Robust Wireless Network Jamming Problems

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Abstract. We extend the results presented in a previous paper which focused on deterministic formulations of the WIRELESS NETWORK JAMMING PROBLEM. This problem seeks to determine the optimal number and placement locations for a set of wireless jamming devices to sufficiently suppress a communication network according to some specified criterion. We now introduce robust variants of those formulations which account for the fact that the exact topology of the network to be jammed may not be known entirely. Particularly, we consider instances in which several topologies are considered likely, and develop robust scenarios for placing jamming devices which are able to suppress the network regardless of which candidate topology is realized. We derive several formulations and include percentile constraints to account for a variety of scenarios. Case studies are presented and the results are analyzed. We conclude with directions of future research.

1 Introduction

Research on suppressing and eavesdropping communication networks has seen a surge recently in the optimization community. Two recent papers by Commander et al. [5,6] represent the current state-of-the-art. These problems have several important military applications and represent a critical area of research as optimization of telecommunication systems improve technological capabilities [12]. In [5], the authors develop lower and upper bounds for the optimal number of wireless jamming devices required to suppress a network contained in a given area such as a map grid. In this work, there were no a priori assumptions made about the topology of the network to be jammed other than the geographical region in which it was contained. This problem is particularly important in the global war on terrorism as improvised explosive devices (IEDs) continue to plague the coalition forces. In fact, IEDs account for approximately 65% of all combat injuries in Iraq [11]. These homemade bombs are almost always detonated by some form of radio frequency device such as cellular telephones, pagers, and garage door openers. The ability to suppress radio waves in a given region will help prevent casualties resulting from IEDs [4].

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In [6] the WIRELESS NETWORK JAMMING PROBLEM (WNJP) was introduced and several formulations derived. In the WNJP, the topology of the network is assumed and various objectives can be considered, from jamming all the communication nodes to constraining the connectivity index of the nodes. In this chapter, we introduce robust variants of those formulations which account for the fact that the exact topology of the network to be jammed may not be known entirely. Particularly, we consider instances in which several topologies are considered likely, and develop robust scenarios for placing jamming devices which are able to suppress the network regardless of which candidate topology is realized. The overarching goal is to develop robust formulations with respect to the uncertainties in the information about the network. These models will provide a more realistic interpretation of combat scenarios in urban and dynamic environments.

The organization of the chapter is as follows. In Section 2, we derive several formulations of the ROBUST WIRELESS NETWORK JAMMING PROBLEM (R-WNJP). In Section 3, we review several percentile measures and incorporate percentile constraints into the models in Section 4. The results of several case studies are presented in Section 5 and the results are analyzed. We conclude with directions of future research.

2 Problem Formulations

Denote a graph $G = (V, E)$ as a pair consisting of a set of vertices V, and a set of edges E. All graphs in this chapter are assumed to be undirected and unweighted. We use the symbol "$b := a$" to mean "the expression a defines the (new) symbol b" in the sense of King [9]. Finally, we will use *italics* for emphasis and SMALL CAPS for problem names.

We assume that the communication network to be jammed comprises a set $\mathcal{M} = \{1, 2, \ldots, m\}$ of radio devices which are outfitted with omnidirectional antennas and function as both transmitters and receivers [6]. Further we assume that the coordinates of the nodes and various parameters such as the frequency range are given by probability distributions. For example, we can assume that a Kalman filter provides some estimate of the locations of the nodes. In a deterministic setup, the topology, which represents the communication pattern, could be represented by a graph in which an edge connects two nodes if they are within a certain communication threshold.

As for the set of jamming devices, we assume that they too are outfitted with omnidirectional antennas with the effectiveness of a jamming device on a communication node being inversely proportional to their squared distance. Suppose that the set of jamming devices is giving by $\mathcal{N} = \{1, 2, \ldots, n\}$, and we are given a set potential locations in which to place them. Figure 1 provides an example of the communication network and the potential jamming device locations. Moreover, each potential location j has an associated cost $c_j, j = 1, 2, \ldots, n$. We can describe the jamming power received by network node

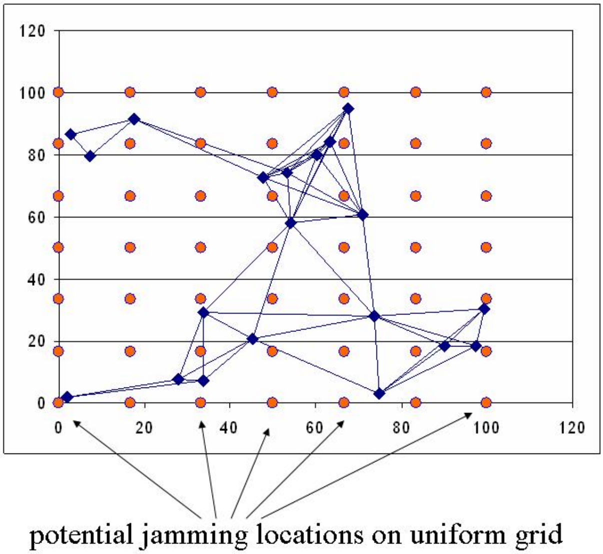

Fig. 1. Example network shown with potential jamming device locations

i located at a point $(\xi_i, \eta_i) \in \mathbb{R} \times \mathbb{R}$, from jamming device $j \in \mathcal{N}$ located at (x, y) as

