The Use of Hilbert-Schmidt Decomposition for Implementing Quantum Gates

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Abstract. It is shown how to realize quantum gates by decomposing the gates into summation of unitary matrices where each of these matrices is given by a tensor multiplication of the unit and Pauli 2x2 spin matrices. It is assumed that each of these matrices is operating on a different copy of the quantum states produced by 'quantum encoders' with a certain probability of success. The use of the present probabilistic linear optics' method for realizing quantum gates is demonstrated by the full analysis given for the control phase shift gate, but the use of the present method for other gates, including the control-not gate, is also discussed.

1 Introduction

A quantum bit (qubit) is a two-level quantum system described by a two-dimensional complex Hilbert space [1,2]. The computational qubit state is described by a superposition of normalized and orthogonal states of a two-level quantum system denoted as $|0>$ and $|1>$. In the present study photonic qubits are used where $|0>$ and $|1>$ represent horizontal $|H>$and vertical $|V>$ polarized photons, respectively. In order to implement general quantum computational processes one needs to apply control operations. In the present work we are interested in the implementation of quantum gates with two input qubits, known as the control qubit and the target qubit, respectively. The control qubit (A) is not changed by the quantum gate, but a certain linear unitary transformation is performed on the target qubit (B) if and only if the control bit is set to $|1>$. Optics seems to be a good candidate for achieving two-qubit quantum gates. Unfortunately, such gates are quite difficult to implement experimentally since the state of the control qubit should affect the second target qubit and this requires strong interactions between single photons. Such interactions need high nonlinearities well beyond what is available experimentally.

Recently it has been shown by Knill, Lafllamme and Milburn [3] that probabilistic quantum logic operations can be performed using only linear optical elements, additional photons (ancilla), and post selection based on the single photon detectors. This idea has been implemented in various studies [4] and in particular Pittman, Jacobs and Franson [5-7] constructed a variety of quantum logic gates by using polarizing beam splitters (PBS) that completely transmit one state of polarization and totally reflect the orthogonal state of polarization. These methods overcome the complications introduced by using non-linear optics for realizing quantum gates, but on the other hand their nature is probabilistic throwing away a part of the measurements.

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Probabilistic 'quantum encoding' processes have been realized experimentally and used for designing various quantum gates transformations [5-7]. The encoder consists primarily of a polarizing beam splitter (PBS) and resource pair of entangled photons in the Bell state $\left|\phi^{+}\right\rangle=\left(1/\sqrt{2}\right)\left(\left|00\right\rangle+\left|11\right\rangle\right)$ [7]. For the quantum encoder the input qubit of a single photon, in a general polarization state $\alpha\left|0\right\rangle+\beta\left|1\right\rangle$, and one member of the entangled resource pair are mixed at the PBS oriented in the HV basis. There are three output ports of the quantum encoder, including two output ports for the PBS and one output port for the second member of the entangled resource pair. Detection of one photon by 'gating detector' in one output port of the PBS signals the fact that the two remaining photons are exiting the device in the other two output ports. Because the PBS transmits $H-$polarized photons and reflects $V-$polarized photons it can be shown [5,7] that the output state is of the form

$$\left|\psi\right\rangle_{out}=\frac{1}{\sqrt{2}}\left(\alpha\left|000\right\rangle+\beta\left|111\right\rangle\right)+\frac{1}{\sqrt{2}}\left|\psi_{\perp}\right\rangle, \tag{1}$$

where $\left|\psi_{\perp}\right\rangle$ represents combinations of states orthogonal to the condition of finding one and only one (1AO1) in the gating detector. In order to implement the quantum encoding process we accept the remaining outputs only when the condition 1AO1 is satisfied. In order to have only the condition 1AO1 and erase any additional information obtained by the gating detector the encoding is completed by accepting the output only when the gating detector measures exactly one photon in a polarization basis rotated by 45^0 from the HV basis [5]. Under these circumstances and ideal conditions, which occur with probability of 1/2 , the device realizes the encoding [7]:

$$\alpha\left|0\right\rangle+\beta\left|1\right\rangle\rightarrow\alpha\left|0\right\rangle_1\left|0\right\rangle_2+\beta\left|1\right\rangle_1\left|1\right\rangle_2 . \tag{2}$$

The subscripts 1 and 2 indicate different copies of each state where the copied wavepackets are located in different places. Under ideal conditions the probability of success of the encoding process is 1/2. The encoding device is described in Fig. 1 of [7]. One should notice that the encoding transformation (2) obtained by post selection is different from the 'cloning' transformation [8]

$$\alpha\left|0\right\rangle+\beta\left|1\right\rangle\rightarrow\left(\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right)_1\left(\alpha\left|0\right\rangle+\beta\left|1\right\rangle\right)_2$$

which is not allowed.

