On Radio Broadcasting in Random Geometric Graphs*

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Abstract. One of the most frequently studied problems in the context of information dissemination in communication networks is the broadcasting problem. In this paper we consider radio broadcasting in random geometric graphs, in which n nodes are placed uniformly at random in $[0, \sqrt{n}]^2$, and there is a (directed) edge from a node u to a node v in the corresponding graph iff the distance between u and v is smaller than the transmission radius assigned to u. Throughout this paper we consider the distributed case, i.e., each node is only aware (apart from n) of its own coordinates and its own transmission radius, and we assume that the transmission radii of the nodes vary according to a power law distribution. First, we consider the model in which any node is assigned a transmission radius $r > r_{\min}$ according to a probability density function $\rho(r) \sim r^{-\alpha}$ (more precisely, $\rho(r) = (\alpha - 1)r_{\min}^{\alpha-1} r^{-\alpha}$), where $\alpha \in (1, 3)$ and $r_{\min} > \delta\sqrt{\log n}$ with δ being a large constant. For this case, we develop a simple radio broadcasting algorithm which has the running time $O(\log \log n)$, with high probability, and show that this result is asymptotically optimal. Then, we consider the model in which any node is assigned a transmission radius $r > c$ according to the probability density function $\rho(r) = (\alpha - 1)c^{\alpha-1} r^{-\alpha}$, where α is drawn from the same range as before and c is a constant. Since this graph is usually not strongly connected, we assume that the message which has to be spread to all nodes of the graph is placed initially in one of the nodes of the giant component. We show that there exists a fully distributed randomized algorithm which disseminates the message in $O(D(\log \log n)^2)$ steps, with high probability, where D denotes the diameter of the giant component of the graph.

Our results imply that by setting the transmission radii of the nodes according to a power law distribution, one can design energy efficient radio networks with low average transmission radius, in which broadcasting can be performed *exponentially* faster than in the (extensively studied) case where all nodes have the same transmission power.

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1 Introduction

In view of recent technological developments in wireless/mobile communication the abstract model of packet radio networks became very popular and received a lot of attention in the algorithms community [2,5,9,12]. Most of the work on time efficient radio broadcasting done so far is devoted to radio networks with an arbitrary (in fact the worst case) topology. Our main intention is to derive efficient distributed algorithms for radio broadcasting in *random geometric graphs*, which are often used to model wireless communication networks.

1.1 Models and Motivation

A radio network is modeled by a directed graph $G = (V, E)$, where V represents the set of nodes of the network, and E contains ordered pairs of distinct nodes such that $(v, w) \in V \times V$ iff node v can directly send a message to node w. The total number of neighbors connected to a node by (in-)coming edges forms its (in-)degree. The *size of the network* is the number of nodes $n = |V|$. The set of nodes directly reachable from a node $v \in V$ is the *range* of v.

One of the radio network properties is that a message transmitted by a node is always sent to all nodes within its range. The communication in the network is synchronous and it consists of a sequence of (communication) steps. During one step, each node v either transmits or listens. If v transmits, then the transmitted message reaches each of its neighbors by the end of this step. However, a node w in the range of v successfully receives this message iff in this step w is listening and v is the only transmitting node which has w in its range. If node w is in the range of a transmitting node but is not listening, or is in the range of more than one transmitting node, then a *collision* (conflict) occurs and w does not retrieve any message in this step. In fact coping with collisions is one of the main challenges in efficient radio communication. A commonly used tool to handle this problem in radio networks with unknown topology is the concept of selected families of transmission sets [5,7,9,19].

The running time of an algorithm is the number of communication steps required to complete the considered communication task. Thus, any internal computation within individual nodes is neglected. In this paper we are mainly interested in the running time of distributed broadcasting algorithms using radio communication protocol. In the broadcasting problem it is assumed that a message is placed in one of the nodes of a radio network, and the goal is to spread this message to all nodes of the network using radio communication. In this paper we assume that each node knows its own position ((x, y) coordinates), its transmission radius, and the number of nodes in the network. However, the location of the other nodes or their transmission radii are not known.

It is of our particular interest to analyze radio communication in ad hoc sensor networks. Ad hoc sensor networks are often modeled by the so called $G(n, r)$ random geometric graph model (e.g. [18,26,28]), i.e., n vertices with radius r are placed within $[0, \sqrt{n}]^2$ uniformly at random[1], and two nodes are connected by an edge in the resulting graph iff their Euclidean distance is smaller than r. This simple model of radio network is applicable to wireless networks where similar stations are randomly distributed in a flat region without large obstacles. In such a terrain, the signal of a transmitter reaches receivers at the same distance in all directions.

In this paper we consider radio broadcasting in two different types of random geometric networks. Due to simplicity reasons, we assume that n points are distributed uniformly at random within $[0, \sqrt{n}]^2$, however, the radii of the nodes may vary according to a power law distribution, i.e., a node is assigned a transmission radius larger than some value r with probability proportional to $r^{1-\alpha}$, where $\alpha \in (1, 3)$ is a fixed constant. Similar graph models are known to have improved fault tolerance [22] and (as we show in this paper) these networks allow very fast broadcasting, in fact exponentially faster than $G(n, r)$ graphs with polylogarithmic transmission radii, while maintaining almost the same average energy consumption parameters as the corresponding $G(n, r)$ model. We should note that the graphs considered in this paper are not necessarily undirected, since a node u with large radius may contain some node v with smaller radius in its range, and thus u might fall outside the range of v. A precise definition of the graph models considered in this paper can be found in Section 1.3.