$$d_j(\xi_i, \eta_i) \equiv d_{ij} := \frac{\lambda}{(x_j - \xi_i)^2 + (y_j - \eta_i)^2}, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a constant. Without the loss of generality, we can let $\lambda = 1$. We say that node $i \in \mathcal{M}$ located at (ξ_i, η_i) is jammed if the total energy received at this point from all jamming devices exceeds some threshold value C_i. That is, node i is jammed if

$$\sum_{j=1}^{n} d_j(\xi_i, \eta_i) \geq C_i. \tag{2}$$

As mentioned above, we are considering robust formulations of the WNJP. Since the exact locations of the network nodes are unknown, we assume that a set of intelligence data has been collected and from that a set $\mathcal{S}$ of the most likely scenarios have been compiled. We assume that for scenario $s \in \mathcal{S}$ both the node locations $\{(\xi_1^s, \eta_1^s), (\xi_2^s, \eta_2^s), \ldots, (\xi_m^s, \eta_m^s)\}$ and the set of jamming thresholds $\{C_1^s, C_2^s, \ldots, C_m^s\}$ are modeled. We make no assumption on the equality of the number of devises to be jammed in the different scenarios. Therefore, we define for each scenario $s \in \mathcal{S}$, the set $\mathcal{M}_s = \{1, 2, \ldots, m_s\}$ which represents the set of nodes to be jammed, where m_s represents the number of nodes in scenario s. For example, the networks shown in Figure 2 represent a set of possible topologies for the network to be jammed.

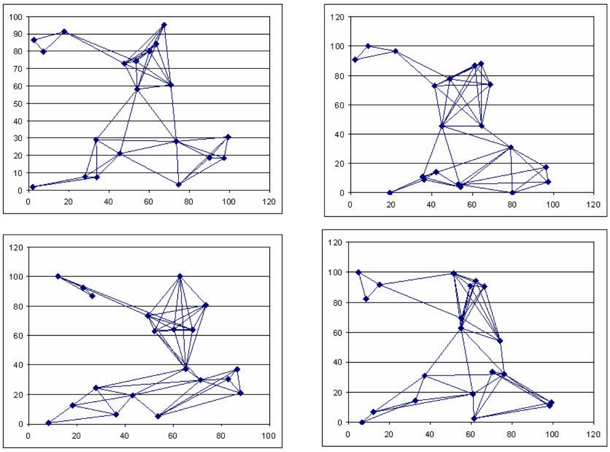

Fig. 2. Four example scenarios are shown

2.1 The Robust Connectivity Index Problem

Given a graph $G = (V, E)$, the connectivity index of a node is defined as the number of nodes reachable from that vertex, i.e., if there exists a path between two nodes. The first formulation of the WNJP we consider imposes constraints on the connectivity indices of the network nodes. The degree to which the connectivity index of a given node is constrained may be determined by its relative importance or how crucial it is for maintaining connectivity among many components. It is at the discretion of the analyst whether to assign arbitrary values to each node or use some heuristic for determining a relative importance. One way to determine the connectivity indices is to identify the so-called *critical nodes* of the graph and impose relatively tighter constraints on these nodes. Critical nodes are those vertices whose removal from the graph induces a set of disconnected components whose sizes are minimally variant [1]. Critical node detection has been recently applied to interdicting wired communication networks [4], to network security applications [2], and most recently to the analysis of protein-protein interaction networks in the context of computational drug design [3].

Suppose for example that for the scenarios shown in Figure 2 the maximum allowable connectivity index is set to 3 for each node. Then the objective of the ROBUST CONNECTIVITY INDEX PROBLEM (RCIP) is to determine the minimum number and locations for the jamming devices so that each node has no more than 3 neighbors in each of the four scenarios presented. Figure 3 provides an example solution for this case.

Suppose that communication between two nodes in the communication graph is said to be severed if at least one of the nodes is jammed. Then the objective of the ROBUST CONNECTIVITY INDEX PROBLEM (RCIP) is to determine the minimum required jamming devices such that the connectivity index of each node i in each scenario s does not exceed some predefined values L_i^s. In order to define

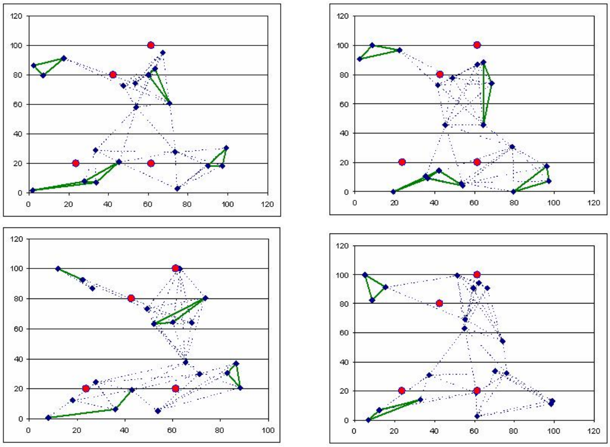

Fig. 3. The optimal solution for the networks in Figure 2 is given for the case when the maximum connectivity index is 3 for all nodes

the corresponding mathematical formulation we must define the following functions. First let $y^s : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ be a surjection where $y^s_{ij} := 1$ if there exists a path from node i to node j in the jammed network according to scenario $s \in \mathcal{S}$. Next let the function $z^s : \mathcal{M}_s \mapsto \{0,1\}$ be a surjective function where z^s_i returns 1 if node i is not jammed in scenario $s \in \mathcal{S}$. Finally, let $x_i, i = 1, \ldots, n$ be a set of decision variables where $x_i := 1$ if a jamming device location i is utilized. If c_k and d_{ij} are as defined in Equation 1, then we can formulate the RCIP as the following optimization problem.

