16

Unified Solution of Shakedown Limit for Circular Plate

16.1 Introduction

The static shakedown theorem (Melan's theorem, the first shakedown theorem, or the lower bound shakedown theorem, Melan, 1936) and the dynamic shakedown theorem (Koiter's theorem, the second shakedown theorem, or the upper bound shakedown theorem, Koiter, 1953; 1956; 1960) and the unified solution of shakedown limit for a thick-walled cylinder have been described in Chapter 15. In this chapter we will deal with the shakedown analysis for a simply supported circular plate and a clamped circular plate. The unified solutions are given for non-SD materials.

Circular plates are used widely in many branches of engineering. They are often subjected to repeated transverse load. Hence it is necessary to conduct the shakedown analysis in order to determine the shakedown load of the plate. Under the varying load the circular plate will deform in elastic and plastic states. The elasto-plastic response of a circular plate to varying loads is a complicated process (Symonds, 1951; König, 1987). The previous shakedown analysis is based on Koiter's upper bound shakedown theorem with the Tresca and Huber-von-Mises yield criteria (Kachanov, 1971; Gokhfeld and Cherniavski, 1980; König, 1978; 1987; Pham, 1996; 1997; 2003). By using numerical methods and based on the static shakedown theorem, the shakedown analysis of perfectly plastic, different kinematic hardening materials is carried out by Stein et al. (1993), Polizzotto (1982; 1993), Ponter and Carter (1997), Maier et al. (2000; 2001).

As discussed by Pham (1997), the shakedown limit will depend on the different yield criteria. From the previous studies we know that the limit load analysis should use different yield criterion for different materials. The Yu unified yield criterion (UYC) will be used in this chapter to investigate the shakedown limit when the circular plates suffer from quasi-static recycle loadings. If the load does not exceed the critical value, the circular plate will behave plastically at first and then elastically and the structure will shakedown due to the repeated loading.

The elastic, plastic and shakedown analysis of a circular plate, which is simply supported or clamped at the edges, will be carried out in this chapter. By choosing proper values for unified yield criterion parameter b, the solution can be applicable to plates made of different materials. In addition, by applying the solution based on the unified yield criterion, the effects of the yield criterion on the shakedown load of the plate are evaluated.

16.2 Unified Solution of Shakedown Limit for Simply Supported Circular Plate

A circular plate with radius a and thickness h is subjected to a uniformly distributed transverse load P, as shown in Fig.16.1, the only non-zero stresses are σ_r, σ_θ and $\tau_{rz} = \tau_{zr}$ in the plate. The generalized stresses can be expressed as

$$\begin{aligned} M_r &= \int_{-h/2}^{h/2} \sigma_r z \mathrm{d}z, \qquad M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z \mathrm{d}z, \\ Q_{rz} &= \int_{-h/2}^{h/2} \tau_{rz} z \mathrm{d}z, \quad M_0 = \int_{-h/2}^{h/2} \sigma_0 z \mathrm{d}z = \sigma_0 h^2/4, \end{aligned} \tag{16.1}$$

where M_r, M_θ and M_0 are the radial, tangential and ultimate (fully plastic) bending moments, respectively, and Q_{rz} is the transverse shear force which is assumed not to influence the plastic yielding.

Defining dimensionless variables, $r = R/a$, $m_r{=}M_r/M_0$, $m_\theta{=}M_\theta/M_0$ and $p{=}Pa^2/M_0$, the equilibrium equation of a circular plate subjected to a constant uniform load is

$$\frac{\mathrm{d}(rm_r)}{\mathrm{d}r} - m_\theta = -\frac{pr^2}{2}. \tag{16.2}$$

When subjected to a uniformly distributed transverse load P, the plate will deform and be in an elastic state, elastic-plastic state and a completely plastic state.

16.2.1 Elastic State

The deformation and the stress state of the plate are in an elastic state when the load p is not big. The dimensionless radial and tangential bending moments m_r and m_θ for a simply supported plate can be written as (Timoshenko and Woinowsky-Krieger, 1959)

$$m_r = \frac{3+\nu}{16} p(1-r^2), \;\; m_\theta = \frac{p}{16}\left[(3+\nu) - (1+3\nu)r^2\right]. \tag{16.3}$$

The elastic limit load p_e can be calculated from Eq.(16.3),

$$p_e = \frac{16}{3+\nu}. \tag{16.4}$$

16.2.2 Elastic-plastic State

The center of the plate ($r = 0$) will firstly go into yield state when $p > p_e$. The plate is in plastic state ranges from 0 to r_e and the plate is in elastic state ranges from r_e to 1. In the plastic zone, if the UYC is chosen as the yield function, the expression of UYC can be written as a piecewise linear function

$$m_\theta = a_i m_r + b_i \qquad (i = 1, ..., 12), \tag{16.5}$$

where the values of parameters a_i and b_i are shown in Table 5.1.

