Chapter 5
Fractions: Continued, Egyptian and Farey

Continued fractions are one of the most delightful and useful subjects of arithmetic, yet they have been continually neglected by our educational factions. Here we discuss their applications as approximating fractions for rational or *irrational numbers* and *functions*, their relations with *measure theory* (and deterministic chaos!), their use in *electrical networks* and in solving the *"squared square"*; and the *Fibonacci* and *Lucas numbers* and some of their endless applications.

We also mention the (almost) useless *Egyptian fractions* (good for designing puzzles, though, including *unsolved* puzzles in number theory) and we resurrect the long-buried *Farey fractions*, which are of considerable contemporary interest, especially for *error-free computing*.

Among the more interesting recent applications of Farey series is the reconstruction of periodic (or nearly periodic) functions from "sparse" sample values. Applied to two-dimensional functions, this means that if a motion picture or a television film has sufficient structure in space and time, it can be reconstructed from a fraction of the customary picture elements ("pixels"). ("Sufficient structure" in spacetime implies that the reconstructions might not work for a blizzard or a similar "snow job".)

5.1 A Neglected Subject

Continued fractions (CFs) play a large role in our journey through number theory [5.1]. A simple continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \dots}}}\quad, \tag{5.1}$$

a typographical nightmare if there ever was one, is usually written as follows: $[b_0; b_1, b_2, b_3, \dots]$. Here the b_m are integers. A finite simple CF then looks like this:

$$[b_0; b_1, \dots, b_n]. \tag{5.2}$$

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If a finite or infinite CF is broken off after $k < n$, then $[b_0;b_1,\dots,b_k] = A_k/B_k$ is called the approximating fraction or convergent of order k. Here A_k and B_k are coprime integers; they obey the recursion

$$A_k = b_k A_{k-1} + A_{k-2}, \tag{5.3}$$

with $A_0 = b_0$, $A_{-1} = 1$ and $A_{-2} = 0$. The B_k are derivable from the same recursion:

$$B_k = b_k B_{k-1} + B_{k-2}, \tag{5.4}$$

with $B_0 = 1$ and $B_{-1} = 0$.

As the order of the approximating fractions increases, so does the degree of approximation to the true value of the fraction, which is approached alternately from above and below.

CFs are unique if we outlaw a 1 as a final entry in the bracket. Thus 1/2 should be written as $[0;2]$ and not $[0;1,1]$. In general, if a 1 occurs in the last place, it can be eliminated by adding it to the preceding entry.

Continued fractions are often much more efficient in approximating rational or irrational numbers than ordinary fractions, including decimals. Thus,

$$r = \frac{964}{437} = [2;\quad 4,\quad 1,\quad 5,\quad 1,\quad 12], \tag{5.5}$$

and its approximating fraction of order 2, $[2;4,1] = 11/5$, approaches the final value within 3 parts in 10^3.

One interesting application of CFs is to answer such problems as "when is the power of the ratio of small integers nearly equal to a power of 2?", a question of interest in designing cameras, in talking about computer memory and in the tuning of musical instruments (Sect. 2.6). For example, what integer number of musical *major thirds* equals an integral number of *octaves*, i. e., when is

$$\left(\frac{5}{4}\right)^n \approx 2^m, \quad \text{or}$$

$$5^n \approx 2^{m+2n}?$$

By taking logarithms to the base 2, we have

$$\log_2 5 \approx \frac{m}{n} + 2.$$

The fundamental theorem tells us that there is no exact solution; in other words, $\log_2 5$ is irrational. With the CF expansion for $\log_2 5$ we find

$$\log_2 5 = 2.3219\cdots = [2;3,9,\dots] \quad \text{or}$$
$$\log_2 5 \approx 2 + \tfrac{1}{3},$$

yielding $m = 1$ and $n = 3$. Check:

$$\left(\frac{5}{4}\right)^3 = 1.953\cdots \approx 2.$$

In other words, the well-tempered third-octave $2^{1/3}$ matches the major third within 0.8 % or 14 musical cents. (The musical cent is defined as 1/1200 of an octave. It corresponds to less than 0.6 Hz at 1 kHz, roughly twice the just noticeable pitch difference.)

The next best CF approximation gives

$$\log_2 5 \approx 2 + \tfrac{9}{28},$$

or $m = 9$, $n = 28$, a rather unwieldy result.

Because $\log_{10} 2$, another frequently occurring irrational number, is simply related to $\log_2 5$:

$$\log_{10} 2 = 1/(1 + \log_2 5),$$

it has a similar CF expansion:

$$\log_{10} 2 = [0; 3, 3, 9, \ldots].$$

It, too, is well approximated by breaking off before the 9. This yields

$$\log_{10} 2 \approx [0; 3, 3] = \tfrac{3}{10},$$

a well-known result (related to the fact that $2^{10} \approx 10^3$).

Some irrational numbers are particularly well approximated. For example, the widely known first-degree approximation to π, namely, $[3; 7] = 22/7$, comes within 4 parts in 10^3. The second-order approximation $[3; 7, 16] = 355/113$, known to the early Chinese, approaches π within 10^{-7}.

Euler [5.2] discovered that the CF expansion of $\mathrm{e} = 2.718281828\ldots$, unlike that of π, has a noteworthy regularity:

$$\mathrm{e} = [2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1,\ 1,\ 8,\ 1,\ \ldots], \tag{5.6}$$

but converges initially very slowly because of the many 1's. In fact, the CF for the *Golden section or Golden ratio* $g = [1; 1, 1, 1, 1 \ldots]$, which contains infinitely many 1's, is the most slowly converging CF. It is therefore sometimes said, somewhat irrationally, that g is the "most irrational" number. In fact, for a given order of rational approximation the approximation to g is worse than for any other number. Because of this property, up-to-date physicists who study what they call "deterministic chaos" in nonlinear systems often pick the Golden ratio g as a parameter (e. g., a frequency ratio) to make the behaviour "as aperiodic as possible". A strange application of number theory indeed!