$$\textbf{(RCIP)} \quad \min \sum_{k=1}^{n} c_k x_k \tag{3}$$

s.t.

$$\sum_{\substack{j=1 \\ j \neq i}}^{m_s} y^s_{ij} \leq L^s_i, \ \forall\ i \in \mathcal{M}_s, \ \forall\ s \in \mathcal{S}, \tag{4}$$

$$M(1 - z^s_i) > \sum_{k=1}^{n} d^s_{ik} x_k - C^s_i \geq -M z^s_i, \ \forall\ i \in \mathcal{M}_s, \ \forall\ s \in \mathcal{S}, \tag{5}$$

$$x_j \in \{0,1\}, \ \forall\ j \in \mathcal{N}, \tag{6}$$

$$z^s_i \in \{0,1\}, \ \forall\ i \in \mathcal{M}_s, \ \forall\ s \in \mathcal{S}, \tag{7}$$

$$y^s_{ik} \in \{0,1\}, \ \forall\ i,k \in \mathcal{M}_s, \ \forall\ s \in \mathcal{S}, \tag{8}$$

where $M \in \mathbb{R}$ is some large constant.

In a manner similar to that shown in [6], we can formulate an equivalent integer programming formulation as follows. First let $v^s : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ and $v^{s'} : \mathcal{M}_s \times \mathcal{M}_s \mapsto \{0,1\}$ be respectively defined as

$$v_{ij}^s := \begin{cases} 1, & \text{if } (i,j) \in E^s, \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

and

$$v_{ij}^{s'} := \begin{cases} 1, & \text{if } (i,j) \text{ exists in the jammed network of scenario } s, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

An equivalent integer program is then given by

$$\textbf{(RCIP-1)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{11}$$

s.t.

$$y_{ij}^s \geq v_{ij}^{s'}, \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{12}$$

$$y_{ij}^s \geq y_{ik}^s y_{kj}^s, \ k \neq i,j; \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{13}$$

$$v_{ij}^{s'} \geq v_{ij}^s z_j^s z_i^s, \ i \neq j; \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{14}$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} y_{ij}^s \leq L_i^s, \ \forall\, i \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{15}$$

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \ \forall\, i \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{16}$$

$$x_j \in \{0,1\}, \ \forall\, j \in \mathcal{N}, \tag{17}$$

$$z_i^s \in \{0,1\}, \ \forall\, i \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{18}$$

$$y_{ij}^s \in \{0,1\} \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{19}$$

$$v_{ij}^s \in \{0,1\}, \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}, \tag{20}$$

$$v_{ij}^{s'} \in \{0,1\}, \ \forall\, i,j \in \mathcal{M}_s, \ \forall\, s \in \mathcal{S}. \tag{21}$$

We establish the equivalence of formulations RCIP and RCIP-1 in the following theorem. The proof follows similarly to a result for the single scenario problem in [6].

Theorem 1. *If* RCIP *has an optimal solution, then* RCIP-1 *has an optimal solution. Furthermore, any optimal solution x^* of the integer programming problem* RCIP-1 *is an optimal solution of the optimization problem* RCIP.

Proof. It is easy to see that if communication nodes i and j are reachable in the jammed network of a given scenario $s \in \mathcal{S}$, then $y_{ij}^s = 1$ in RCIP-1. Indeed if i and j are reachable, then there exists a sequence of pairwise adjacent vertices

$$\{(i_0, i_1), \ldots, (i_{m-1}, i_m)\}, \tag{22}$$

where $i_0 = i$ and $i_m = j$. By inducting along the vertices, we can establish the fact that $y^s_{i_0,i_{k+1}} = 1$ for all $k = 1, \ldots, m$. To do this, first note that from (12) we have that $y^s_{i_k,i_{k+1}} = 1$. Then if $y^s_{i_0,i_k} = 1$, then by (13) we have that

$$y^s_{i_0,i_{k+1}} \geq y^s_{i_0,i_k} y^s_{i_k,i_{k+1}} = 1. \tag{23}$$

This completes the induction step. Thus far we have shown that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y^s_{ij} \geq \text{ connectivity index of node } i.$$