Substituting Eq.(16.5) into Eq.(16.2) and then integrating Eq.(16.2), m_r falling on the segments L_i (the lines shown in Fig.5.3) is obtained as follows:

$$m_r = \frac{b_i}{1-a_i} - \frac{pr^2}{2(3-a_i)} + c_i r^{-1+a_i} \qquad (i = 1, ..., 5), \tag{16.6}$$

where c_i are the constants and can be derived from the continuous and boundary conditions. For a simply supported circular plate going into a plastic state, the bending moments at every point in the plate are located on the sides AB and BC for the normality requirement of plasticity (Ma et al., 1999). Therefore, i should be 1 and 2 in the plastic zone when the plate is in an elastic-plastic state (Fig.16.1).

In elastic zone, the bending moments can be expressed as (Timoshenko and Woinowsky-Krieger, 1959)

$$m_r = \frac{B}{r^2} - C - \frac{3+\nu}{16}pr^2, \ m_\theta = -\frac{B}{r^2} - C - \frac{1+3\nu}{16}pr^2, \tag{16.7}$$

where B and C are the constants. They can be derived from the continuous and boundary conditions.

The boundary and continuous conditions are:

(a) $m_r\ (r = 0) = m_\theta\ (r = 0) = 1$;

(b) $m_r\ (r = r_1)$ and $m_\theta\ (r = r_1)$ are continuous, and $m_r\ (r = r_1) = (1 + b)/(2+b)$, where r_1 is the non-dimensional radius of a ring where the moments correspond to point B in Fig.5.2 where UYC is expressed by generalized stresses;

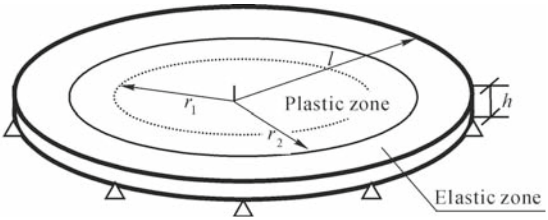

Fig. 16.1. Elastic-plastic state for a simply supported circular plate

(c) m_r $(r = r_2)$ and m_θ $(r = r_2)$ are continuous, where r_2 is the non-dimensional radius of plastic zone;

(d) m_r $(r = 1) = 0$.

The integration coefficients c_i $(i = 1,\ 2)$ can be derived from Eqs.(16.5) and (16.6) and the boundary and continuous conditions as follows:

$$c_1 = 0, \quad c_2 = \frac{2(1+b)}{3+2b} r_1^{\frac{1}{1+b}}, \tag{16.8}$$

and the relation between load p and r_1 is

$$p = \frac{6+2b}{2+b} \frac{1}{r_1^2}. \tag{16.9}$$

The relations between B, C, r_2 and r_1 are obtained from Eq.(16.5) to Eq.(16.7) and the boundary and continuous conditions (c) and (d) as follows:

$$\begin{aligned} \frac{B}{r_2^2} - C = (1+b) + \left(\frac{3+\nu}{8} \cdot \frac{3+b}{1+b} - \frac{3+b}{2+b} \cdot \frac{1+b}{3+2b} \right) \left(\frac{r_2}{r_1} \right)^2 \\ - \frac{2b(1+b)}{3+2b} \left(\frac{r_1}{r_2} \right)^{\frac{1}{1+b}}, \end{aligned} \tag{16.10}$$

$$B - C = \frac{3+\nu}{8} \cdot \frac{3+b}{2+b} \frac{1}{r_1^2}, \tag{16.11}$$

$$\begin{aligned} \frac{B}{r_2^2}(1+2b) + C + \left[\frac{(1+3\nu)(1+b)(3+b)}{8(2+b)} - \frac{(3+\nu)b(3+b)}{8(1+b)} \right] \left(\frac{r_2}{r_1} \right)^2 \\ + (1+b) = 0. \end{aligned} \tag{16.12}$$

The constants B, C, and plastic zones r_2 and r_1 can be derived from Eq.(16.8) to Eq.(16.11). Then the moment field in elastic and plastic zones can be obtained for a given load p.