CFs are also useful for approximating *functions*. Thus, in a generalization of our original bracket notation, permitting noninteger entries,

$$\tan z = \left[\frac{z}{1-} \frac{z^2}{3-} \frac{z^2}{5-} \dots \right] := \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5}} \dots} \tag{5.7}$$

yields the second-order approximation [5.3]

$$\tan z = z \frac{15 - z^2}{15 - 6z^2}. \tag{5.8}$$

For $z = \pi/4$, this is about 0.9998 (instead of 1). By contrast, the three-term power series for $\tan z$, $\tan z = z + z^3/3 + z^5/5$, makes an error that is 32 times larger. The reason for the superiority of the CF over the power-series expansion is quite obvious. As we can see from (5.8), the CF expansion makes use of polynomials not only in the numerator but also in the *denominator*. (Not making use of this degree of freedom is as if a physicist or engineer tried to approximate the behaviour of a *resonant system* by zeros of analytic functions only, rather than by zeros and *poles*: it is possible, but highly inefficient.)

Equally remarkable is the approximation of the error integral by a CF. The third-order approximation

$$\int_0^z e^{-x^2} dx \approx \frac{49140 + 3570z^3 + 739z^5}{49140 + 19950z^2 + 2475z^4} \tag{5.9}$$

makes an error of only 1.2 % for $z = 2$, as opposed to a power series including terms up to z^9 which overshoots the true value by 110 %.

Incidentally, the fact that e has so regular a CF representation as (5.6), while π does not, does not mean that there is no regular relationship between CFs and some relative of π. In fact, the (generalized) CF expansion of $\arctan z$ for $z = 1$ leads to the following neat CF representation:

$$\frac{\pi}{4} = \left[\frac{1}{1+} \frac{1}{2+} \frac{9}{2+} \frac{25}{2+} \frac{49}{2+} \dots \right].$$

Gauss, the prodigious human calculator, used CFs profusely; even on the first page of his new notebook begun on the occasion of the discovery of the regular 17-gon, CFs make their appearance (Fig. 5.1).

Why are CFs treated so negligently in our high (and low) schools? Good question, as we shall see when we study their numerous uses.

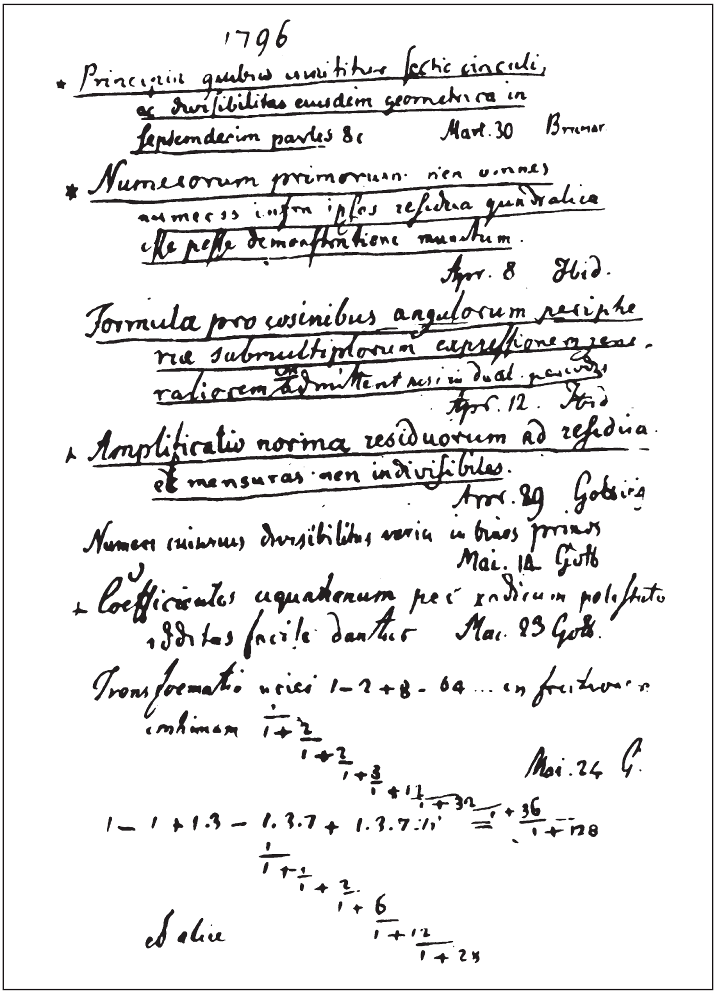

Fig. 5.1 First page of Gauss's notebook, begun in his native city of Brunswick when he was only 18. The first entry concerns the epochal "geometrical" construction of the regular 17-gon which convinced him that he should become a mathematician. The last entry on this page, written like the three preceding ones in Göttingen, shows his early interest in continued fractions