1.2 Related Work

The broadcasting problem has attracted a great deal of attention in the context of radio networks with an arbitrary topology. For networks with linearly bounded labels, in which the nodes do not possess any global knowledge about the topology of the network, the trivial $O(n^2)$ upper bound on deterministic broadcasting was first improved by Chlebus et al. [6] to $O(n^{11/6})$. The subsequent improvements included an $\tilde{O}(n^{5/3})$ time algorithm proposed by De Marco and Pelc [12], an $O(n^{3/2})$ time algorithm proposed by Chlebus et al. [5], and an $O(n \log^2 n)$ time algorithm developed by Chrobak et al. [7]. Clementi et al. [9] presented a deterministic broadcasting algorithm for *ad-hoc* radio networks which works in time $\tilde{O}(D\Delta)$, where D is the diameter of the network (the number of edges on the longest shortest path) and Δ is the maximum in-degree of a node. The $O(n \log^2 n)$ and $\tilde{O}(D\Delta)$ algorithms, presented in [7] and [9], respectively, can easily be adapted for polynomially bounded node labels. Brusci and Del Pinto [2] showed that for any deterministic broadcasting algorithm $\mathcal{A}$ in ad-hoc radio networks, there are networks on which $\mathcal{A}$ requires $\Omega(n \log n)$ time. Later, Czumaj and Rytter proposed a randomized algorithm which achieves with high probability linear broadcasting time on arbitrary networks [10]. Under the assumption that the network diameter is known, they presented a broadcasting algorithm which has a running time of $O(D \log(n/D) + \log^2 n)$. Independently, Kowalski and Pelc introduced a similar algorithm with the same running time [24].

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[1] In the general model the vertices are placed in $[0, 1]^d$ for some $d > 0$, however, in this paper we only consider placement of n points on the plane $[0, \sqrt{n}]^2$.

In the model where the network topology is known to all nodes in advance Gaber and Mansour [17] proposed a centralized broadcasting procedure completing the task in time $O(D + \log^5 n)$. Elkin and Kortsarz improved this bound to $D+O(\log^4 n)$ in general graphs and to $D+O(\log^3 n)$ in planar graphs [14]. Gąsieniec et al. proposed an alternative solution with times $D + O(\log^3 n)$ and $O(D)$ respectively [20]. Very recently, the constructive upper bounds w.r.t. broadcasting in general graphs have been improved to $D + O(\log^3 n/\log\log n)$ and $O(D + \log^2 n)$ in [8] and [25], respectively. Note that computing an optimal (radio) broadcast schedule for an arbitrary network is NP-hard [4,31].

In [15] the authors considered radio broadcasting in the traditional Erdős-Rényi random graph model. In this model, given a set of n nodes a graph $G_{n,p}$ is constructed by letting any two pair of vertices be connected with probability p, independently. They presented centralized as well as fully distributed procedures for the broadcasting problem in such graphs, and showed that these algorithms are asymptotically optimal. In [1] Berenbrink et al. considered efficient radio broadcasting algorithms w.r.t. running time and energy consumption in these types of random graphs.

In [13] Dessmark and Pelc analyzed radio broadcasting in geometric networks. They showed that if each node knows its neighbors, then broadcasting can be performed in $O(D)$ steps. If each node knows only its own position, then broadcasting can be performed in $O(n)$ steps, and, if the nodes are not able to detect collisions, this result cannot be improved.

In [16] Emek et al. considered the broadcasting problem in geometric graphs in which each node has the same transmission radius (UDG model). They determined the broadcasting time depending on the diameter D and the granularity g, which is the inverse of the minimum distance between any two nodes. First, it was shown that if the nodes other than the source are initially idle and cannot transmit until they hear a message for the first time, then broadcasting can be accomplished in time $O(Dg)$. For the case, in which all nodes may transmit messages from the beginning, an optimal broadcasting algorithm with running time $O(\min\{D + g^2, D\log g\})$ was presented.

Radio communication in the $G(n,r)$ model has been analyzed by Lotker and Navarra in [27]. In order to cope with radio broadcasting or gossiping on the $G(n,r)$ graph, these problems have first been solved on the grid. Then, Lotker and Navarra emulated the corresponding grid protocol on the $G(n,r)$ model, and obtained asymptotically optimal algorithms for the broadcasting and gossiping problem. That is, if $r = \Omega(\sqrt{\log n})$, then the time needed to spread a message is $O(D)$, with high probability, where $D = \Theta(\sqrt{n}/r)$ is the diameter of the graph, with probability $1 - o(1)$.

Recently, Czumaj and Wang considered radio gossiping under different locality assumptions in the $G(n,r)$ graph and generalized the results mentioned before [11]. However, these algorithms cannot be extended to random geometric graphs in which the distribution of the transmission radii varies according to some (e.g. power law) distribution.

1.3 Our Results

In this paper, we consider distributed radio broadcasting algorithms in random geometric graphs in which the transmitting radii of the nodes vary according to a power law distribution. More precisely, we consider the following graph models:

1. Let n vertices be placed uniformly at random within $[0, \sqrt{n}]^2$. In this case, a node is assigned transmission radius $r > r_{\min}$ according to the probability density function $\rho(r) = (\alpha - 1) r_{\min}^{\alpha-1} r^{-\alpha}$, independently, where $\alpha \in (1, 3)$ is a constant and $r_{\min} > \delta\sqrt{\log n}$ with δ being a (large) constant. In the resulting graph $G_{\geq r_{\min}}$ a node v is in the range of a node u if the Euclidean distance between u and v is smaller than the radius of u. The choice of δ implies that the graph is strongly connected with very high probability[2] (e.g. [29]).
2. Let n vertices be placed uniformly at random within $[0, \sqrt{n}]^2$. Here, a node is assigned radius $r > c$ according to the probability density function $\rho(r) = (\alpha - 1) c^{\alpha-1} r^{-\alpha}$, independently, where c is some (large) constant. The ranges of the nodes in the resulting graph $G_{\geq c}$ are defined by the same rules as in the previous model.

