baker and Heyman, 1969) and the yield line theory for slabs.

The theorems of limit analysis were first presented by Gvozdev in 1938 and independently proved by Hill in 1950 for rigid-perfectly-plastic materials and by Drucker et al. in 1952 for elastic-perfectly-plastic materials. The general forms of the theorems of limit analysis are described in the following sections.

## 4.2 Perfectly Plastic Solid

A perfectly-plastic solid refers to the material undergoing unlimited plastic deformation under a constant yield stress $\sigma_Y$. Fig.4.1 schematically shows the difference among elastic, perfectly-plastic (ideal plastic), hard and soft behavior of material. The value of $\sigma_Y$ is different for different materials, and even for the same material in different environmental conditions. In the following context, strains and strain rates refer to the plastic quantities unless it is explicitly stated otherwise.

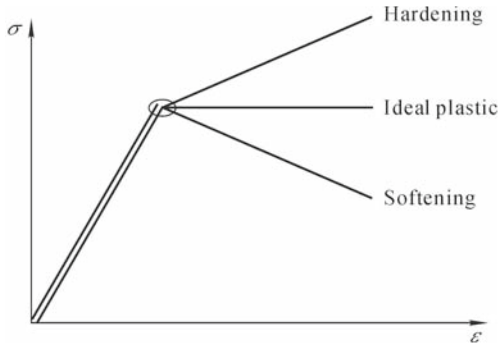


**Fig. 4.1.** Behavior of elastic, strain hardening, and perfect-plastic materials

At the incipience of plastic flow, it is assumed that strains are very small. Hence strains and displacement are related through Eqs.(2.19) and (2.20), whereas strain rates are derived from displacement rates (or velocities) through Eqs.(2.19) and (2.20).

## 4.3 Power of Dissipation

At the incipience of plastic flow for a specific point in the stress space, where the stress state is described by $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$, and the strain rate by $(\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z, \dot{\gamma}_{xy}, \dot{\gamma}_{yz}, \dot{\gamma}_{xy})$, the power of the stress per unit volume of material is

$$d = \sigma_x \dot{\varepsilon}_x + \sigma_y \dot{\varepsilon}_y + \sigma_z \dot{\varepsilon}_z + \tau_{xy} \dot{\gamma}_{xy} + \tau_{yz} \dot{\gamma}_{yz} + \tau_{xz} \dot{\gamma}_{xz}. \tag{4.1}$$

For purely plastic strains this power is dissipated as heat during plastic flow. Therefore, it is called “power of dissipation”, which is essentially positive.

Eq.(4.1) can be put into vector form as

$$d = \{\sigma\}^{\mathrm{T}}\{\dot{\varepsilon}\}. \tag{4.2}$$

If elastic strain rates are neglected so that $\{\dot{\varepsilon}\}$ represents the plastic strain rate, the scalar product in Eq.(4.2) is the specific rate of energy dissipation.

The yield surface has the expression

$$\sigma_R(\sigma_x, \cdots, \tau_{xy}, \cdots) - \sigma_y = 0. \tag{4.3}$$

Any stress state at the yield limit is represented by a stress point on this surface. For perfectly plastic materials, $\sigma_R$ depends on only the stress state instead of the strain state because these materials do not exhibit workhardening. The yield surface is therefore a fixed surface in the six-dimensional space.

The yield surface can be represented by the equations of $\sigma_R(\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{yz})=\sigma_Y$, where $\sigma_R(\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy},\ \tau_{yz},\ \tau_{yz})$ is a potential function for the strain rates because normality of $\{\dot{\varepsilon}\}$ to the surface at the stress point $P$ gives

$$\dot{\varepsilon}_x = \lambda\frac{\partial\sigma_R}{\partial\sigma_x}, \cdots, \gamma_{xy} = \lambda\frac{\partial\sigma_R}{\partial\sigma_{xy}}, \cdots, \qquad (x, y, z), \tag{4.4}$$

where $\lambda$ is a positive scalar factor. When generalized to vertices and flats, Eq.(4.4) is also called the plastic potential flow law.