5.2 Relations with Measure Theory

Consider the CF

$$\alpha = [0; a_1, a_2, \ldots]. \tag{5.10}$$

In 1828, Gauss established that for almost all α in the open interval $(0,1)$ the probability

$$W\left([0;a_n,a_{n+1},\ldots] < x\right) \tag{5.11}$$

tends to $\log_2(1+x)$ as n goes to infinity. Gauss also showed that the probability

$$W(a_n = k) \to \log_2\left[1+\frac{1}{k(k+2)}\right], \tag{5.12}$$

i. e., the probabilities for $a_n = 1,2,3,\ldots$ decline as 0.42, 0.17, 0.09, ..., in contrast to the equal probabilities of the 10 digits for "most" decimal digits. *Khinchin* [5.4] showed in 1935 that for almost all real numbers the geometric mean

$$(a_1 a_2 \ldots a_n)^{\frac{1}{n}} \to \prod_{k=1}^{\infty}\left[1+\frac{1}{k(k+2)}\right]^{\log_2 k} = 2.68545\ldots, \tag{5.13}$$

and that the denominators of the approximating fractions

$$(B_n)^{\frac{1}{n}} \to \mathrm{e}^{\pi^2/12\ln 2} = 3.27582\ldots. \tag{5.14}$$

These strange constants are reminiscent of the magic numbers that describe period doubling for strange attractors in deterministic chaos. And perhaps there is more than a superficial connection here.

5.3 Periodic Continued Fractions

As with periodic decimals, we shall designate (infinite) periodic CFs like $[1;2,2,2,\ldots]$ by a bar over the period:

$$[1;\overline{2}]. \tag{5.15}$$

Incidentally, $[1;\overline{2}]$ has the value $\sqrt{2}$. In general, periodic CFs have values in which square (but no higher) roots appear.

An integer that is a nonperfect square, whose square root has a periodic, and therefore *infinite*, CF, has an irrational square root. However, there are simpler proofs that $\sqrt{2}$, say, is irrational without involving CFs. Here is a simple indirect proof: suppose $\sqrt{2}$ is rational:

$$\sqrt{2} = \frac{m}{n}, \tag{5.16}$$

where m and n are coprime:

$$(m,n) = 1, \tag{5.17}$$

i. e., the fraction for $\sqrt{2}$ has been "reduced" (meaning the numerator m and the denominator n have no common divisor). Squaring (5.16) yields

$$2n^2 = m^2. \tag{5.18}$$

Thus, m must be even:

$$m = 2k, \tag{5.19}$$

or, with (5.18),

$$2n^2 = 4k^2, \tag{5.20}$$

which implies that n is also even. Consequently,

$$(m,n) > 1, \tag{5.21}$$

contradicting (5.17). Thus, there are *no* integers n,m such that $\sqrt{2} = m/n$; in other words: $\sqrt{2}$ is irrational. Q.E.D.

An even shorter proof of the irrationality of $\sqrt{2}$ goes as follows. Suppose $\sqrt{2}$ is rational. Then there is a *least* positive integer n such that $n\sqrt{2}$ is an integer. Set $k = (\sqrt{2} - 1)n$. This is a positive integer *smaller* than n, but

$$k\sqrt{2} = \left(\sqrt{2} - 1\right)n\sqrt{2} = 2n - \sqrt{2}n$$

is the difference of two different integers and so is a positive integer. Contradiction: n was supposed to be the smallest positive integer such that multiplying it by $\sqrt{2}$ gives an integer! Using this kind of proof, for which s can one show that $\sqrt{s}$ is irrational? What modification(s) does the proof require?

Another exhibition example for CF expansion which we have already encountered is

$$[1;\overline{1}] = \tfrac{1}{2}\left(1 + \sqrt{5}\right), \tag{5.22}$$

the famous Golden ratio g: if a distance is divided so that the ratio of its total length to the longer portion equals g, then the ratio of the longer portion to the shorter one also equals g. By comparison with (5.13) we see that the expansion coefficients in the continued fraction of g, being all 1, are 2.68... times smaller than the geometric mean over (almost) all numbers.

Golden *rectangles* have played a prominent role in the pictorial arts, and Fig. 5.2 illustrates the numerous appearances of g in a painting by Seurat. Figure 5.3 shows an infinite sequence of "golden rectangles" in which the sides have ratio g. To construct this design, lop off a square from each golden rectangle to obtain the next smaller golden rectangle.

The Golden ratio, involving as it does the number 5 – a Fermat prime – is also related, not surprisingly, to the regular pentagon, as illustrated in Fig. 5.4. It is easily verified (from Pythagoras) that $\overline{AB}/\overline{AT} = g$. Thus, the Golden ratio emerges as the ratio of the diagonal of the regular pentagon to its side.

Finally, the Golden ratio g emerged as the noblest of "noble numbers", the latter being defined by those (irrational) numbers whose continued fraction expansion

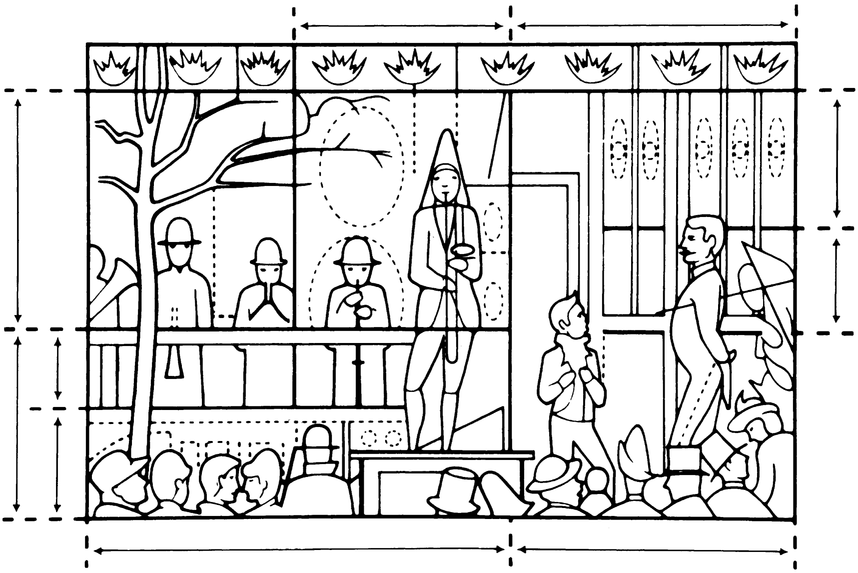

Fig. 5.2 Golden ratios in a painting by Seurat

ends in infinitely many 1's. In fact, the CF of g, see (5.22), has *only* 1's, whence also its nickname "the most irrational number" (because no irrational has a CF approximation that converges more slowly than that for g).