Let F be the objective function in RCIP-1 and RCIP. Furthermore, suppose (x^*, y^*) and $(\hat{x}^*, \hat{y}^*)$ represent optimal solutions for each formulation respectively. Then so far, we have confirmed that

$$F(x^*) \geq F(\hat{x}^*). \tag{24}$$

It is easy to verify that $(\hat{x}^*, \hat{y}^*)$ is feasible in RCIP-1. This follows from the definition of y^s_{ij} in RCIP and the fact that $(\hat{x}^*, \hat{y}^*)$ satisfies the feasibility constraints in RCIP. This proves the first statement of the theorem. Hence from RCIP-1, we have that

$$F(x^*) \leq F(\hat{x}^*). \tag{25}$$

Therefore using (24) and (25), we have

$$F(x^*) = F(\hat{x}^*). \tag{26}$$

Now define y^s such that

$$y^s_{ij} := 1 \Leftrightarrow j \text{ is reachable from } i \text{ when scenario } s \text{ is jammed by } x^*. \tag{27}$$

Using the above results, we know that (x^*, y^s) is feasible in RCIP-1, and hence optimal. Also from the construction of y^s it follows that (x^*, y^s) is feasible in RCIP. According to (26), we can conclude that x^* is also optimal for RCIP. Thus the theorem is proved.

With the previous theorem, we have established a one-to-one correspondence between the two formulations. By using some standard techniques, we can now reformulate RCIP-1 into the following integer linear program

$$\textbf{(RCIP-2)} \quad \min \sum_{k=1}^{n} c_k x_k \tag{28}$$

s.t.

$$y^s_{ij} \geq v^{s'}_{ij}, \ \forall \ i, j = 1, \ldots, \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{29}$$

$$y^s_{ij} \geq y^s_{ik} + y^s_{kj} - 1, \ k \neq i, j; \ \forall \ i, j \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{30}$$

$$v^{s'}_{ij} \geq v^s_{ij} + z^s_j + z^s_i - 2, \ i \neq j; \ \forall \ i, j \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{31}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} y_{ij}^s \leq L_i^s, \ \forall \ i \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{32}$$

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \ \forall \ i \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{33}$$

$$x_j \in \{0, 1\}, \ \forall \ j \in \mathcal{N}, \tag{34}$$

$$z_i^s \in \{0, 1\}, \ \forall \ i \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{35}$$

$$y_{ij}^s \in \{0, 1\} \ \forall \ i, j \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{36}$$

$$v_{ij}^s \in \{0, 1\}, \ \forall \ i, j \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}, \tag{37}$$

$$v_{ij}^{s'} \in \{0, 1\}, \ \forall \ i, j \in \mathcal{M}_s, \ \forall \ s \in \mathcal{S}. \tag{38}$$

Theorem 2. *If* RCIP-1 *has an optimal solution, then* RCIP-2 *has an optimal solution. Further, any optimal solution x^* of* RCIP-2 *is an optimal solution of* RCIP-1*.*

Proof. For binary variables, notice that the following equivalence holds

$$y_{ij}^s \geq y_{ik}^s y_{kj}^s \Leftrightarrow y_{ij}^s \geq y_{ik}^s + y_{kj}^s - 1. \tag{39}$$

Then the only other difference between the formulations are the two constraints:

$$v_{ij}^{s'} = v_{ij}^s z_j^s z_i^s \tag{40}$$

$$v_{ij}^{s'} \geq v_{ij}^s + z_i^s + z_j^s - 2 \tag{41}$$

In a manner similar to (39) above, it is easy to verify that (40) implies (41). Therefore the feasible region of RCIP-2 includes the feasible region of RCIP-1. With this we have the first statement of the theorem.

As in the previous proof, let F represent the objective function in RCIP-1 and RCIP-2 and let x^* and $\hat{x}^*$ represent respective optimal solutions. Then it follows that

$$F(x^*) \geq F(\hat{x}^*). \tag{42}$$

Let $(x^*, y^{s*}, v^{'s*}, z^{s*})$ be an optimal solution for formulation RCIP-2. Now, define $v^{''s*}$ as follows:

$$v_{ij}^{''s*} := \begin{cases} 1, & \text{if } v_{ij}^s + z_i^{s*} + z_j^{s*} - 2 = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

Notice that if $v_{ij}^{'s*} \geq v_{ij}^{''s*}$ then $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is feasible in RCIP-2 according to constraint (29) (i.e., $y_{ij}^s \geq v_{ij}^{''s*}$). Furthermore, $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is optimal in RCIP-2 as the the objective value is $F(x^*)$, which is optimal by definition. Using (43), $(v^{''s*}, z^{s*})$ satisfies:

$$v_{ij}^{''s*} = v_{ij}^s z_j^{s*} z_i^{s*}. \tag{44}$$

Using this we have that $(x^*, y^{s*}, v^{''s*}, z^{s*})$ is feasible for RCIP-1. If $\hat{x}^*$ is an optimal solution of RCIP-1 then it follows that

$$F(\hat{x}^*) \leq F(x^*) \tag{45}$$

On the other hand, we have shown in (42) above, that

$$F(x^*) \leq F(\hat{x}^*). \tag{46}$$

(45) and (46) together imply $F(x_1) = F(x^*)$. The last equality proves that x^* is an optimal solution of RCIP-1. Thus, the theorem is proved.

Finally, we have the following theorem which establishes the equivalence between the optimization problem RCIP and the integer linear programming formulation RCIP-2 [6].