## 4.4 Lower-bound Theorem

If a stress distribution balances the applied load, and is below yield or at yield throughout the structure, the structure will not collapse or will just be at the point of collapse. This gives a lower bound of the limit load and is called the lower bound theorem. The maximum lower bound is the limit load.

We can define a statically admissible stress field as being in internal equilibrium when in balance with the external load $\lambda_p$, and not exceeding the yield limit anywhere. A multiplier $\lambda$ is used to define the load magnitude acting on the structure. The multiplier $\lambda$ corresponding to a statically admissible stress field is called a statically admissible multiplier. The lower bound theorem can be stated by saying that the limit load factor $\lambda^0$ is the largest statically admissible multiplier $\lambda^-$, i.e.,

$$\lambda^- \leqslant \lambda^0. \tag{4.5}$$

## 4.5 Upper-bound Theorem

The structure will collapse if there is any compatible pattern of plastic deformation for which the rate of the external forces at work is equal to or exceeds the rate of internal dissipation. It gives the upper bound of the limit or collapse load, and thus is called the upper bound theorem. The minimum upper bound is the limit load.

The upper bound theorem can be stated in view of the admissible multiplier as follows: the limit load factor $\lambda^0$ is the smallest kinematically admissible multiplier $\lambda^+$, i.e.,

$$\lambda^+ \geqslant \lambda^0. \tag{4.6}$$

The above theorems define the upper and lower bounds for the limit load. They can be summarized as

$$\lambda^- \leqslant \lambda^0 \leqslant \lambda^+ \ . \tag{4.7}$$

## 4.6 Fundamental Limit Theorems

When considering a structure which is subjected to a system of loads that start from zero and increase quasi-statically and proportionally, the term "quasi-static" indicates that the loading process is sufficiently slow for all dynamic effects to be disregarded. The term "proportionally" implies that the ratios of the stresses at the same locations of any two different loads are constant throughout the structure. A specific type of loading is determined by the loading location, the direction, and the ratios of stresses at different locations. Choosing one of the loads, we use its magnitude $P$ as a measure for the loading intensity. The variable $P$ is then called the loading parameter.

For beams and frames the transition from a purely elastic region through restricted plastic deformation to unrestricted plastic flow has been extensively studied. For complex structures however it is not straightforward. Emphasis has been put on the direct determination of the limit state in which the plastic deformation in the plastic zones is no longer restricted by the adjacent non-plastic zones and the structure begins to flow under constant loads. The intensity of loading for this limit state is called the limit load, which is usually denoted as $P_l$.

Limit analysis of a structure is concerned with the limit states of structures under loads. The incipience of the limit state of unrestrained plastic flow is characterized by two phenomena:

a) The stresses are in equilibrium state with the applied loads $P$, and satisfy the yield condition $\sigma_R = \sigma_Y$ all over the domain. Such a stress field is called statically admissible.

b) The flow mechanism satisfies the kinematical boundary conditions, and for energy balance the power of the applied loads $P$ is equal to the power dissipated in the plastic flow. Such a flow mechanism is called kinematically admissible.

For a given type of loading there may be numerous statically admissible stress fields. Each of the fields corresponds to a certain intensity of loading, which is denoted as $P_-$. Similarly, for a given kinematically admissible mechanism and a given type of loading, an intensity of loading $P_+$ can be defined in such a manner that the power of the loads at this intensity of loading is equal to the power of dissipation in the yield mechanism.

The fundamental theorems for limit analysis can then be stated as follows:

a) Static (or lower bound) theorem: the limit load $P_l$ is the largest of all loads $P_-$ corresponding to statically admissible stress fields.

b) Kinematical (or upper bound) theorem: the limit load $P_l$ is the smallest of all loads $P_+$ corresponding to kinematically admissible mechanisms.