Throughout this paper we assume full synchronization, i.e., all nodes share a global clock. In the first model, the graph is (strongly) connected w.v.h.p. [29]. In the second model, the graph has a strongly connected giant component containing $\Theta(n)$ vertices, w.v.h.p. [30]. We develop for the graph model $G_{\geq r_{\min}}$ an efficient randomized broadcasting algorithm[3] which is able to distribute a message, placed initially in one of the nodes of the graph, to all nodes within $O(\log \log n)$ steps. Concerning the $G_{\geq c}$ model, we show that any message placed initially in one of the nodes of the giant component of the graph can be distributed to all nodes within $O(D(G_{\geq c})(\log \log n)^2)$ steps, w.v.h.p., where $D(G_{\geq c})$ denotes the diameter of the giant component of the graph. Notice that the nodes of the giant component can reach *any* node in the graph within $O(D)$ steps, w.v.h.p. (cf. Section 3).

A main implication of our results is that by setting the transmission radii in a set of nodes placed uniformly at random in the plane according to a power law distribution, we obtain a radio network which supports very fast broadcasting by keeping the energy consumption almost as low as in a $G(n, r)$ graph with the same average transmission radius. More precisely, in a graph $G(n, r)$ with $r = \log^{c'} n$, where $c' > 1/2$, a message is broadcasted to all nodes of the graph within $\tilde{\Theta}(\sqrt{n})$ steps, w.h.p., where $\tilde{\Theta}$ is the Θ-function omitting polylogarithmic terms [13]. The total energy consumption needed for transmission during the broadcasting process is $\tilde{\Theta}(n)$. In the $G_{\geq r_{\min}}$ graph with $r_{\min} = \log^{c'} n$, where $c' > 1/2$, a message can be broadcasted within $\Theta(\log \log n)$ steps, w.h.p., while the total energy consumption and the average transmission radius remain almost the same as in the corresponding $G(n, r)$ graph.

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[2] When we write "with very high probability" or "w.v.h.p.", we mean with probability $1 - o(n^{-1})$.

[3] The running time of this algorithm is guaranteed with high probability. "With high probability" or "w.h.p." means with probability $1 - o(1)$.

2 Broadcasting in $G_{\geq r_{\min}}$

In this section, we consider the geometric random graph model $G_{\geq r_{\min}} = (V, E)$ defined in the previous section. In this graph, a vertex u has an outgoing edge to a vertex v in $G_{\geq r_{\min}}$ iff the corresponding Euclidean distance is smaller than the radius assigned to u. We assume that $r_{\min} \geq \delta\sqrt{\log n}$, where δ is a large constant. Then, $G_{\geq r_{\min}}$ is connected with very high probability [30]. In the rest of the paper $S((x, y), (x', y'))$ denotes the rectangle delimited by the points $(x, y), (x, y'), (x', y)$, and (x', y'), where $0 \leq x \leq x' \leq \sqrt{n}$ and $0 \leq y \leq y' \leq \sqrt{n}$. The distance between two nodes (x, y) and (x', y') means the Euclidean distance between them and is denoted by $dist((x, y), (x', y'))$. The number of hops from a node u to a node v represents the length of a shortest path from u to v in the resulting graph. The set of points in $[0, \sqrt{n}]^2$ lying within the transmission radius of at least one of the nodes of some subset $S \subseteq V$ is called the area covered by S. In the sequel (x_0, y_0) represents the node in which the message which has to be spread to all nodes is placed at time 0.

In order to show that a message can efficiently be spread to all nodes of such a graph, we first state the following proposition.

Proposition 1. *In a graph $G_{\geq r_{\min}}$ (or $G_{\geq c}$) there are $\Omega(n/r^{\alpha-1})$ nodes with radius at least r, with probability $1 - o(n^{-2})$, for any $r \geq r_{\min}$ (or $r \geq c$).*

Proof. We know that in this graph a node has been assigned radius r according to the probability density function $\rho(r) = (\alpha - 1)r_{\min}r^{-\alpha}$, independently of all other nodes. This implies that a node has radius larger than some r with probability $\int_r^\infty (\alpha - 1)r_{\min}^{\alpha-1}x^{-\alpha}\mathrm{d}x = r_{\min}^{\alpha-1}r^{-(\alpha-1)}$. Hence, using the Chernoff bounds [3,21] we conclude that there are less than $\epsilon n/r^{\alpha-1}$ nodes, which have radius at least r, with probability at most

$$\sum_{i=n-\epsilon n/r^{\alpha-1}}^{n} \binom{n}{i} \left(1 - \left(\frac{r_{\min}}{r}\right)^{\alpha-1}\right)^i \left(\frac{r_{\min}}{r}\right)^{(\alpha-1)(n-i)}$$

$$\leq \left(\frac{1 - (r_{\min}/r)^{\alpha-1}}{1 - \epsilon/r^{\alpha-1}}\right)^{n(1-\epsilon/r^{\alpha-1})} \left(\frac{(r_{\min}/r)^{\alpha-1}}{\epsilon/r^{\alpha-1}}\right)^{n\epsilon/r^{\alpha-1}} \quad (1)$$

$$= \left(1 - \frac{r_{\min}^{\alpha-1} - \epsilon}{r^{\alpha-1} - \epsilon}\right)^{n(1-\epsilon/r^{\alpha-1})} \left(\frac{r_{\min}^{\alpha-1}}{\epsilon}\right)^{n\epsilon/r^{\alpha-1}}$$

which equals $o(n^{-2})$ whenever $r_{\min} = \Omega(1)$ and ϵ is small enough. □

Proposition 1 implies that in a graph $G_{\geq r_{\min}}$ there are $\Omega(1)$ nodes with radius at least $2\sqrt{n}$, with probability $1 - o(n^{-2})$. We should note that in the case of $G_{\geq c}$ we may replace Ω by Θ in the statement of Proposition 1.