16.2.3 Completely Plastic State

If the plate goes completely into a plastic state, r_2 will equal 1, and the plastic limit load p_p derived from Eq.(16.9) is

$$p_p = \frac{6+2b}{2+b}\frac{1}{r_1^2}, \tag{16.13}$$

where r_1 satisfies the following equation:

$$-(3+2b)(2+b)+(3+b)r_1^{-2}+2b(2+b)r_1^{1/(1+b)}=0. \tag{16.14}$$

16.2.4 Shakedown Analysis

If the circular plate is unloaded from the initial elastic-plastic state, i.e. $p \to 0$, it will be left with residual stresses. Here we assume that the residual stresses will not produce inverse yielding. In that case the unloading process is purely elastic. So from the elastic solution of the circular plates, we obtain the changes in m_r and m_θ as

$$\Delta m_r = -\frac{p}{16}(3+\nu)(1-r^2),\ \Delta m_\theta = -\frac{p}{16}\left[(3+\nu)+r^2(1+3\nu)\right]. \tag{16.15}$$

The residual stresses in the plate are

$$\begin{aligned} m_r^r &= \frac{b_i}{1-a_i} - \frac{pr^2}{2(3-a_i)} + c_i r^{-1+a_i} - \frac{p}{16}(3+\nu)(1-r^2), \\ m_\theta^r &= a_i m_r + b_i - \frac{p}{16}[(3+\nu)+r^2(1+3\nu)], \qquad (i=1,2). \end{aligned} \tag{16.16}$$

Observing the residual stress, it can be seen that the reverse yielding would begin first at the center of the plate, that is,

$$m_r^r|_{r=0} = m_\theta^r|_{r=0} = -1. \tag{16.17}$$

Eqs.(16.16) and (16.17) lead to the minimum uniformly distributed transverse load p_s for reverse yielding to occur in the plate

$$p_s = \frac{32}{1+\nu} = 2p_e. \tag{16.18}$$

Evidently, as long as the applied transverse load p does not exceed p_s, the residual stresses will not result in reverse plastic deformation in the circular plate.

When the transverse load p, not exceeding the original value, acts on the circular plate and is then removed, the loading-unloading process will not result in a new plastic deformation in the plate. From the above analysis it can been found that if a simply supported circular plate is subjected to cyclic pressure that ranges from $0 \to p \to 0 \to p \to 0 \to \cdots$, and p does not exceed p_p (for the first loading from $0 \to p$) and $2p_e$ (for the other loading from $p \to 0 \to p \to 0 \to \cdots$), yielding will not occur in the circular plate during the loading-unloading process and the circular plate is safe. When this happens the circular plate is said to be in shakedown. Hence the shakedown limit for a circular plate subjected to a load p is

$$p_s = \min\{2p_e, p_p\}. \tag{16.19}$$

16.2.5 Discussion

There is always $2p_e > p_p$ in Eq.(16.19) for ν and parameter b. So p_s is equal to p_p or $(6+2b)/(2+b)/r_1^2$, where r_1 is satisfied with Eq.(16.14). The parameter b shows the effect of intermediate principal stress and the difference of various yield criteria. The influences of parameter b on the shakedown limit p_s and r_1 are analyzed. Figs.16.2 and 16.3 indicate that the unified yield criterion parameter b will influence both p_s and r_1. The shakedown limit p_s is the smallest for b=0 (corresponding to the Tresca criterion) and is the biggest for b=1.0 (corresponding to the twin-shear yield criterion). The difference of p_s for these two cases of b=0 and b=1.0 is about 14%. Fig.16.3 shows that the radius of $m_{\theta\max}$ decreases with the increase in the unified yield criterion parameter b.