## 4.7 Important Remarks

### 4.7.1 Exact Value of the Limit Load (Complete Solution)

Assuming that a statically admissible stress field and a kinematically admissible mechanism that correspond to the same load $P$ have been identified, there are $P \leqslant P_l$ and $P \geqslant P_l$ according to the aforementioned two fundamental theorems. Hence, $P = P_l$ is the exact limit load. It very often happens that it is possible to associate a statically admissible stress field and a kinematically admissible mechanism by the plastic potential flow law. The work equation defining $P_+$ can then be regarded as a virtual work equation expressing the equilibrium of the associated stress field. Consequently $P_+ = P_-$, and denoting this common value as $P$, we have $P = P_l$.

A combined theorem is thus derived when it is possible to associate a statically admissible stress field and a kinematically admissible mechanism by the plastic potential flow law and the load $P$ corresponding simultaneously to both fields is the exact limit load $P_l$.

The two fields of above form are called a complete solution of the limit analysis of a structure. For practical application, one usually starts from a mechanism or from a statically admissible stress field and then searches for the other field.

### 4.7.2 Elastic-plastic and Rigid-plastic Bodies

With the elastic-plastic idealization, the limit state matches the incipient unrestrained plastic flow. For an example of a relevant displacement $\delta$ versus the applied load $P$ (Fig.4.2 (a)), the first part is a ray $OA$ (elastic range), the second part a curve $AB$ (elastic-plastic range, restricted plastic flow) followed by the part parallel to the axis, which indicates unrestrained plastic flow.

With the rigid-plastic idealization, all deformations up to the onset of unrestrained plastic flow at the limit load $P_l$, sometimes also called the yield-point load, vanish (Fig.4.2 (b)).

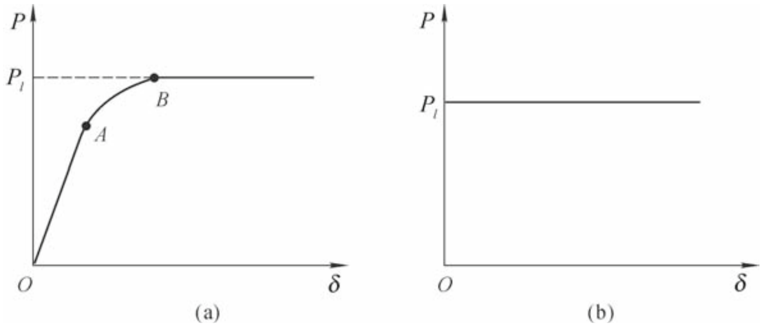


**Fig. 4.2.** Elasto-plastic (a) and rigid-plastic (b) idealization cases

On the other hand, the fundamental theorems of limit analysis are identical for both idealizations. They are based exclusively on the concepts of statically admissible stress fields and kinematically admissible plastic strain rate fields, irrespective of the elastic or rigid nature of the material before yielding. Thus the lower bound $P_-$, the upper bound $P_+$, and the complete solutions are valid for both idealizations.

### 4.7.3 Load-bearing Capacity

To derive the load-bearing capacity of a structure it requires:

a) an equilibrium stress field satisfying $\sigma_R \leqslant \sigma_Y$;

b) a field of plastic strains at impending unrestrained plastic flow.

From the limit analysis point of view, the stress field is statically admissible. The strain field specifies a kinematically admissible mechanism that corresponds to the stress field by the plastic potential flow law. Consequently, a “load-bearing capacity” determined by the deformation theory is the exact limit load for limit analysis.

### 4.7.4 Uniqueness

The limit load for proportional loading $P_l$ is unique, since it simultaneously matches a statically admissible stress field ($\{\sigma\}$) and a kinematically admissible strain rate field ($\{\dot{\varepsilon}\}$). If there exist several limit loads, the fundamental theorems indicate that $P^*$ must be equal to any one of them. Hence $P_l$ is unique and coincides with $P^*$.

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