Theorem 3. *If* RCIP *has an optimal solution, then* RCIP-2 *has an optimal solution. Moreover, any optimal solution of* RCIP-2 *is an optimal solution of* RCIP.

Proof. The proof follows directly from Theorem 1 and Theorem 2.

2.2 Robust Node Covering Problem

What follows is a robust formulation of the OPTIMAL NODE COVERING problem presented in WNJP. As before, we are given $\mathcal{M}_s$, the set of nodes to be jammed. We are also given the set of potential locations for the jamming devices, $\mathcal{N}$. The objective of the ROBUST NETWORK COVERING PROBLEM (RNCP) is to minimize the number of jamming devices required to suppress communication on all of the nodes for each of the scenarios. Recall from Equation (2) that a node in a given scenario is said to be suppressed if the cumulative amount of energy received by that node from all jamming devices exceeds some threshold level. Let c_k, d_{ik}^s, and C_i^s be as defined previously. Also, recall that the decision variable $x_k := 1$ if a jamming device is installed at location $k \in \mathcal{N}$. With this, we can formulate the RNCP as follows.

$$\textbf{(RNCP)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{47}$$

$$\text{s.t.}$$

$$\sum_{k=1}^{n} d_{ik}^s x_k \geq C_i^s, \; i = 1, 2, \ldots, m_s, s = 1, 2, \ldots, S, \tag{48}$$

$$x_k \in \{0, 1\}, \; k = 1, 2, \ldots, n, \tag{49}$$

The RNCP is $\mathcal{NP}$-hard which can be easily shown by a reduction from the MULTIDIMENSIONAL KNAPSACK PROBLEM [7]. With this in mind, we recognize that solving large-scale instances is unreasonable, thus we must seek alternative solution methods. One possible way of doing this is accepting the fact that jamming a sufficient percentage of the network nodes will suffice given the intractability of the problem. We now examine the R-WNJP with the inclusion of percentile constraints.

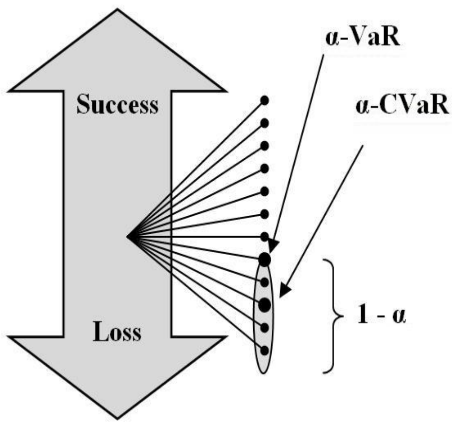

Fig. 4. A graphical depiction of VaR and CVaR

3 Percentile Constraints

As demonstrated in the seminal paper on deterministic network jamming problems [6], it is often the case that a network can be sufficiently neutralized by suppressing communication on a fraction of the total nodes. This can be accomplished by the inclusion of percentile constraints into a mathematical model. In this section, we review two commonly used risk measures and derive formulations of the RCIP and a RNCP.

3.1 Review of Value at Risk (VaR) and Conditional Value at Risk (CVaR)

The first percentile constraint we examine is the simplest risk measure used in optimization of robust systems and is known as the Value at Risk (VaR) measure [8]. VaR provides an upper bound, or percentile on a given loss distribution. For example, consider an application in which a constraint must be satisfied within a specific confidence level $\alpha \in [0, 1]$. Then the corresponding α-VaR value is the lowest value ζ such that with probability α, the loss does not exceed ζ [10]. In economic terms, VaR is simply the maximum amount at risk to be lost from an investment. VaR is the most widely applied risk measure in stochastic settings primarily because it is conceptually simple and easy to incorporate into a mathematical model [6]. However with this ease of use come several complicating factors as we will soon see. Some disadvantages of VaR are that the inclusion of VaR constraints adds to the number of discrete variables in a problem. Also, VaR is not a so-called *coherent* risk measure, implying among other things that it is not sub-additive.

Another commonly applied risk measure is the so-called Conditional Value-at-Risk (CVaR) developed by Rockafellar and Uryasev [13]. CVaR is a more conservative measure of risk, defined as the expected loss under the condition that VaR is exceeded. A graphical representation of the relationship between CVaR and VaR is shown in Figure 4. In order to define CVaR and Var we need to determine the cumulative distribution function for a given decision vector subject to some uncertainties. Suppose $f(x, y)$ is a performance (or loss) function associated with a decision vector $x \in X \subseteq \mathbb{R}^n$, and a random vector $y \in \mathbb{R}^m$ which is the uncertainties that may affect the performance. Assume that y is governed by a probability measure P on a Borel set, say Y [6]. The loss $f(x, y)$ for each $x \in X$ is a random variable having a distribution in $\mathbb{R}$ induced by y. Then the probability of $f(x, y)$ not exceeding some value ζ is defined as

$$\psi(x, \zeta) := P\{y|f(x, y) \leq \zeta\}. \tag{50}$$

By fixing x, the cumulative distribution function of the loss associated with the decision x is thus given by $\psi(x, \zeta)$ [15].