The designation *noble numbers* stems from the fact that in many nonlinear dynamical systems "winding numbers" (the frequency ratios of orbits in phase space) that equal noble numbers are the most resistant against the onset of chaotic motion, which is ubiquitous in nature. (Think of turbulence – or the weather, for that matter.)

Cassini's divisions in the rings of Saturn are a manifestation of what happens when, instead of noble numbers, base numbers reign: rocks and ice particles constituting the rings, whose orbital periods are in simple rational relation with the periods of other satellites of Saturn, are simply swept clean out of their paths by the resonance effects between commensurate orbital periods. In fact, the very stability of the entire solar system depends on the nobility of orbital period ratios.

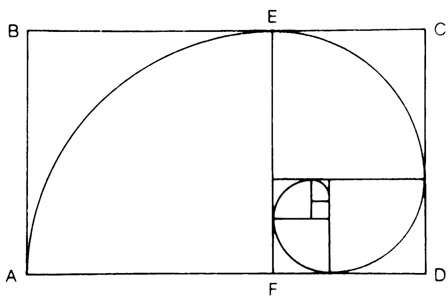

Fig. 5.3 The beginning of an infinite sequence of Golden rectangles. Some sea shells are said to use this construction

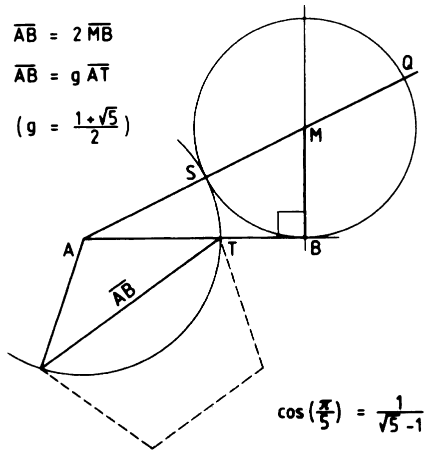

Fig. 5.4 The Golden ratio and the construction of the regular pentagon

A double pendulum in a gravitational field is a particularly transparent nonlinear system. As the nonlinearity is increased (by slowly "turning on" the gravitation), the last orbit to go chaotic is the one with a winding number equal to $1/(1+g) = [0;2,1,1,1,\ldots]$, a *very* noble number!

For physical systems a winding number $w < 1$ is often equivalent to the winding number $1-w$. Suppose $w = [0;a_1,a_2,\ldots]$, what is the CF for $1-w$? The reader will find it easy to show that

$$1-[0;a_1,a_2,\ldots] = [0;1,a_1-1,a_2,\ldots]. \tag{5.23}$$

Thus, if w is noble, so is $1-w$. Note that if $a_1-1=0$, we need to invoke the rule

$$[\ldots,a_m,0,a_{m+2},\ldots] = [\ldots,a_m+a_{m+2},\ldots],$$

which assures that the CF for $1-(1-w)$ equals that for w.

Among the most exciting nonlinear systems where CF expansions have led to deep insights are the *fractional quantization* in a two-dimensional electron gas [5.5] and "Frustrated instabilities" in active optical resonators (lasers) [5.6].

5.4 Electrical Networks and Squared Squares

One of the numerous practical fields where CFs have become entrenched – and for excellent reasons – are electrical networks.

What is the input impedance Z of the "ladder network" shown in Fig. 5.5 when the R_k are "series" impedances and the G_k are "shunt" admittances? A moment's thought will provide the answer in the form of a CF:

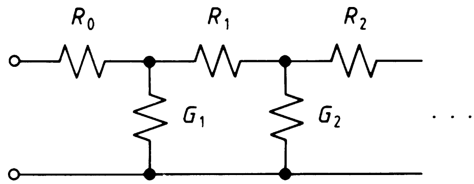

Fig. 5.5 Electrical ladder network. Input impedance is given by continued fraction $[R_0; G_1, R_1, G_2, \ldots]$

$$Z = [R_0; G_1, R_1, G_2, R_2, \ldots].$$

Here, in the most general case, the R_k and G_k are complex-valued rational functions of frequency.

If all R_k and G_k are 1-ohm resistors, the final value of Z for an infinite network, also called the characteristic impedance Z_0, will equal

$$g = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \text{ ohm.}$$

The application of CFs to electrical networks has, in turn, led to the solution of a centuries-old teaser, the so-called Puzzle of the Squared Square, i. e., the problem of how to divide a square into unequal squares with integral sides. This problem had withstood so many attacks that a solution was widely believed impossible [5.7]. Thus, the first solution, based on network theory, created quite a stir when it appeared (Fig. 5.6).

In the meantime Littlewood has given a solution for a 112 by 112 square, which is the smallest squared square found so far.