Now we consider broadcasting in the $G_{\geq r_{\min}}$ graph. Here, we only consider the case $r_{\min} < 2^{\log^\epsilon n}$, where ϵ may be any constant smaller than 1, and show that for these graphs broadcasting can be performed in time $O(D(G_{\geq r_{\min}}))$, w.h.p., where $D(G_{\geq r_{\min}})$ denotes the diameter of the graph. The same results can also be shown for any $G_{\geq r_{\min}}$ with $r_{\min} = n^{o(1)}$, however, the case $r_{\min} > 2^{\log^\epsilon n}$ for *any* $\epsilon < 1$ would require an elaborate case analysis which is omitted in this extended abstract.

Now we concentrate on a lower bound on the diameter of $G_{\geq r_{\min}}$.

Theorem 1. *If $r_{\min} < 2^{\log^\epsilon n}$ for some constant $\epsilon < 1$, then the diameter of $G_{\geq r_{\min}}$ is $\Omega(\log\log n)$, w.v.h.p.*

The proof of this theorem is omitted due to space limitations. Intuitively, with some constant probability a node v with radius r_v can only reach nodes with radius at most $r_v^{\Theta(1)}$, and hence, there is a node with radius $r_{\min}$ which needs at least $\Omega(\log\log n)$ hops to reach a node with radius $\Theta(\sqrt{n})$, w.v.h.p.

Now we show that there exists an optimal distributed broadcasting algorithm in $G_{\geq r_{\min}}$. The idea behind the algorithm is that, with sufficient probability, each node u has an edge to a node v with a somewhat larger radius. Among the several such nodes v, one can be selected by having all such nodes v reply with a probability inversely proportional to their expected number, after which the chosen node can replace u and repeat the procedure. Then, after O(log log n) steps, the broadcast message reaches a node with an edge to every other node. A precise description of the algorithm is given in the next two paragraphs.

Let (x_0, y_0) denote the vertex which has the broadcast message at the beginning and assume that its radius r_0 is smaller than $\log^3 n$. In the first round this node transmits the message, and its transmission range r_0, together with a control bit set to 1. The succeeding rounds consist of several steps. In the second round the informed nodes which have their radii in the range $[3r_0, 6r_0]$ transmit in each odd step with probability $1/(r_{\min}^{\alpha-1} r_0^{3-\alpha})$ a control bit set to 0. If in some odd step (x_0, y_0) receives the control bit, i.e., exactly one of the informed nodes with the properties described above was transmitting, then (x_0, y_0) sends in the next (even) step a control bit set to 1. In the next *even* step the node that sent the control bit, received by (x_0, y_0) three steps before, transmits the message and its transmission range r_1, together with the control bit set to 1.

Generally, in some round $i > 1$ we consider two cases. If the radius r_{i-2} of the node (x_{i-2}, y_{i-2}) is smaller than $\log^{4/\epsilon} n$, where $\epsilon < 6 - 2\alpha$ is some constant, then in each odd step of this round, the nodes which received the message in the last step of round $i-1$ from the node (x_{i-2}, y_{i-2}) **and** have their radius in the range $[3r_{i-2}, 6r_{i-2}]$ transmit with probability $1/(r_{\min}^{\alpha-1} r_{i-2}^{3-\alpha})$ a control bit set to 0. If $r_{i-2} > \log^{4/\epsilon} n$, then the nodes which received the message in the last step of round $i-1$ from the node (x_{i-2}, y_{i-2}) and have their radius in the range $[r_{i-2}^{(4-\epsilon)/(2(\alpha-1))}, 2r_{i-2}^{(4-\epsilon)/(2(\alpha-1))}]$ transmit with probability $1/(r_{i-2}^{\epsilon/2} r_{\min}^{\alpha-1})$ the control bit set to 0. In both cases if in some odd step the node (x_{i-2}, y_{i-2}) receives the control bit, i.e., only one of the nodes in its range with the properties described above has sent a message in the most recent step, then (x_{i-2}, y_{i-2}) transmits in the next (even) step the control bit set to 1. In the following *even* step, the single node which transmitted the control bit three steps before transmists the message and its transmission range r_{i-1}, together with the control bit set to 1. This transmitting node is denoted after this step by (x_{i-1}, y_{i-1}), and round $i+1$ begins.

Theorem 2. *Let $G_{\geq r_{\min}}$ be the graph defined at the beginning of this section, where $r_{\min} \geq \delta\sqrt{\log n}$. Furthermore, let a message be placed in one of the nodes of $G_{\geq r_{\min}}$. Then, the randomized distributed radio broadcasting algorithm described above spreads the message to all nodes of $G_{\geq r_{\min}}$ in $O(\log\log n)$ steps, w.h.p.*

Proof. In order to show that the algorithm described above informs a node with radius $2\sqrt{n}$ within $O(\log\log n)$ rounds, w.h.p., we first prove that any node with some radius $r \in [\delta\sqrt{\log n}, \log^{4/\epsilon} n]$ reaches $\Theta(r_{\min}^{\alpha-1} r^{3-\alpha})$ nodes which have their radii in $[3r, 6r]$, w.v.h.p. As in the proof of Proposition 1, we can show that a node has its radius in the range $[3r, 6r]$ with probability $\int_{3r}^{6r}(\alpha-1)r_{\min}x^{-\alpha}\mathrm{d}x = (6^{\alpha-1} - 3^{\alpha-1})/18^{\alpha-1} \cdot r_{\min}r^{-(\alpha-1)}$, independently. Applying now the Chernoff bounds [3,21] we obtain that with probability $1-o(n^{-2})$ there are $\Theta(nr_{\min}^{\alpha-1}r^{-(\alpha-1)})$ nodes which have their radii in the range $[3r, 6r]$. These nodes fall into the range of a fixed node with radius r with probability $\pi r^2/n$, independently. Hence, the Chernoff bounds imply that there are $\Theta(r_{\min}^{\alpha-1}r^{3-\alpha})$ nodes in the range of a fixed node with radius r, w.v.h.p., whenever δ is large enough.