Given the loss random variable $f(x, y)$ and any $\alpha \in (0, 1)$, we can use equation (50) to define α-VaR as

$$\zeta_\alpha(x) := \min\{\zeta \in \mathbb{R} : \psi(x, \zeta) \geq \alpha\}. \tag{51}$$

From this we see the probability that the loss $f(x, y)$ exceeds $\zeta_\alpha(x)$ is $1 - \alpha$. Using the definition above, CVaR is the conditional expectation that the loss according to the decision vector x dominates $\zeta_\alpha(x)$ [13]. Thus we have α-CVaR denoted as $\phi_\alpha(x)$ defined as

$$\phi_\alpha(x) := E\{f(x, y)|f(x, y) \geq \zeta_\alpha(x)\}. \tag{52}$$

In order to include CVaR and VaR constraints in optimization models we must characterize $\zeta_\alpha(x)$ and $\phi_\alpha(x)$ in terms of a function $F_\alpha : X \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$F_\alpha(x, \zeta) := \zeta + \frac{1}{(1 - \alpha)} E\{\max\{f(x, y) - \zeta, 0\}\}. \tag{53}$$

In the seminal paper on CVaR [13], Rockafellar and Uryasev prove that as a function of ζ, $F_\alpha(x, \zeta)$ is convex and continuously differentiable. Moreover, they show that α-CVaR of the loss associated with any $x \in X$, i.e., $\phi_\alpha(x)$, is equal to the global minimum of $F_\alpha(x, \zeta)$, over all $\zeta \in \mathbb{R}$. Further, if $A_\alpha(x) = \operatorname{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x, \zeta)$ is the set consisting of the values of ζ for which F is minimized, then $A_\alpha(x)$ is a non-empty, closed and bounded interval and $\zeta_\alpha(x)$ is the left endpoint of $A_\alpha(x)$. In particular, it is always the case that $\zeta_\alpha(x) \in \operatorname{argmin}_{\zeta \in \mathbb{R}} F_\alpha(x, \zeta)$ and $\psi_\alpha(x) = F_\alpha(x, \zeta_\alpha(x))$ [13].

This result gives a linear optimization algorithm for computing α-CVaR. It is a result of the convexity of $F_\alpha(x, \zeta)$, that we are able to minimize CVaR for $x \in X$ without having to numerically calculate $\phi_\alpha(x)$ for every x. This has been shown by Rockafellar and Uryasev in [14]. Further, it has been shown in [14] that for any probability threshold α and loss tolerance ω, that constraining $\phi_\alpha(x) \leq \omega$ is equivalent to constraining $F_\alpha(x, \zeta) \leq \omega$.

3.2 Robust Jamming with Percentile Constraints

Now that we have theoretical groundwork for the VaR and CVaR percentile measures, we develop formulations of the robust jamming problems incorporating these risk constraints. We begin with the ROBUST NODE COVERING PROBLEM. Since we are given a set of possible network scenarios, we would like to develop a formulation for the RNCP in which the optimal solution will be guaranteed to cover some predetermined fraction $\alpha \in (0,1)$ of the network nodes, regardless of the scenario realized. To do this, we can include α-VaR constraints in the RNCP as follows. First we define the surjection $\rho^s : \mathcal{M}_s \mapsto \{0,1\}$ by

$$\rho_i^s := \begin{cases} 1, & \text{if node } i \text{ is jammed in scenario } s, \\ 0, & \text{otherwise.} \end{cases} \tag{54}$$

Then we can formulate the ROBUST NODE COVERING PROBLEM with Value-at-Risk constraints as

$$\textbf{(RNCP-VaR)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{55}$$

s.t.

$$\sum_{k=1}^{n} d_{ik}^s x_k \geq C_i^s \rho_i^s, \ \forall\, s \in \mathcal{S}, \forall\, i \in \mathcal{M}_s, \tag{56}$$

$$\sum_{i=1}^{m_s} \rho_i^s \geq \alpha m_s, \ \forall\, s \in \mathcal{S}, \tag{57}$$

$$x_k \in \{0,1\}, \ \forall\, k \in \mathcal{N}, \tag{58}$$

$$\rho_i^s \in \{0,1\}, \ \forall\, s \in \mathcal{S}, \forall\, i \in \mathcal{M}_s, \tag{59}$$

Notice that to include the VaR constraints an additional m_s binary variables are required for each scenario $s \in \mathcal{S}$.

In a similar manner, we can incorporate VaR constraints into the RCIP by introducing $\omega^s : \mathcal{M}_s \mapsto \{0,1\}$ as

$$\omega_i^s := \begin{cases} 1, & \text{connectivity index of node } i \text{ is constrained on scenario } s, \\ 0, & \text{no requirement on connectivity index of node } i. \end{cases} \tag{60}$$

Using this we have

$$\textbf{(RCIP-VaR)} \quad \min \sum_{k=1}^{n} c_k x_k \tag{61}$$

s.t.