One should notice that the state $\frac{\left|0\right\rangle_1\left|0\right\rangle_2+\left|1\right\rangle_1\left|1\right\rangle_2}{\sqrt{2}}$ is the well known entangled state which includes certain quantum correlations ,i.e, if the first photon is in the state |0> (horizontal polarized photon) then the second photon is also in the state |0> while if the first photon is in the state |1> (vertical polarized photon) then the second photon is also in the state |1>. As is well known quantum entanglement is a fundamental resource for quantum computation processes [1,2]. The encoding transformation (2) produces a general entangled state and we would like to exploit such entangled states for implementing quantum gates.

A new method is developed in the present work for implementing quantum gates, based on the use of quantum encoders, which is basicly different from that presented in the previous works [5-7]. As is well known any unitary matrix can be decomposed into summation of tensor products of Pauli and unit spin matrices [2]. The application of such decomposition for the realization of quantum gates is quite problematic since in quantum computation we should use multiplications of unitary operators and not summations of them. However, there is a certain trick by which such decomposition can implement quantum gates. By using quantum encoders [5-7] we can 'copy' each state in the qubit superposition. Then each matrix in the above decomposition of the unitary gate operates on a different copy and by *adding* the results in the different copies we implement the corresponding gate. It should be apparent that the quantum encoders which are based on probabilistic detection procedures [6,7] and which have been realized experimentally are *different* from 'cloning' of the qubits which is prohibited by the quantum 'no-cloning' theorem [8]. The present new method is analyzed explicitly for the $CPHASE(\theta)$ gate but the options of using it for other quantum gates are also discussed.

The present paper is arranged as follows: In Section 2 we analyze the decomposition of two-qubit gates into summation of tensor products of Pauli and unit spin matrices. In Section 3 we analyze the use of the present method for implementing the $CPHASE(\theta)$ gate. In Section 4 we discuss and summarize our results and conclusions.

2 Two Qubit Gates Described by Tensor Products of Pauli and Unit Spin Matrices

For using matrix representations of quantum gates the qubits $|0\rangle$ and $|1\rangle$ are described by the following column vectors

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

The linear transformations operating on these single-qubit column vectors are given by multiplying them by unitary matrices of dimension 2x2. These matrices can be represented by linear combination of the four spin matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

where I is the two-dimensional unit matrix, and σ_1 , σ_2 , σ_3 are the Pauli spin matrices.

The two-qubit state can be given by four dimensional column vectors

$$|00\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad ; \quad |01\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad ; \quad |11\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \tag{5}$$

where in the ket states on the left handside of these equations the first and second number denote the state of the first and second qubit, respectively. The sign $\otimes$ represents tensor product where the two-qubit states can be described by tensor products of the first and second qubit column vectors.

$CNOT$ gate is given by

$$CNOT\,|x, y\rangle = |x, x \oplus y\rangle \quad , \quad (x = 0,1\,;\, y = 0,1). \tag{6}$$

and $\otimes$ indicates addition modulo 2 . This gate flips the state of the target qubit y if the control qubit x is in the state |1> and does nothing if the control qubit is in the state |0>. $CNOT$ can be represented by a unitary 4x4 dimensional matrix operating on the above four dimensional vectors :

$$CNOT = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \quad . \tag{7}$$

The 4x4 unitary matrix $CPHASE(\theta)$ gate is given by

$$CPHASE(\theta) = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & e^{i\theta}\end{pmatrix} \tag{8}$$

This gate (which does not have a classical analog [1]) applies a phase shift for the state $|1,1\rangle$ giving

$$CPHASE(\theta)|1,1\rangle = e^{i\theta}|1,1\rangle, \tag{9}$$

and does nothing if it operates on other states.