Next we show that any node with radius $r \geq \log^{4/\epsilon} n$ reaches $\Theta(r^{\epsilon/2}r_{\min}^{\alpha-1})$ nodes which have their radii in $\left[r^{(4-\epsilon)/(2(\alpha-1))}, 2r^{(4-\epsilon)/(2(\alpha-1))}\right]$, w.v.h.p. As before, we conclude that there are $\Theta(nr_{\min}^{\alpha-1}r^{-(4-\epsilon)/2})$ nodes which have their radii in the range $\left[r^{(4-\epsilon)/(2(\alpha-1))}, 2r^{(4-\epsilon)/(2(\alpha-1))}\right]$, w.v.h.p. Since any node falls into the range of a fixed node with radius r with probability $\pi r^2/n$ (we ignore border effects), independently, applying the Chernoff bounds we obtain that there are $\Theta(r_{\min}^{\alpha-1}r^{\epsilon/2})$ nodes in the range of a fixed node with radius r, w.v.h.p. Combining the results of the previous two paragraphs, we conclude that the diameter of $G_{\geq r_{\min}}$ is $O(\log\log n)$.

In order to conclude the proof, let $X_{i,j}$ be a random variable which is 1 if in the jth odd step of the ith round only one node transmits the control bit set to 0, and 0 otherwise. Furthermore, let $A_{i,j}$ denote the event that $E[X_{i,j}] = \Theta(1)$. Then, due to the choice of the nodes, $\Pr[X_{i,j}|A_{i,j}] = \Theta(1)$ for any i, j. We denote by Y_l a random variable which is 1 if exactly one node transmits the control bit set to 0 in the lth odd step (the odd steps are now counted over the whole time period), and A_l is the event that $E[X_l] = \Theta(1)$. We are looking now for some T such that $\Pr[\sum_{l=1}^{T} Y_l \geq \phi \cdot D(G_{\geq r_{\min}})| \cup_{i=1}^{T} A_l] = 1 - o(1/D(G_{\geq r_{\min}}))$, where $D(G_{\geq r_{\min}})$ is the diameter of $G_{\geq r_{\min}}$ and ϕ is some (large) constant. Since $\Pr[Y_l = 1|A_l] = \Omega(1)$, independently, we can use the Chernoff bounds [3,21], and obtain that $T = \Theta(\log\log n)$. Since A_l occurs with very high probability, applying the Union bound over $O(\log\log n)$ steps we obtain that a node with radius at least $2\sqrt{n}$ gets the message within $O(\log\log n)$ steps, w.h.p. Such a node node transmits the message alone in a time step with constant probability. This implies that within additional $O(\log\log n)$ steps all nodes receive the message, w.h.p., and the theorem follows. □

Applying similar arguments as in the previous proof, one can show that if the algorithm presented above is run for $O(\log n)$ steps, one can disseminate a message to all nodes of $G_{\geq r_{\min}}$ with *very high probability* (instead of probability $1-o(1)$). Using the so called echo procedure from [23], we can derandomize the algorithm described in the proof of Theorem 2 (as well as the algorithm described in Theorem 5), and obtain the same results as before. The result of Theorem 2 can also be extended to random geometric graphs obtained from a homogeneous Poisson point process with some intensity which exceeds the connectivity threshold value. Please refer to [30] for details.

3 Broadcasting in $G_{\geq c}$

In this section we consider the $G_{\geq c}$ model defined in the introduction. Due to the choice of c, this graph is not necessarily strongly connected, however, it contains a strongly connected giant component of size $\Theta(n)$, w.v.h.p. [30]. Then, we can state the following theorem.

Theorem 3. *If $q_2 = 1/(3-\alpha)$, then the diameter of the giant component in $G_{\geq c}$ is $O(\log^{2q_2} n)$, w.v.h.p.*

Proof. In this proof we only show (due to simplicity reasons) that for any (slow-growing) function $f(n) \in [\omega(1), O(\log\log n)]$ the diameter of the giant component of $G_{\geq c}$ is $O(f(n)\log^{2q_2} n)$. To simplify the analysis, let the graph $G_{\geq c}$ be constructed in two steps. First, construct a graph G'_c by placing the nodes with radius $r \leq f^{2/5}(n)\log^{q_2} n$ in $[0,\sqrt{n}]^2$, uniformly at random. In a second step, place the remaining nodes and obtain the graph $G_{\geq c}$.

Let u be a node of G'_c, and let v be another node which is $f(n)\log^{2q_2} n$ hops away from u in G'_c (whenever such a node exists). Furthermore, let $P(u,v) = (u, u_1, u_2, \ldots, u_{f(n)\log^{2q_2} n-1}, v)$ denote a shortest path between u and v in G'_c. We show in the following that the nodes $u_1, \ldots, u_{f(n)\log^{2q_2} n-1}$ cover an area of $\Omega(f(n)\log^{2q_2} n)$.