$$M(1 - z_i^s) > \sum_{k=1}^{n} d_{ik}^s x_k - C_i^s \geq -M z_i^s, \ \forall\, s \in \mathcal{S}, \ \forall\, i \in \mathcal{M}_s, \tag{62}$$

$$\sum_{\substack{j=1\\j\neq i}}^{m_s} y_{ij}^s \leq L_i^s \omega_i^s + M(1-\omega_i^s),\ \forall\ s \in \mathcal{S},\ \forall\ i \in \mathcal{M}_s, \tag{63}$$

$$\sum_{j=1}^{m_s} \omega_j^s \geq \alpha m_s,\ \forall\ s \in \mathcal{S}, \tag{64}$$

$$x_k \in \{0,1\},\ \forall\ k \in \mathcal{N}, \tag{65}$$

$$z_i^s, \omega_i^s \in \{0,1\},\ \forall\ s \in \mathcal{S},\ \forall\ i \in \mathcal{M}_s, \tag{66}$$

$$y_{ik}^s \in \{0,1\},\ \forall\ s \in \mathcal{S},\ \forall\ i,k \in \mathcal{M}_s, \tag{67}$$

where $M \in \mathbb{R}$ is some large constant. As with the **RNCP-VaR** formulation, an additional m_s binary variables are required for each scenario $s \in \mathcal{S}$. We will see in the following section the dramatic effect that the inclusion of VaR constraints has on the computability of optimal solutions.

In order to develop formulations incorporating CVaR constraints, we must derive an appropriate loss function for each problem. We will begin with the RNCP. As in [6], we introduce the function $f^s : \{0,1\}^n \times \mathcal{M}_s \mapsto \mathbb{R}$ defined by

$$f^s(x,i) := C_i^s - \sum_{j=1}^{n} x_j d_{ij}^s. \tag{68}$$

Given a decision vector x representing the placement of the jamming devices, the loss function is defined as the difference between the energy required to jam network node i, namely C_i^s, and the cumulative amount of energy received at node i due to x over each scenario [6]. With this we can formulate the RNCP with CVaR constraints as follows.

$$\textbf{(RNCP-CVaR)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{69}$$

s.t.

$$\zeta^s + \frac{1}{(1-\alpha)m_s} \sum_{i=1}^{m_s} \max\left\{ C_{min}^s - \sum_{k=1}^{n} d_{ik}^s x_k - \zeta^s, 0 \right\} \leq 0,\ \forall\ s \in \mathcal{S}, \tag{70}$$

$$x_k \in \{0,1\},\ \forall\ k \in \mathcal{N}, \tag{71}$$

$$\zeta^s \in \mathbb{R},\ \forall\ s \in \mathcal{S}. \tag{72}$$

The CVaR constraint (70) implies that for the $(1-\alpha)\cdot 100\%$ of the worst (least) covered nodes, the average value of $f(x)$ is less than or equal to 0.

In a similar manner, we can formulate the ROBUST CONNECTIVITY INDEX PROBLEM with the addition of CVaR constraints. As before, we need to define an appropriate loss function. We define the loss function f_s' for a network node i in scenario s as the difference between the connectivity index of i and the maximum allowable connectivity index L_i^s which occurs as a result of the placement of the jamming devices according to x. That is, let $f' : \{0,1\}^n \times \mathcal{M}_s \mapsto \mathbb{Z}$ be defined by

$$f'_s(x,i) := \sum_{j=1, j\neq i}^{m_s} y^s_{ij} - L^s_i. \tag{73}$$

With this, the CIP-CVaR formulation is given as follows.

$$\textbf{(RCIP-CVaR)} \quad \min \sum_{k=1}^{n} c_k x_k, \tag{74}$$

s.t.

$$\zeta^s + \frac{1}{(1-\alpha)m_s} \sum_{i=1}^{m_s} \max \left\{ \sum_{\substack{j=1 \\ j\neq i}}^{m_s} y^s_{ij} - L^s_{max} - \zeta^s, 0 \right\} \leq 0, \ \forall\, s \in \mathcal{S}, \tag{75}$$

$$x_k \in \{0,1\}, \ \forall\, k \in \mathcal{N}, \tag{76}$$

$$y^s_{ik} \in \{0,1\}, \ \forall\, s \in \mathcal{S}, \ \forall\, i,k \in \mathcal{M}_s, \tag{77}$$

$$\zeta^s \in \mathbb{R}, \ \forall\, s \in \mathcal{S}, \tag{78}$$

where L^s_{max} is a maximum allowable connectivity index under scenario s which occurs as a result of the placement of the jamming devices. The constraint on CVaR provides that for the $(1-\alpha)\cdot 100\%$ of the worst cases, the average connectivity index will not exceed $L_{\max}$. Notice that to include the CVaR constraint, we only add real variables to the problem. The continuous nature of CVaR variables versus the discrete nature of the VaR variables will explain the vast difference in the computation times in the case studies presented in the following section.

4 Case Studies

In this section, we present some preliminary numerical results comparing the performance and solution quality of the proposed formulations. The experiments were performed on a PC equipped with a 1.4MHz Intel Pentium R 4 processor with 1GB of RAM, working under the Microsoft Windows R XP SP2 operating system. The optimal solutions for the case studies were calculated using CPLEX 9.0.