The CNOT gate is a standard component in computational circuits analysis [1,2]. Quantum computational circuits in which the *control phase shift gate*- $CPHASE(\theta)$ is inserted as one component has been analyzed in the literature (see e.g. [2], Figures (3.5) and (3.6) on page 116).

Any two-qubit gate can be expressed in the Hilbert-Schmidt (HS) representation [9] as

$$U_2 = \sum_{j,k=0}^{3} t_{j,k} \sigma_j \otimes \sigma_k \ , \tag{10}$$

where by taking into account the properties of the spin matrices we find

$$t_{l,m} = \frac{1}{4} Tr\left[U_2 \bullet \left(\sigma_l \otimes \sigma_m\right)\right] \tag{11}$$

The point $\bullet$ represents the ordinary matrix multiplication, the sign $\otimes$ denotes tensor product and the notation Tr represents the trace operation. $4t_{l,m}$ is given by the trace of the ordinary matrix multiplication of the four-dimensional matrix U_2 by the four dimensional matrix $\left(\sigma_l \otimes \sigma_m\right)$. In deriving (11) we use the relations

$$Tr\left[\left(\sigma_j \otimes \sigma_k\right) \bullet \left(\sigma_l \otimes \sigma_m\right)\right] = 4\delta_{j,l}\delta_{k,m} \tag{12}$$

While for a general two-qubit gate 16 elements of $t_{j,k}$ might be different from zero, for the main basic two-qubit gates only 4 elements $t_{j,k}$ are different from zero. We find by straightforward calculations for the CNOT unitary matrix of (7):

$$CNOT = \frac{1}{2}\left[\sigma_3 \otimes \left(I - \sigma_1\right)\right] + \frac{1}{2}\left[I \otimes \left(\sigma_1 + I\right)\right]. \tag{13}$$

For the $CPHASE(\theta)$ gate of (8) we get:

$$CPHASE(\theta) = \kappa\left(I \otimes I\right) + \lambda\left(\sigma_3 \otimes \sigma_3\right) + \mu\left(I \otimes \sigma_3\right) + \nu\left(\sigma_3 \otimes I\right) \tag{14}$$

where

$$\lambda = \frac{1}{4}\left(e^{i\theta} - 1\right) \quad , \quad \mu = \nu = -\lambda \quad , \quad \kappa = 1 + \lambda \quad . \tag{15}$$

Each of Eqs. (13,14) includes four tensor products where the first and second 2x2 matrix in each tensor product operates on the first and second qubit column vectors , respectively. For each qubit we can use the relations:

$$\begin{array}{l} \sigma_3|0\rangle=|0\rangle \quad ; \quad \sigma_3|1\rangle\equiv-|1\rangle \quad ; \quad \sigma_1|0\rangle=|1\rangle \quad ; \quad \sigma_1|1\rangle=|0\rangle \quad ; \\ \sigma_2|0\rangle=i|1\rangle \quad ; \quad \sigma_2|1\rangle=-i|0\rangle \quad ; \quad I|0\rangle=|0\rangle \quad ; \quad I|1\rangle=|1\rangle \end{array} \tag{16}$$

In polarization optics the states represented by the column vectors of (3) are known as Jones vectors [10]. By using Jones calculus it is quite easy to implement the unitary transformations of (16) operating on single qubit column vectors.(See polarization optics transformations obtained by Jones calculus, including the effects of *half- and quarter-wave retardation plate*s , and the general 2x2 unitary matrices transformation (1.5-11) of [10]).

As explained in the introduction the application of the decompositions (13,14) for the realization of quantum gates is quite problematic but they can implement quantum gates by the use of quantum encoders, as described in the following analysis.