Assume for simplicity that $\sqrt{n}$ is an integer, c is even, and $c/2$ divides $\sqrt{n}$. Let $C(i,j)$ denote the square $S((ic/2, jc/2), ((i+1)c/2, (j+1)c/2))$. Now we show that any such square contains at most two nodes which lie on $P(u,v)$. Let us assume that there is some square $C(i,j)$ which contains three nodes u_{s_1}, u_{s_2}, and u_{s_3} lying on $P(u,v)$. Since the diameter of $C(i,j)$ is $\sqrt{2}c/2 < c$ every node in this square reaches any other node within $C(i,j)$. Then, u_{s_1} has u_{s_2} and u_{s_3} in its range, and $P'(u,v) = (u, \ldots u_{s_1}, u_{s_3}, u_{s_3+1}, \ldots, v)$ is a valid path from u to v. Since $|P'(u,v)| < |P(u,v)|$, $P(u,v)$ cannot be a shortest path from u to v, which contradicts our assumption. Summarizing, the nodes of $P(u,v)$ cover an area of at least $f(n)\log^{2q_2} nc^2/8$.

According to Proposition 1 there are $\Omega(n/(f^{2/5}(n)\log^{q_2} n)^{\alpha-1})$ with radius larger than $f^{2/5}(n)\log^{q_2} n$, with probability $1-o(n^{-2})$. Given that there are $\Omega(n/(f^{2/5}(n)\log^{q_2} n)^{\alpha-1})$ nodes with radius larger than $f^{2/5}(n)\log^{q_2} n$ in $G_{\geq c}$, the area covered by $P(u,v)$ contains no node having radius $r > f^{2/5}(n)\log^{q_2} n$ with probability

$$\left(1 - \frac{\Omega(f(n)\log^{2q_2} n)}{n}\right)^{\Omega\left(\frac{n}{(f^{2/5}(n)\log^{q_2} n)^{\alpha-1}}\right)} \leq o(e^{-\Omega(\sqrt[5]{f(n)}\log n)}) \leq o(n^{-3}).$$

Therefore, there is some node with radius $r > f^{2/5}(n)\log^{q_2} n$ placed in the area covered by $P(u,v)$ with probability $1 - o(n^{-3})$. This implies that u reaches a node which has radius larger than $f^{2/5}(n)\log^{q_2} n$ in $O(f(n)\log^{2q_2} n)$ steps, with probability $1 - o(n^{-3})$. Applying now the Union bound over all nodes of G'_c, we conclude that all nodes, which are connected to some other node via $f(n)\log^{2q_2} n$ hops, can reach a node with radius larger than $f^{2/5}(n)\log^{q_2} n$ in $O(f(n)\log^{2q_2} n)$ steps, with probability $1 - o(n^{-2})$. If for some node w isn't any node w' at $f(n)\log^{2q_2} n$ hops from w in G'_c, but w is in the giant component of $G_{\geq c}$, then w must reach a node with radius $r > f^{2/5}(n)\log^{q_2} n$ in $O(f(n)\log^{2q_2} n)$ hops. This holds since w reaches every node in its strong component in G'_c within less than $O(f(n)\log^{2q_2} n)$ hops, and this component joins the giant component of $G_{\geq c}$ via a node of $G_{\geq c} \setminus G'_c$. According to the definition of G'_c, a node of $G_{\geq c} \setminus G'_c$ has radius larger than $f^{2/5}(n)\log^{q_2} n$.

Now we show that in the range of any node which has radius $r > f^{2/5}(n)\log^{q_2} n$ there is at least one node with radius larger than

$$f^{1/5}(n)\log^{q_2} n \cdot (r/(f^{1/5}(n)\log^{q_2} n))^{1+(3-\alpha)/(\alpha-1)},$$

with probability $1 - o(n^{-2})$. Given that there are $\Omega(n/r^{\alpha-1})$ nodes which have radii larger than r, there is no node with a radius larger than $f^{1/5}(n)\log^{q_2} n \cdot (r/(f^{1/5}(n)\log^{q_2} n))^{1+(3-\alpha)/(\alpha-1)}$ in the range of a node having radius r with probability

$$\left(1 - \frac{\pi r^2}{n}\right)^{\Omega\left(\frac{n}{\left(f^{1/5}(n)\log^{q_2} n\left(\frac{r}{f^{1/5}(n)\log^{q_2} n}\right)^{1+\frac{3-\alpha}{\alpha-1}}\right)^{\alpha-1}}\right)} \leq e^{-\omega(\log n)} = o(n^{-3}).$$

We conclude by applying the Union bound over all nodes with radius larger than $f^{2/5}(n)\log^{q_2} n$. Iterating this procedure $O(\log n)$ times we obtain that one can reach a node with radius $2\sqrt{n}$ within $O(\log n)$ additional hops.

Summarizing, any node of the giant component reaches within $O(f(n)\log^{2q_2} n)$ hops a node with radius $2\sqrt{n}$, w.v.h.p., and the theorem follows. □

We might ask whether the upper bound given in Theorem 3 is asymptotically tight. A related open question was formulated in [30] about the second largest component in $G_{\geq c}$, namely whether the second largest component of the traditional $G(n,r)$ model with $r = c$ is of size $\Theta(\log^2 n)$. Concerning the diameter of the giant component in $G_{\geq c}$ we can only prove a lower bound of $\Omega(\log n)$.

Theorem 4. *The diameter of the giant component of $G_{\geq c}$ is $\Omega(\log n)$, w.v.h.p.*

The proof of this theorem uses similar techniques as Theorem 3. Due to lack of space, we do not prove Theorem 4 here.