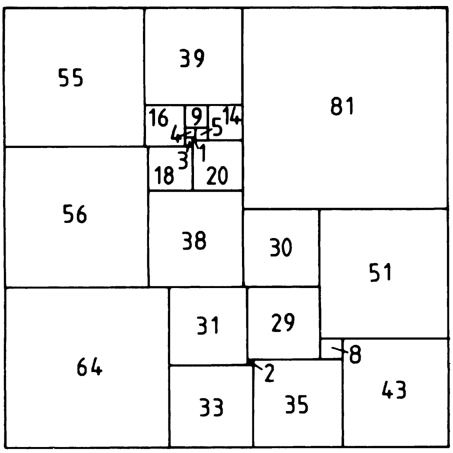

Fig. 5.6 The first squared square, a solution based on the theory of electrical networks and continued fractions (courtesy E. R. Wendorff)

5.5 Fibonacci Numbers and the Golden Ratio

Another close relative of CFs are the Fibonacci numbers [5.8], defined by the recursion

$$F_n = F_{n-1} + F_{n-2}, \quad \text{with} \quad F_0 = 0 \quad \text{and} \quad F_1 = 1, \tag{5.24}$$

which is identical with the CF recursion for the case $b_k = 1$. The first Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... where each number is the sum of its two predecessors.

The ratio of two successive F_n approaches the Golden ratio $g = (1+\sqrt{5})/2$, which is easily verified in terms of CFs. From the recursion (5.24) it follows that

$$F_{n+1}/F_n = [1;\ \underbrace{1,\ldots,1}_{n-1\ 1's}\,] \qquad (n > 1), \tag{5.25}$$

where the right-hand side of (5.25) is the approximating fraction to the Golden ratio. Equation (5.25) also implies that successive F_n are coprime:

$$(F_n, F_{n+1}) = 1, \qquad n > 0. \tag{5.26}$$

Also, the product of F_n $(n > 1)$ and its predecessor differs by ± 1 from the product of their two neighbours:

$$F_{n-1}F_n - F_{n-2}F_{n+1} = (-1)^n. \tag{5.27}$$

Examples: $21 \cdot 34 = 13 \cdot 55 - 1$; $34 \cdot 55 = 21 \cdot 89 + 1$.

The reader may wish to prove his or her prowess by proving these simple statements. Equations such as (5.27) often provide quick answers to a certain class of problems such as the “banking” puzzle described in Sect. 5.11.

A simple alternative recursion for F_n is

$$F_n = 1 + \sum_{k=1}^{n-2} F_k. \tag{5.28}$$

Because of the internal structure of the F_n, which relates each F_n to its *two* predecessors, the odd-index F_n can be obtained from the even-index F_n alone:

$$F_{2n+1} = 1 + \sum_{k=1}^{n} F_{2k}. \tag{5.29}$$

It is sometimes said that there is no *direct* (nonrecursive) formula for the F_n, meaning that all predecessors F_k, with $k < n$, have to be computed first. This statement is true, however, only if we restrict ourselves to the integers. If we extend our number field to include square roots, we get the surprising direct formula, discovered by A. de Moivre in 1718 and proved ten years later by *Nicolas Bernoulli*:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}(1+\sqrt{5})\right)^n - \left(\frac{1}{2}(1-\sqrt{5})\right)^n\right], \tag{5.30}$$

and if we admit even complex transcendental expressions, we obtain a very compact formula:

$$F_n = \mathrm{i}^{n-1}\frac{\sin(nz)}{\sin z}, \quad z = \frac{\pi}{2} + \mathrm{i}\ln\left(\frac{1+\sqrt{5}}{2}\right). \tag{5.31}$$

In (5.30), the first term grows geometrically, while the second term alternates in sign and decreases geometrically in magnitude because

$$-1 < \tfrac{1}{2}(1-\sqrt{5}) < 0.$$

In fact, the second term is so small, even for small n, that it can be replaced by rounding the first term to the nearest integer:

$$F_n = \left\lfloor \frac{1}{\sqrt{5}}\left[\frac{1}{2}(1+\sqrt{5})\right]^n + \frac{1}{2}\right\rfloor. \tag{5.32}$$

The result (5.30) is most easily obtained by solving the homogeneous *difference equation* (5.24) by the *Ansatz*

$$F_n = x^n. \tag{5.33}$$

This converts the difference equation into an algebraic equation:

$$x^2 = x + 1. \tag{5.34}$$

(This is akin to solving *differential* equations by an *exponential Ansatz*.)

The two solutions of (5.34) are

$$x_1 = \frac{1}{2}(1+\sqrt{5}) \quad \text{and} \quad x_2 = \frac{1}{2}(1-\sqrt{5}). \tag{5.35}$$

The general solution for F_n is then a linear combination:

$$F_n = ax_1^n + bx_2^n, \tag{5.36}$$

where with the initial conditions $F_0 = 0$ and $F_1 = 1$,

$$a = -b = \frac{1}{\sqrt{5}}. \tag{5.37}$$

Equations (5.35–5.37) taken together yield the desired nonrecursive formula (5.30).

Equation (5.30) can be further compacted by observing that $x_1 = -1/x_2 = g$, so that

$$\sqrt{5}\,F_n = g^n - (-g)^{-n}. \tag{5.38}$$

The right side of (5.38) can be converted into a trigonometric function by setting

$$t = \mathrm{i} \ln g, \tag{5.39}$$

yielding

$$\sqrt{5} F_n = 2\mathrm{i}^{n-1} \sin\left(\frac{\pi}{2} + t\right) n, \tag{5.40}$$

which is identical with (5.31) because

$$\sin\left(\frac{\pi}{2} + \mathrm{i} \ln g\right) = \frac{1}{2}\sqrt{5}, \tag{5.41}$$

a noteworthy formula in itself.