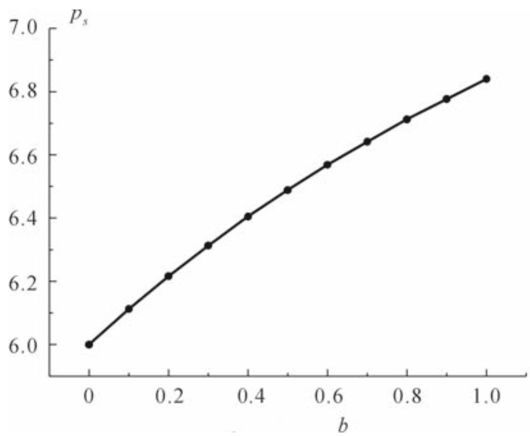

Fig. 16.2. Unified solution of shakedown limit p_s for simply supported circular plate

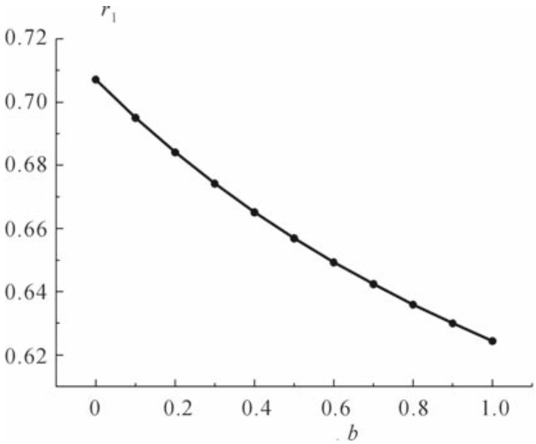

Fig. 16.3. Effect of the unified strength theory parameter b on the radius of $m_{\theta\max}$

16.3 Unified Solution of Shakedown Limit for Clamped Circular Plate

16.3.1 Elastic State

In an elastic state the moment fields of a clamped circular plate satisfy $m_r = m_\theta$ at the center of the plate ($r = 0$), $m_\theta = \nu m_r$ at the clamped edge and $m_\theta > m_r$ at other points on the plate. The dimensionless radial and tangential bending moments m_r and m_θ for a clamped plate can be written as (Timoshenko and Woinowsky-Krieger, 1959)

$$m_r = \frac{p}{16}[(1+\nu) - r^2(3+\nu)],\ m_\theta = \frac{p}{16}[(1+\nu) - r^2(1+3\nu)], \tag{16.20}$$

the elastic limit load p_e can be calculated from Eq.(16.20),

$$p_e = \frac{16}{1+\nu}. \tag{16.21}$$

16.3.2 Elastic-plastic State

When the plate is in an elastic-plastic state, the boundary and continuous conditions are

(a) $m_r\ (r = 0) = m_\theta\ (r = 0) = 1$;

(b) $m_r\ (r = r_j)$ are continuous, where $j = 1, ..., 4$;

(c) $m_r\ (r = r_e)$ and $m_\theta\ (r = r_e)$ are continuous, where r_e is the dimensionless radius of plastic zone and $r_j < r_e$;

(d) $m_r = m_\theta/\nu$ at $r = 1$.

In the plastic zone ($0 \leqslant r \leqslant r_e$) the generalized stresses satisfy the UYC and have the same form of Eq.(16.6). The moment fields of the entire clamped plate lie on the five sides corresponding to AB, BC, CD, DE and EF (Ma et al., 1999).

In the elastic zone ($r_e \leqslant r \leqslant 1$) the generalized stresses can be expressed as the same form of Eq.(16.7), where r ranges from r_e to 1 and the constants B and C can be derived from the continuous and boundary conditions (c) and (d).

16.3.3 Completely Plastic State

When the plate is in a completely plastic state, there is $r_e = 1$ and $m_r(\ r = 1) = m_\theta/\nu = -(1+b)/(1+b-\nu b)$. The load-carrying capacity of a clamped plate in this state has been obtained by Ma et al. (1999),

$$p_p = \frac{6+2b}{2+b}\frac{1}{r_1^2}, \tag{16.22}$$

where r_1 can be solved from the boundary and continuous conditions.

16.3.4 Shakedown Analysis

Using the similar analyzing method, the residual stresses can be obtained when the plate is unloaded from the initial load of the elasto-plastic state to zero, i.e. $p \to 0$.

$$\begin{aligned} m_r^r &= \frac{b_i}{1-a_i} - \frac{pr^2}{2(3-a_i)} + c_i r^{-1+a_i} - \frac{p}{16}[(1+\nu) - r^2(3+\nu)], \\ m_\theta^r &= a_i m_r + b_i - \frac{p}{16}\left[(1+\nu) - r^2(1+3\nu)\right], \quad (i = 1, ..., 5). \end{aligned} \tag{16.23}$$