The results of Theorems 3 and 4 can be extended to further random geometric graph models. Consider for example the graph $G_{\geq c}$, in which we enlarge the radius of a node in any strongly connected component so that the graph becomes

DISTRIBUTED ALGORITHM FOR BROADCASTING IN $G_{\geq c}$

for $t = 1$ to $O(D(G_{\geq c}))$ **do**
    for $s = 1$ to $1024(c \log\log n)^2$ **do**
        for every vertex $v = (x', y')$ **in parallel do**
            $s' \leftarrow (s-1) - ((s-1) \mod 256c^2)$
            $i' - 1 \leftarrow \frac{s'}{256c^2} \mod 2\log\log n$
            $i - 1 \leftarrow \frac{1}{2\log\log n} \cdot (\frac{s'}{256c^2} - (i'-1))$
            $j \leftarrow \lfloor (x' \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$
            $j' \leftarrow \lfloor (y' \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$
            if $r'(v) \in [c^{i'}, c^{i'+1}]$ **and** $j = \lfloor \frac{s-1}{16c} \rfloor \mod 16c$ **and** $j' = (s-1) \mod 16c$ **then**
                v transmitts with probability $\frac{1}{c^i}$
            end if
        end for
    end for
end for

Fig. 1. Algorithm used in the proof of Theorem 5. Here $r'(v)$ denotes the transmission radius of node v.

strongly connected. Another model is the extension of the point Poisson process on $[0, \sqrt{n}]^2$ with intensity c, whereas the radii are distributed as in the $G_{\geq c}$ model. In all these models it is possible to broadcast any message, placed initially in one of the nodes of the giant component, to all nodes of the graph within $O(\log^{2/(3-\alpha)} n)$ steps, w.v.h.p.

Before we start with the analysis of radio broadcasting in $G_{\geq c}$ we first give a high level description of our broadcasting algorithm. The algorithm consists of two main phases. In the first phase (cf. Figure 1) the goal is to let the message generated at a source node reach a node with radius larger than $c^{2\log\log n}$, w.v.h.p. In the second phase the message reaches a node with radius $2\sqrt{n}$, w.v.h.p. The second phase performs similarly to the algorithm presented for $G_{\geq r_{\min}}$, and thus, we omit the analysis of this phase in the paper. For the first phase, we show that the message traverses a shortest path from the source of the message to a node with radius larger than $c^{2\log\log n}$, w.v.h.p. In order to ensure that each node on this path transmits the message to the next node on the path, the algorithm consists of $O(D(G_{\geq c}))$ phases, and each phase is executed over $O((\log\log n)^2$ time steps. During these time steps, each informed node of radius r, where $r \in [c, c^{2\log\log n}]$, transmits at least once with some probability in the range $[r^{3-\alpha}/c, cr^{3-\alpha}]$. By ensuring that interferences can only occur if several nodes lying in the same square $I^{i',j,j'}$ (see Figure 2) transmit at the same time, one can show that the message will traverse the shortest path mentioned above within $O(D(G_{\geq c}))$ phases, w.v.h.p.

Formaly, the distributed algorithm that guarantees the running time given in the theorem below consists of $O(D(G_{\geq c}))$ initial rounds. In each round we have $1024(c\log\log n)^2$ steps. In step $256c^2(2(i-1)\log\log n + (i'-1)) + 16cj + j' + 1$ with $1 \leq i, i' \leq 2\log\log n$ and $0 \leq j, j' \leq 16c - 1$ any informed node

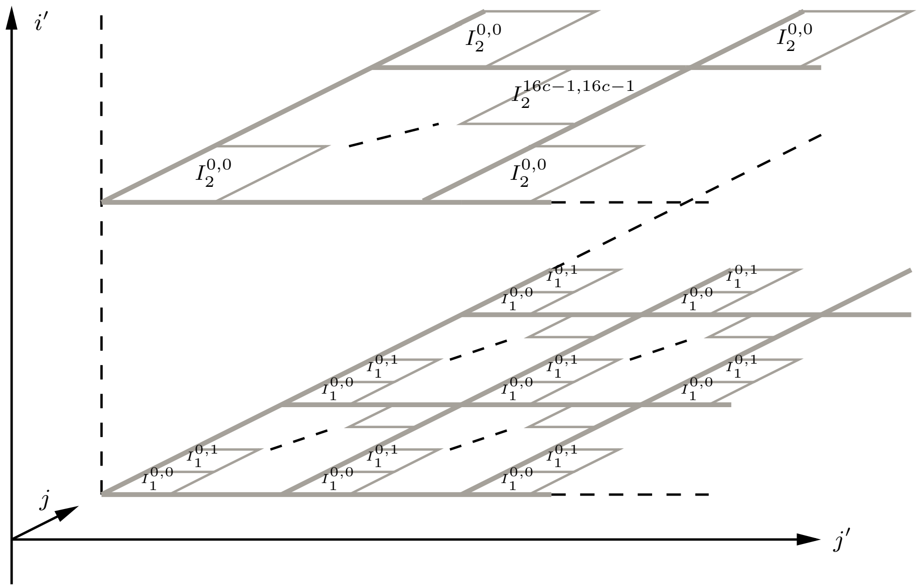

Fig. 2. The nodes with radius in the range $[c^{i'}, c^{i'+1}]$ placed in the squares denoted by $I_{i'}^{j,j'}$ transmit in step $256c^2(2(i-1)\log\log n + (i'-1)) + 16cj + j' + 1$ with probability $1/c^i$. The two planes consisting of the squares $I_1^{*,*}$ and $I_2^{*,*}$, respectively, are both embedded into $[0, \sqrt{n}]^2$ and contain the same set of points. Here, we have drawn two parallel planes for a better visualization.

(x', y') with radius $r' \in [c^{i'}, c^{i'+1}]$ and $j = \lfloor (x' \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$, $j' = \lfloor (y' \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$ transmits with probability $1/c^i$ (cf. Figure 1). For a pseudo code of these $O(D(G_{\geq c}))$ initial rounds see Figure 1.