3 Realization of $CPHASE(\theta)$ Gate by Quantum Encoders

An input two-qubit state can be written as

$$|\psi\rangle_{in}=\{\alpha|0\rangle_A+\beta|1\rangle_A\}\{\gamma|0\rangle_B+\delta|1\rangle_B\}, \tag{17}$$

where the subscripts A and B refer to two separated qubits. The complex amplitudes for the first and second qubit are given by α and β , and γ and δ, respectively. The $CPHASE(\theta)$ is defined as leading to the output state

$$|\psi\rangle_{out}=\alpha\gamma|0\rangle_A|0\rangle_B+\alpha\delta|0\rangle_A|1\rangle_B+\beta\gamma|1\rangle_A|0\rangle_B+\beta\delta e^{i\theta}|1\rangle_A|1\rangle_B \tag{18}$$

The first qubit (A) acts as a control and its value is unchanged on the output. In case that the first control qubit is in the |0> state nothing happens to the second target qubit (B). In case that the first control qubit is in the |1> additional phase θ is inserted between the |0> state and the |1> state of the second target qubit, and we *define* this additional phase to be inserted in the |1) state (but take into account that only the relative phase is important). In quantum computational circuits the case $CMINUS = CPHASE(\pi)$ is especially important [2]. In the following analysis it is shown how to implement the transformation (18) by using quantum encoders.

By using quantum encoders [7], as explained in the introduction, we can copy two times each input state transforming (17) into

$$|\psi\rangle_{in}=\left[\alpha\{|0\rangle_{A1}|0\rangle_{A2}\}+\beta\{|1\rangle_{A1}|1\rangle_{A2}\}\right]\times\left[\gamma\{|0\rangle_{B1}|0\rangle_{B2}\}+\delta\{|1\rangle_{B1}|1\rangle_{B2}\}\right] \tag{19}$$

In (19) we get multiplications of four states since each of the two-states denoted by the subscripts A and B has been copied twice by the quantum encoders and these copies are indicated by adding the subscripts one and two.

The input state $|\psi\rangle_{in}$ of (19) can be rearranged as

$$|\psi\rangle_{in} = \alpha\gamma\{|0\rangle_{A1}|0\rangle_{B1}\}\{|0\rangle_{A2}|0\rangle_{B2}\} + \alpha\delta\{|0\rangle_{A1}|1\rangle_{B1}\}\{|0\rangle_{A2}|1\rangle_{B2}\}$$
$$+\beta\gamma\{|1\rangle_{A1}|0\rangle_{B1}\}\{|1\rangle_{A2}|0\rangle_{B2}\} + \beta\delta\{|1\rangle_{A1}|1\rangle_{B1}\}\{|1\rangle_{A2}|1\rangle_{B2}\} \quad (20)$$

For each four-state multiplication of (20) the two-states given in the first curled bracket which are indicated by the subscripts A_1 and B_1 are copied into equivalent two-states given in the second curled bracket which are indicated by the subscripts A_2 and B_2.

Eq. (14) can also be written as a summation of two unitary matrices

$$CPHASE(\theta) = \left(I \otimes (\kappa I + \mu\sigma_3)\right) + \left(\sigma_3 \otimes (\lambda\sigma_3 + \nu I)\right) \quad (21)$$

By using the decomposition of (21) into the summation of two 4X4 unitary matrices and the relations (16), we will assume that the unitary matrix $\left(I \otimes (\kappa I + \mu\sigma_3)\right)$ will operate on the two-states given in the first curled brackets of (20) with subscripts A_1 and B_1 leading to the transformations:

$$\{|0\rangle_{A1}|0\rangle_{B1}\} \to |0\rangle_{A1}\{(\kappa+\mu)|0\rangle_{B1}\}; \quad \{|0\rangle_{A1}|1\rangle_{B1}\} \to |0\rangle_{A1}\{(\kappa-\mu)|1\rangle_{B1}\}$$
$$\{|1\rangle_{A1}|0\rangle_{B1}\} \to |1\rangle_{A1}(\kappa+\mu)|0\rangle_{B1}; \quad \{|1\rangle_{A1}|1\rangle_{B1}\} \to |1\rangle_{A1}\{(\kappa-\mu)|1\rangle_{B1}\} \quad (22)$$

and that the unitary matrix $\left(\sigma_3 \otimes (\lambda\sigma_3 + \nu I)\right)$ will operate on the two-states given in the second curled brackets of (20) with subscripts A_2 and B_2 leading to the transformations:

$$\{|0\rangle_{A2}|0\rangle_{B2}\} \to |0\rangle_{A2}\{(\lambda+\nu)|0\rangle_{B2}\}; \quad \{|0\rangle_{A2}|1\rangle_{B2}\} \to |0\rangle_{A2}\{(\nu-\lambda)|1\rangle_{B2}\}$$
$$\{|1\rangle_{A2}|0\rangle_{B2}\} \to -|1\rangle_{A2}\{(\lambda+\nu)|0\rangle_{B2}\}; \quad \{|1\rangle_{A2}|1\rangle_{B2}\} \to |1\rangle_{A2}\{(\lambda-\nu)|1\rangle_{B2}\} \quad (23)$$