There are also numerous relations between the binomial coefficients and the Fibonacci numbers. The reader might try to prove the most elegant of these:

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}. \tag{5.42}$$

In other words, summing diagonally upward in Pascal's triangle yields the Fibonacci numbers. (Horizontal summing, of course, gives the powers of 2.)

There is also a suggestive matrix expression for the Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{5.43}$$

which is obviously true for $n = 1$ and is easily proved by induction. Since the determinant on the left equals -1, it follows immediately that

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n, \tag{5.44}$$

which generalizes to

$$F_{n+k} F_{m-k} - F_n F_m = (-1)^n F_{m-n-k} F_k, y \tag{5.45}$$

where any negative-index F_n are defined by the "backward" recursion

$$F_{n-1} = F_{n+1} - F_n, \tag{5.46}$$

giving

$$F_{-n} = -(-1)^n F_n. \tag{5.47}$$

Summation leads to some interesting relationships, for example

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1},$$

which is easily proved by induction.

Products of reciprocals, too, have noteworthy sums:

$$\sum_{k=2}^{n} \frac{1}{F_{k-1}F_{k+1}} = 1 - \frac{1}{F_n F_{n+1}}.$$

Ratios of successive Fibonacci numbers have very simple continued fractions. Instead of (5.25) we may write:

$$\frac{F_n}{F_{n+1}} = [1,1,\ldots,1,2],$$

where the number of 1's equals $n-2$.

By adding Fibonacci numbers, the positive integers can be represented *uniquely*, provided each F_n $(n > 1)$ is used at most once and no two adjacent F_n are ever used. Thus, in the so-called *Fibonacci number system*,

$$\begin{aligned}
3 &= 3\\
4 &= 3+1\\
5 &= 5\\
6 &= 5+1\\
7 &= 5+2\\
8 &= 8\\
9 &= 8+1\\
10 &= 8+2\\
11 &= 8+3\\
12 &= 8+3+1\\
1000 &= 987+13 \quad \text{etc.}
\end{aligned}$$

A simple algorithm for generating the Fibonacci representation of m is to find the largest F_n not exceeding m and repeat the process on the difference $m - F_n$ until this difference is zero.

The Fibonacci number system answers such questions as to where to find 0's or 1's or double 1's in the following family of binary sequences:

0
1
1 0
1 0 1
1 0 1 1 0 etc.,

where the next sequence is obtained from the one above by appending the one above *it*. (See Sect. 32.1.)

Another application of the Fibonacci number system is to nim-like games: from a pile of n chips the first player removes any number $m_1 \neq n$ of chips; then the second player takes $0 < m_2 \leq 2m_1$ chips. From then on the players alternate, never taking less than 1 or more than twice the preceding "grab". The last grabber wins.

What is the best first grab? We have to express n in the Fibonacci system:

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}.$$

The best initial move is then to take

$$F_{k_j} + \cdots + F_{k_r}$$

chips for some j with $1 \leq j \leq r$, provided $j = 1$ or

$$F_{k_{j-1}} > 2(F_{k_j} + \cdots + F_{k_r}).$$

Thus, for $n = 1000$, the first player should take 13 chips – the *only* lucky number in this case: only for $m_1 = 13$ can he force a victory by leaving his opponent a Fibonacci number of chips, making it impossible for the second player to force a win.

5.6 Fibonacci, Rabbits and Computers

Fibonacci numbers abound in nature. They govern the number of leaves, petals and seed grains of many plants (see Fig. 5.7 [5.9, 10]), and among the bees the number of ancestors of a drone n generations back equals F_{n+1} (Fig. 5.8).

Rabbits, not to be outdone, also multiply in Fibonacci rhythm if the rules are right: offspring beget offspring every "season" except the first after birth – and they never die (Fig. 5.9). As already mentioned, this was the original Fibonacci problem [5.11] considered in 1202 by Fibonacci himself.

Leonardo da Pisa, as Fibonacci was also known, was a lone star of the first magnitude in the dark mathematical sky of the Middle Ages. He travelled widely in Arabia and, through his book *Liber Abaci*, brought the Hindu-Arabic number system and other superior methods of the East to Europe. He is portrayed in Fig. 5.10.

Fibonacci numbers also tell us in how many ways a row of n squares can be covered by squares or "dominoes" (two squares side-by-side). Obviously for $n = 2$, there are two ways: either 2 squares or 1 domino. For $n = 11$ there are 144 ways. What is the general rule?

n squares can be covered by first covering $n-1$ squares and then adding another square or by first covering $n-2$ squares and then adding 1 domino. Thus, calling the number of different coverings of n squares f_n, we have

$$f_n = f_{n-1} + f_{n-2},$$

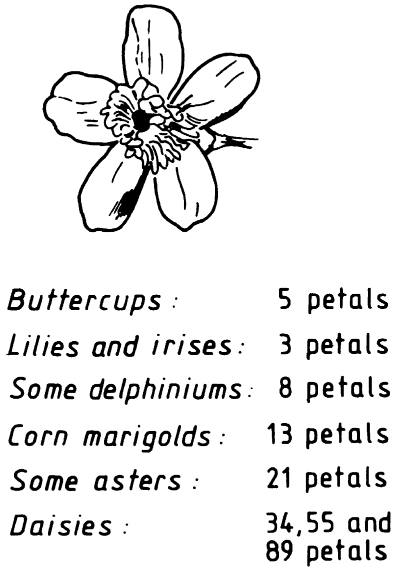

Fig. 5.7 Flowers have petals equal to Fibonacci numbers

i.e., the familiar recursion for the Fibonacci numbers F_n. With the initial values $f_1 = 1$ and $f_2 = 2$, we thus see that $f_n = F_{n+1}$.