Observing the residual stress, it can be seen that the reverse yielding would begin first at the center of the plate, that is,

$$m_r^r|_{r=0} = m_\theta^r|_{r=0} = -1. \tag{16.24}$$

Eqs.(16.23) and (16.24) lead to the minimum uniformly distributed transverse load p_s for reverse yielding to occur in the plate

$$p_s = \frac{32}{1+\nu} = 2p_e. \tag{16.25}$$

So the shakedown limit for a clamped circular plate subjected to a load p is

$$p_s = \min\{2p_e, p_p\}. \tag{16.26}$$

16.3.5 Discussion

There is always $2p_e > p_p$ in Eq.(16.26) for every ν and parameter b. So p_s is equal to p_p or $(6 + 2b)/(2 + b)/r_1^2$. When $b = 0$, the UYC becomes the Tresca criterion and the load-bearing capacity of the circular plate with respect to the Tresca criterion is 11.258 in the case of ν=0.25 which is in good agreement with the analyzing result p_s=11.26 (ν=0.25) in the reference (Pham, 1997) with error at approximately 0.018%. In the same case of the Poisson's ratio, the shakedown solution using UYC is 12.23 when b=0.5 (near to the von-Mises criterion), while the shakedown result using Mises material (Pham, 1997) is 12.23 with M_0 being substituted by $2/3^{0.5}M_0$. The difference between these two results is only about 7.3%.

The relations between the unified strength theory parameter b with the shakedown limit p_s and the radius r_1 of $m_{\theta\max}$ are illustrated in Fig.16.4 and 16.5. Both the figures show that the parameter b affects the values of shakedown limit p_s and the radius r_1. It can be seen in Fig.16.4 that for a given kind of Poisson's ratio, the shakedown limit p_s is the smallest in the case of b=0, and p_s is the biggest in the case of b=1. For three kinds of Poisson's ratio, the p_s-b curve increases most slowly when ν=0.

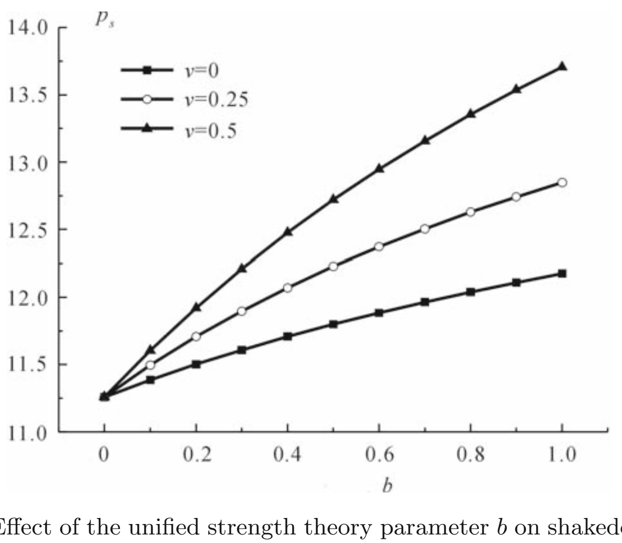

Fig. 16.4. Effect of the unified strength theory parameter b on shakedown limit p_s

When b=1, the difference of the shakedown limit p_s for the case of ν=0 and ν=0.5 is about 1.631. For a given Poisson's ratio in Fig.16.5, the radius of $m_{\theta\max}$ decreases with an increase in the unified strength theory parameter b. For a given parameter b, the shakedown limit p_s for ν=0.5 is the biggest, while p_s for ν=0 is the smallest.

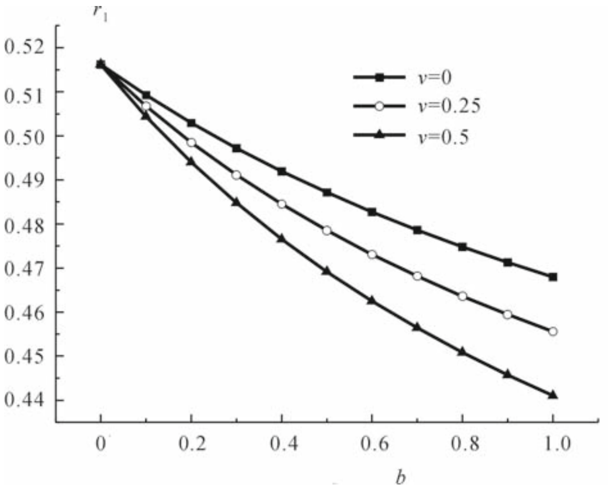

Fig. 16.5. Effect of the unified strength theory parameter b on the radius of $m_{\theta\max}$

16.4 Comparison between Shakedown Solution and Limit Results

Based on the unified yield criterion, a shakedown analysis of a circular plate under a uniformly distributed transverse load is carried out and the unified solution of a shakedown load for a circular plate is derived in this chapter. The solution encompasses the existing classical solution as a special case and a series of new results.