Such processes can be implemented experimentally due to different locations of the two-states so that the operation of the unitary matrix $CPHASE(\theta)$ has been decomposed here into the summation of two unitary processes each operating on a different copy of the two-states.

Performing the transformations (22-23) on the input state (20) we get :

$$\begin{aligned} |\psi\rangle_{out} &= \alpha\gamma\left\{|0\rangle_{A1}\left[(\kappa+\mu)|0\rangle_{B1}\right]\right\}\left\{|0\rangle_{A2}\left[(\lambda+\nu)|0\rangle_{B2}\right]\right\} \\ &+\alpha\delta\left\{|0\rangle_{A1}\left[(\kappa-\mu)|1\rangle_{B1}\right]\right\}\left\{|0\rangle_{A2}\left[(\nu-\lambda)|1\rangle_{B2}\right]\right\} \\ &+\beta\gamma\left\{|1\rangle_{A1}\left[(\kappa+\mu)|0\rangle_{B1}\right]\right\}\left\{|1\rangle_{A2}\left[-(\lambda+\nu)|0\rangle_{B2}\right]\right\} \\ &+\beta\delta\left\{|1\rangle_{A1}\left[(\kappa-\mu)|1\rangle_{B1}\right]\right\}\left\{|1\rangle_{A2}\left[(\lambda-\nu)|1\rangle_{B2}\right]\right\} \end{aligned} \tag{24}$$

Here the phases of the states with subscripts A_1 and A_2 are assumed to be positive and relative to them the phases of the qubits with subscribts B_1 and B_2 are given. One should take into account that by the copying procedure the input and correspondingly the output states were doubled.

We can consider (24) as a certain implementation of the $CPHASE(\theta)$ gate where the control operation of this gate has been decomposed into two equal control qubits. We find that the states with subscript A_1 are equal to those with subscript A_2, both can be considered as equal to the control qubit which is not changed by the quantum gate. The target states have been decomposed here into two *different* target states denoted by the subscripts B_1 and B_2. We get a *relative* phase of the target state denoted by subscript B_1 relative to the control state denoted by subscript A_1, and we get a relative phase of the target state denoted by subscript B_2 relative to the control state denoted by subscript A_2. When we add these two relative phases, which can be obtained in two separated experiments, the $CPHASE(\theta)$ gate is realized as described by the following correspondences:

$$\begin{aligned} &|0\rangle_{A1}|0\rangle_{A2} \to 2|0\rangle_{A} \quad ; \quad |1\rangle_{A1}|1\rangle_{A2} \to 2|1\rangle_{A} \quad ; \\ &\left[(\kappa+\mu)|0\rangle_{B1}\right]\left[(\lambda+\nu)|0\rangle_{B2}\right] \to (\kappa+\mu+\lambda+\nu)|0\rangle_{B} \quad ; \\ &\left[(\kappa-\mu)|1\rangle_{B1}\right]\left[(\nu-\lambda)|0\rangle_{B2}\right] \to (\kappa-\mu+\nu-\lambda)|0\rangle_{B} \quad ; \\ &\left[(\kappa+\mu)|0\rangle_{B1}\right]\left[-(\nu+\lambda)|0\rangle_{B2}\right] \to (\kappa+\mu-\nu-\lambda)|0\rangle_{B} \quad ; \\ &\left[(\kappa-\mu)|1\rangle_{B1}\right]\left[(\lambda-\nu)|1\rangle_{B2}\right] \to (\kappa-\mu+\lambda-\nu)|0\rangle_{B} \end{aligned} \quad . \tag{25}$$