Fibonacci numbers also crop up in computer science and artificial languages. Suppose there is a "language" with variables $A, B, C, \ldots$ and functions of one or two variables $A(B)$ or $A(B,C)$. If we leave out the parentheses, how many ways can a string of n letters be parsed, i. e., grammatically decomposed without repeated multiplication? For a string of three letters, there are obviously two possibilities: $A \cdot B(C)$ and $A(B,C)$. In general, the answer is F_n ways, or so says Andrew Koenig of Bell Laboratories [5.12].

Another area in which Fibonacci numbers have found useful application is that of efficient sequential search algorithms for unimodal functions. Here the kth interval for searching is divided in the ratio of Fibonacci numbers F_{n-k}/F_n, so that after the $(n-1)$st step, the fraction of the original interval (or remaining uncertainty)

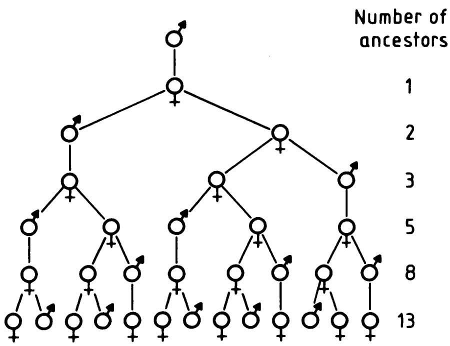

Fig. 5.8 Bees have Fibonacci-number ancestors

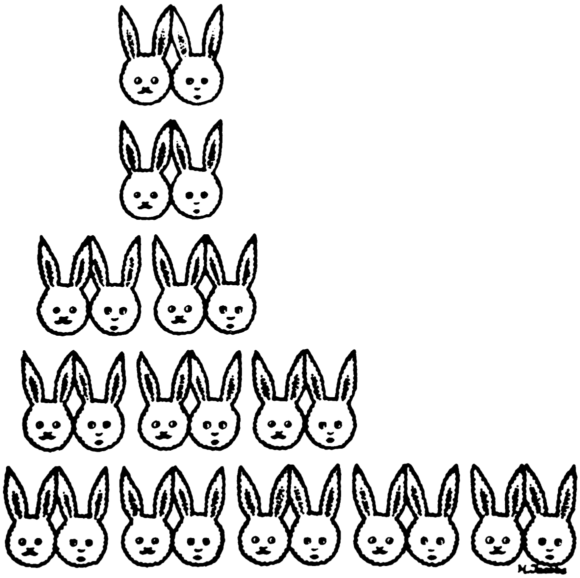

Fig. 5.9 Pairs of rabbits under certain (artifical) birth constraints, multiply according to a Fibonacci series

is $1/F_n \simeq \sqrt{5}/[(1+\sqrt{5})/2]^n$ as opposed to $(1/\sqrt{2})^n$ for "dichotomic" sequential search. After 20 steps, the precision of the Fibonacci-guided search is 6.6 times higher than the dichotomic one [5.13]. For an extensive treatment of applications of number theory in numerical analysis, see [5.14].

Fig. 5.10 Leonardo da Pisa, widely known as Fibonacci ("blockhead"), the great mathematical genius of the Middle Ages – a mathematical dark age outside the Middle East (and the Middle Kingdom!)

5.7 Fibonacci and Divisibility

It can be proved by induction that

$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n, \tag{5.48}$$

which for $m = 2$ is our fundamental recursion (5.24).

By choosing m as a multiple of n, one can further infer that F_{nk} is a multiple of F_n (and of F_k).

Example: $F_{30} = 832040$, which is divisible by $F_{15} = 610$, $F_{10} = 55$, $F_6 = 8$, $F_5 = 5$, etc.

In other words, every third F_n is even, every fourth F_n is divisible by $F_4 = 3$, every fifth F_n by $F_5 = 5$, etc. As a consequence, all F_n for composite n (except $n = 4$) are composite. However, not all F_p are prime. For example, $F_{53} = 953 \cdot 55945741$.

In 1876 Lucas showed even more, namely that, magically, the two operations "take GCD" and "compute Fibonacci" commute:

$$(F_m, F_n) = F_{(m,n)}, \tag{5.49}$$

a "magic" that can be proved with the help of Euclid's algorithm.

Example: $(F_{45}, F_{30}) = (1134903170, 832040) = 610 = F_{15}$.

One of the most interesting divisibility properties of the Fibonacci numbers is that for *each* prime p, there is an F_n such that p divides F_n. More specifically, $p \neq 5$ divides either F_{p-1} [for $p \equiv \pm 1 \pmod 5$] or F_{p+1} [for $p \equiv \pm 2 \pmod 5$]. And of course, for $p = 5$ we have $p = F_p$. In fact, *every integer* divides some F_n (and therefore infinitely many).

Also, for odd prime p,

$$F_p \equiv 5^{\frac{p-1}{2}} \pmod p \tag{5.50}$$

holds.

Many intriguing identities involve powers of Fibonacci numbers, e.g.

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2[F_n^4 + F_{n+1}^4 + F_{n+2}^4]$$

5.8 Generalized Fibonacci and Lucas Numbers

By starting with initial conditions different from $F_1 = F_2 = 1$, but keeping the recursion (5.24), one obtains the generalized Fibonacci sequences, which share many properties with the Fibonacci sequences proper. The recursion for the generalized

Fibonacci sequence G_n, in terms of its initial values G_1 and G_2 and the Fibonacci numbers, is

$$G_{n+2} = G_2 F_{n+1} + G_1 F_n. \tag{5.51}$$

Of course, for $G_1 = G_2 = 1$ the original Fibonacci sequence is obtained. For some initial conditions, there is only a shift in the index, as for example with $G_1 = 1$ and $G_2 = 2$.