From the above analysis it is noted from Eqs.(16.19) and (16.26) that the shakedown solutions for simply supported and clamped plates are almost the same, that is, both are the minimum of the twice elastic limit load and the plastic limit load,

$$p_s = \min\{2p_e, p_p\}.$$

Because the unified strength parameter b ranges from 0 to 1 and the Poisson's ratio is from 0 to 0.5 no matter what the value of parameter b and the Poisson's ratio, there is always $2p_e > p_p$. Therefore the shakedown solution p_s of a circular plate is equal to its plastic limit load p_p; i.e. $p_s = p_p = (6+2b)/(2+b)/r_1^2$, which is related to the parameter b for the simply supported circular plate and both the parameter b and the Poisson's ratio for the clamped circular plate, as shown in Figs.16.2 and 16.4.

It is also found that the special solutions for shakedown analysis of circular plates when b=0 are equal to the plastic limit load of the Tresca criterion; i.e. p_s=6.0 for the simply supported circular plate and p_s=11.258 for the clamped circular plate. Meanwhile, when b=0 the shakedown solution of clamped circular plate is the same as the result in the references. Besides the shakedown solution for b=0, the other solutions for different parameter b when it ranges from $b > 0$ to b=1 can also be calculated, as can be seen in Fig.16.2 and Fig.16.4. The following table shows the shakedown limit load for both the simply supported and clamped plates when the unified yield criterion parameter b changes.

Table 16.1. Shakedown limit load p_p with the changes in parameter b

b	0	0.3	0.5	0.6	0.8	1.0
Simply supported plate	6.0	6.31	6.49	6.57	6.71	6.8
Clamped plate (ν=0.25)	11.26	11.9	12.23	12.37	12.63	12.85

16.5 Summary

The unified yield criterion is used to analyze the shakedown limit of a circular plate. The results are applicable for a wide range of materials and structures. The shakedown analysis of the circular plate shows the effect of yield criterion on the plastic limit loads and shakedown loads.

For both a simply supported and a clamped circular plate, the shakedown limit p_s increases with the growth of the unified yield criterion parameter b and the shakedown limit p_s for the simply supported plate is smaller than that for the clamped plate. When b=0, the analyzed result is in good agreement with the result in the references. The study also shows that the radius of $m_{\theta\max}$ decreases with the growth of the unified yield criterion parameter b. The shakedown limit p_s for the clamped circular plate is influenced by the Poisson's ratio, while p_s for the simply supported circular plate will not be affected by the Poisson's ratio.

By comparison with the limit load, the shakedown solutions p_s of the simply supported and clamped circular plates are both equal to the plastic limit load. p_s of the simply supported circular plate varies only with the unified yield criterion parameter b, while p_s of the clamped circular plate changes with the unified yield criterion parameter b and the Poisson's ratio.

16.6 Problems

Problem 16.1 Compare the solutions of limit analysis and shakedown analysis.

Problem 16.2 Compare the solutions of shakedown analysis of a simply supported and a clamped circular plate.

Problem 16.3 Determine the shakedown load of a simply supported circular plate by using the Tresca yield criterion (b=0).

Problem 16.4 Determine the shakedown load of a clamped circular plate by using the Tresca yield criterion (b=0).

Problem 16.5 Determine the shakedown load of a simply supported circular plate by using the twin-shear yield criterion (b=1).

Problem 16.6 Determine the shakedown load of a clamped supported circular plate by using the twin-shear yield criterion (b=1).

Problem 16.7 Determine the shakedown load of a simply supported circular plate by using the unified yield criterion with b=0.5.

Problem 16.8 Determine the shakedown load of a clamped circular plate by using the unified yield criterion with b=0.5.

Problem 16.9 Determine the shakedown load of a simply supported circular plate by using the unified yield criterion with b=0.8.

Problem 16.10 Determine the shakedown load of a clamped circular plate by using the unified yield criterion with b=0.8.

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