The problem set considered is relatively small, but provides some insights into the solutions obtained using VaR and CVaR constraints. The network consists of 20 nodes which must be jammed. For this problem, we consider five network scenarios. We note here that the jamming thresholds of the nodes do not depend upon the scenarios. As for the placement of the jamming devices, we use the same approach as in [6], which consists of 36 potential locations located on the vertices of a uniform grid placed over the region containing the network. One network scenario showing the potential locations of the jamming devices is shown in Figure 1. The remaining scenarios are depicted in Figure 2.

4.1 Node Covering Problems

We begin by examining the ROBUST NODE COVERING PROBLEM. For the first case study, we consider the RNCP with Value-at-Risk constraints. The loss threshold is .9 which implies that the covering constraints must be satisfied for at least 90% of the network nodes. The optimal solution requires 9 jamming devices. CPLEX computed the solution for this problem in 18 seconds. The results of this instance are provided in Table 1. The table shows the total jamming level as a percentage of the jamming threshold received by each node in each scenario. Notice that all but 7 (over all scenarios) were totally jammed; however, for those nodes not totally jammed VaR constraints provide no guarantee that they will receive any jamming energy whatsoever. Though not an important factor in this case, this fact could potentially lead to problems in large-scale instances of the problem.

Next, we examine the same problem only replacing the VaR constraints with Conditional Value-at-Risk constraints. As before, the loss threshold is .90, implying that the maximum allowable losses (uncovered nodes) exceeding VaR must be no greater than 10%. Interestingly, the optimal solution in this case also requires 9 jamming devices. However this solution was computed in 0.922 seconds. The results from this study are shown in Table 2. Notice in this case that with the same number of jamming devices all but 2 nodes (across all scenarios) were totally jammed. We see that not only is this solution better in terms of the total number of jammed nodes, but it was also computed in an order of magnitude less time.

Table 1. Network coverage with VaR constraints. The total jamming level (%) for each scenario is shown.

Node	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
1	100	100	100	100	100
2	100	100	100	100	100
3	100	100	100	100	100
4	100	100	100	100	100
5	100	100	100	100	100
6	100	100	100	100	100
7	100	100	100	100	100
8	100	100	100	100	100
9	100	100	100	100	100
10	100	100	100	100	100
11	100	100	100	100	100
12	100	100	100	100	100
13	100	100	100	100	79.31
14	100	100	100	100	100
15	100	75.74	100	100	100
16	86.45	81.86	100	57.84	100
17	100	100	100	100	100
18	100	100	100	100	100
19	100	100	100	100	100
20	86.47	100	100	65.24	100

Table 2. Network coverage with CVaR constraints. The total jamming level (%) for each scenario is shown.

Node	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
1	100	100	100	100	100
2	100	100	100	100	100
3	100	100	100	100	100
4	100	100	100	100	100
5	100	100	100	100	100
6	100	100	100	100	100
7	100	100	100	100	100
8	100	100	100	100	100
9	100	100	100	100	100
10	100	100	100	100	100
11	100	100	100	100	100
12	100	100	100	100	100
13	100	100	97.25	100	100
14	100	100	100	100	100
15	100	100	100	100	100
16	100	100	100	93.93	100
17	100	100	100	100	100
18	100	100	100	100	100
19	100	100	100	100	100
20	100	100	100	100	100

4.2 Connectivity Index Problems

Now we discuss the results of the case study for the RSCIP with VaR and CVaR constraints. In this case, both maximum connectivity indices are set to $L = 3$. Again, the VaR threshold is .90. The optimal solution for this problem (without percentile constraints) is shown in Figure 3. This solution requires 4 jamming

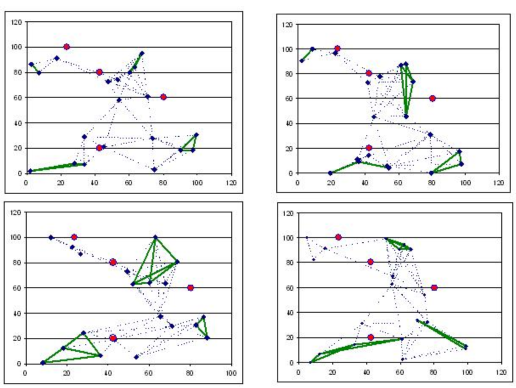

Fig. 5. The optimal solution with VaR constraints for the networks in Figure 2 is given for the case when the maximum connectivity index is 3 for all nodes

devices and was computed in 3 minutes, 58 seconds. The solution using VaR constraints is shown in Figure 5. This solution also required 4 jamming devices, but took 8 hours, 49 minutes, 43 seconds to compute. The same solution was also obtained using CVaR constraints in a time comparable to the original formulation. Even for this small example, we see that including VaR constraints in an optimization model often leads to drastic increases in computation times. This provides more evidence that using CVaR constraints instead is usually more efficient and provides appropriate solutions.

5 Conclusion

In this chapter, we develop models for jamming communication networks under uncertainty. This work extends prior work by the authors in which deterministic cases of the problems were considered [4,6]. In particular, we have developed formulations for jamming wireless networks when the exact topology of the underlying network is unknown. We have used scenario based techniques which provide robust solutions to the problems considered. Future areas of research include investigating the required number of scenarios to accurately model the statistical properties of the data. Due to the complexity of the problems considered, heuristics and advanced cutting plane techniques should also be investigated.

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