After these $O(D(G_{\geq c}))$ initial rounds we reach a node with a radius in the range $[c^{2\log\log n}, 2c^{2\log\log n}]$, and then we apply a similar procedure as in Theorem 2. We only consider the first phase, which requires $O(D(G_{\geq c})(\log\log n)^2)$ steps. The second phase requires only $O(\log n)$ steps.

Now we state the main theorem of this section.

Theorem 5. *Let $G_{\geq c}$ be the graph defined at the beginning of this section, where c is a large constant. Furthermore, let a message be placed in one of the nodes of the giant component of $G_{\geq c}$. Then, the randomized distributed radio broadcasting algorithm described above spreads the message to all nodes of $G_{\geq c}$ (even to nodes outside of the giant component) in $O(D(G_{\geq c})(\log\log n)^2)$ steps, w.v.h.p., where $D(G_{\geq c})$ denotes the diameter of the giant component of $G_{\geq c}$.*

Proof. We show that within $O(D(G_{\geq c})(\log\log n)^2)$ steps any (x, y) receives the message, w.v.h.p. Obviously, two nodes (x_1, y_1) and (x_2, y_2) with radii $r_1, r_2 \in [c^{i'}, c^{i'+1}]$, where $i' \leq 2\log\log n$, cannot produce an interference at any node whenever $\lfloor (x_1 \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor \neq \lfloor (x_2 \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$ or $\lfloor (y_1 \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor \neq \lfloor (y_2 \mod (4c^{i'+1}))/(c^{i'}/4) \rfloor$. Let now $P = (v_0, v_1, \ldots, v_k)$

be a shortest path from $(x_0, y_0) = v_0$ to $(x, y) = v_k$. We know that an informed node transmits at most $2 \log \log n$ times in a round, each time with a different probability. Let $t_{q,i,l}$ denote the time step in the lth round, in which v_q transmits with probability $1/c^i$. Furthermore, denote by $X_{q,i,l}$ a random variable which is 1 if the message reaches v_{q+1} in step $t_{q,i,l}$ and 0 otherwise. Now, v_q can produce an interference with at most $O(c^{(3-\alpha)i'} + \log n)$ other nodes, with probability $1 - o(n^{-2})$, where the radius r_q of v_q is in the range $[c^{i'}, c^{i'+1}]$. Thus, there is at least one i such that $\Pr[X_{q,i,l} = 1 \mid v_q \text{ is informed before round } l] = \Omega(1)$. Let $Y_l = X_{q,i,l}$, with $q = \max_{q'}\{v_{q'} \text{ is informed before round } l\}$, and let i be chosen such that $\Pr[X_{q,i,l} = 1] = \Omega(1)$. Then, $\Pr[Y_l = 1] = \Omega(1)$, independently. As in the proof of Theorem 2 we can show that there is some $T = O(|P| + \log n)$ such that $\Pr[\sum_{l=1}^{T} Y_l \geq |P|] = 1 - o(n^{-2})$. Since each round consists of $O((\log \log n)^2)$ steps, (x, y) becomes informed within $O((|P| + \log n)(\log \log n)^2)$ steps, with probability $1 - o(n^{-2})$. Applying now the Union bound over all nodes of the graph, we obtain that within $O(D(G_{\geq c})(\log \log n)^2)$ steps a node with radius in the range $[c^{2 \log \log n}, 2c^{2 \log \log n}]$ receives the message. If now c is large enough, using the same arguments as in the proof of Theorem 2 we conclude that within additional $O(\log n)$ steps the message reaches any node of the graph, w.v.h.p. □

As in the case of Theorem 2, the result of Theorem 5 can also be extended to random geometric graphs obtained from a homogeneous Poisson process with a corresponding intensity.

We know that a message, placed on one of the nodes of a $G(n, r)$ graph, can be spread to all other nodes within $\tilde{\Theta}(\sqrt{n})$ steps [13,30], v.w.h.p., where $r = \log^{c'} n$ with $c' \geq 1/2$ and $\tilde{\Theta}$ is the Θ-fuction omitting polylogarithmic terms. The total energy consumption needed for transmission during the broadcasting process in the network is bounded by $\tilde{\Theta}(n)$. However, if we consider our results for α being a constant in the range $(2, 3)$, then we may perform broadcasting in time $\tilde{\Theta}(\log n)$, and the total energy consumption needed for transmissions is still bounded by $\tilde{\Theta}(n)$. Moreover, the average transmission radius is asymptotically the same as in the corresponding $G(n, r)$ graph. Thus, our results imply that if we are given n radio transmitters, and we are allowed to set the transmission radius of each of these devices before they are placed uniformly at random in $[0, \sqrt{n}]^2$, then we are able to design a radio network, which supports broadcasting in (poly)logarithmic time and keeps the energy consumption in the network very low. Furthermore, our results can also be extended to the case when the transmission radii of the nodes vary in time, independently, according to a power law distribution with some exponent $\alpha \in (1, 3)$.

4 Conclusion

As described in the introduction, our main intention was to derive efficient algorithms for radio broadcasting in wireless networks which are modeled by random geometric graphs containing nodes with different transmission radii. The results presented here can only be viewed as a first step in this direction, and there are still several interesting open problems in this field which are worth to be analyzed. In the case of the $G_{\geq c}$ model for example there is still a gap of $\log^{\Theta(1)} n$ between the upper and lower bound w.r.t. the diameter of the giant component of the graph, and it would be of great interest to close this gap. Another open problem is whether it is possible to broadcast a piece of information in G_c within $O(D(G_{\geq c}))$ steps.

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