Using the relations (15) we get

$$\begin{aligned} &(\kappa+\mu+\lambda+\nu)=1 \quad ; \quad (\kappa-\mu+\nu-\lambda)=1 \quad ; \\ &(\kappa+\mu-\nu-\lambda)=1 \quad ; \quad (\kappa-\mu+\lambda-\nu)=1+4\lambda=e^{i\theta} \end{aligned} \tag{26}$$

Substituting (25,26) into (24) we get

$$\frac{|\psi\rangle_{out}}{2} = \alpha\gamma|0\rangle_A|0\rangle_B + \alpha\delta|0\rangle_A|1\rangle_B + \beta\gamma|1\rangle_A|0\rangle_B + \beta\delta e^{i\theta}|1\rangle_A|1\rangle_B \quad (27)$$

which is equivalent to the transformation given by (18) (up to the unimportant factor 2). The transformation in the doubled space of (20) to the output (24) with the above correspondences leads in a certain special way to implementation of the $CPHASE(\theta)$ gate.

One should take into account that $CPHASE(\theta)$ has been implemented by one to one correspondence of (20) to (24) so that such implementation is mainly in 'principle'. One might also perform the addition of the relative phases in interference experiments leading to relations (26) and then the implementation will be also in 'practice'. The relations (25,26) are given by the addition of the amplitudes like those given by interference experiments and are basicly different from the addition of logic states numbers [2]. The transformation in the doubled space of the input state $|\psi\rangle_{in}$ of (19) to the output state $|\psi\rangle_{out}$ of (24) realizes the quantum gate since we can transform back the doubled output state to the ordinary $CPHASE(\theta)$ output two-qubit state.

4 Summary, Discussion and Conclusion

The use of probabilistic logic operations has been developed for implementing quantum gates. It has been shown that quantum gates can be decomposed into summation of tensor products of unit and Pauli 2x2 spin matrices. Such HS decomposition has been applied for the $CPHASE(\theta)$ gate leading to a summation of two unitary matrices of dimension 4x4. In the present method each 4x4 matrix operates on a different copy of the two-states produced by quantum encoders. By adding the relative phases in the two two-states' copies the $CPHASE(\theta)$ gate is realized.

The same technique that has been described in the present work for implementing the $CPHASE(\theta)$ gate can be used also for implementing the $CNOT$ gate. The encoding process for the quantum states is the same for the two cases. The only difference is that the decomposition of the $CPHASE(\theta)$ gate by (14) is replaced by the decomposition of the $CNOT$ by (13). Replacing the operations of the unitary matrices of (14) on the quantum states, by those of (13) one can use a similar procedure for implementing the $CNOT$ gate. While the $CPHASE(\theta)$ gate is realized by adding the relative phases of the two target states, the $CNOT$ gate can be realized by adding the polarization states of the two target states. Thus, we have shown therefore a new method for implementing both $CNOT$ and $CPHASE(\theta)$ gates [1,2].

In the present work we have used the HS decomposition by which the quantum gate is decomposed into a summation of unitary matrices where each of these matrices is given by tensor products of Pauli and unit spin matrices. It has been shown that by operating with each of these matrices on a different copy of the states in each superposition and by adding the results for the different copies the quantum gate is realized. The analyses for $CPHASE(\theta)$ and $CNOT$ gates are relatively simple due to the fact that for these gates the decomposition can include only two such matrices. For other gates the HS decomposition might include the summation of more unitary matrices so that the corresponding copying processes by quantum encoders should be more complicated. However, the present analysis becomes quite general if we consider the fact that any quantum gate can be obtained by the combinations of single-qubit gates and two-qubit $CPHASE(\theta)$ and $CNOT$ gates [1,2].

Quantum encoding processes which are obtained by using probabilistic transformations have been already applied successfully in the experiments reported in [5-7]. In the present work the use of quantum encoders has been developed for implementing quantum gates by new methods using the HS decomposition which are different from those used previously and these methods should therefore be of interest both theoretically and experimentally.

Any quantum computational process is described by a certain circuit assuming any initial input state and its end to be measured. The quantum circuits described in the present analysis seem to be different from the conventional ones. However, the initial state assumed in our analysis and its end to be measured are equivalent to the corresponding conventional two-qubit gates. Therefore we find that the present method has developed certain realizations of the quantum gates.

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