However, for $G_1 = 2$ and $G_2 = 1$, one obtains a different sequence:

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ldots, \tag{5.52}$$

the so-called Lucas sequence [5.8]. Of course, obeying the same recursion as the Fibonacci numbers, the ratio of successive Lucas numbers also approaches the Golden ratio. However, they have "somewhat" different divisibility properties.

For example, statistically only *two out of three primes* divide some Lucas number. This result is deeper than those on the divisibility of Fibonacci numbers that we mentioned; it was observed in 1982 by Jeffrey Lagarias [5.15].

A closed form for the Lucas numbers is

$$L_n = g^n + \left(-\frac{1}{g}\right)^n, \quad \text{with} \quad g = \frac{1}{2}\left(1+\sqrt{5}\right), \tag{5.53}$$

where the second term is again alternating and geometrically decaying. This suggests the simpler formula obtained by rounding to the nearest integer:

$$L_n = \left\lfloor g^n + \frac{1}{2} \right\rfloor, \qquad n \geq 2. \tag{5.54}$$

Like the ratios of Fibonacci numbers, the ratios of successive Lucas numbers have very simple continued fractions:

$$\frac{L_n}{L_{n+1}} = [1,1,\ldots,1,3],$$

where the number of 1's equals $n-2$.

Equation (5.53) leads to an intriguing law for the continued fractions of the odd powers of the Golden ratio g. With

$$g^{2n+1} = L_{2n+1} + g^{-2n-1},$$

we get

$$g^{2n+1} = \left[L_{2n+1}; \overline{L_{2n+1}}\right],$$

i. e., a periodic continued fraction of period length 1 with all partial quotients equal to the corresponding Lucas number.

What are the continued fractions of the *even* powers of g? What other irrationals have similarly simple continued fractions? What is the eighth root of the infinite continued fraction:

$$2207 - \cfrac{1}{2207 - \frac{1}{2207} - \dots}?$$

(The latter question was asked as problem B-4 in the 56th Annual William Lowell Putnam Mathematical Competition; see *Mathematics Magazine* **69**, 159 (April 1996), where a somewhat tortuous solution was given.)

Lucas numbers can be used to advantage in the calculation of large even Fibonacci numbers by using the simple relation

$$F_{2n} = F_n L_n \tag{5.55}$$

to extend the accuracy range of limited-precision (noninteger arithmetic) calculators. Similarly, we have for the even Lucas numbers

$$L_{2n} = L_n^2 - 2(-1)^n. \tag{5.56}$$

For odd-index F_n, one can use

$$F_{2n+1} = F_n^2 + F_{n+1}^2 \tag{5.57}$$

to reach higher indices.

The "decimated" Lucas sequence

$$\tilde{L}_n = L_{2^n}, \quad \text{i.e.,}$$

$$3,\ 7,\ 47,\ 2207,\ 4870847, \dots,$$

for which the simple recursion (5.56) $\tilde{L}_{n+1} = \tilde{L}_n^2 - 2$ holds, plays an important role in the primality testing of Mersenne numbers M_p with $p = 4k+3$ (see Chap. 3 for the more general test).

It is not known whether the Fibonacci or Lucas sequences contain infinitely many primes. However, straining credulity, R. L. Graham [5.16] has shown that the generalized Fibonacci sequence with

$$G_1 = 1786\,772701\,928802\,632268\,715130\,455793$$
$$G_2 = 1059\,683225\,053915\,111058\,165141\,686995$$

contains no primes at all!

Interesting results are obtained by introducing random signs into the Fibonacci recursion:

$$f_n = f_{n-1} \pm f_{n-2},$$

or a "growth factor" b:

$$f_n = f_{n-1} + b f_{n-2},$$

or combining both random signs and $b \neq 1$. For $b = \frac{1}{2}$ and random signs (± 1 with equal probabilities) the series doesn't grow but converges on 0. For which value of b does the series neither grow indefinitely nor decay to 0?

Another generalization of Fibonacci numbers allows more than two terms in the recursion (5.24). In this manner kth order Fibonacci numbers $F_n^{(k)}$ are defined that are the sum of the k preceding numbers with the initial conditions $F_0^{(k)} = 1$ and $F_n^{(k)} = 0$ for $n < 0$. For $k = 3$, the 3rd order Fibonacci numbers sequence starts as follows (beginning with $n = -2$): 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, ...

Generalized Fibonacci numbers have recently made their appearance in an intriguing railroad switch yard problem solved by Ma Chung-Fan of the Institute of Mathematics in Beijing (H. O. Pollak, personal communication). In a freight classification yard, a train arrives with its cars in more or less random order, and before the train leaves the yard the cars must be recoupled in the order of destination. Thus, the cars with the nearest destination should be at the front of the train so they can simply be pulled off when that destination is reached, those with the second stop as destination should be next, etc. Recoupling is accomplished with the aid of k spur tracks, where usually $4 \leq k \leq 8$. The initial sequence of cars is decomposed into $\leq k$ subsequences by backing successive cars onto the various spurs, and the subsequences can then be recombined in an arbitrary order as segments in a new sequence. For any initial sequence, the desired rearrangement should be accomplished with a minimum number of times that a collection of cars is pulled from one of the spurs. These are called "pulls". For example, if 10 cars with possible destinations 1 to 7 are given in the order 6324135726, we wish to get them into the order 1223345667. On two tracks, this can be done by first backing

341526 onto the first track,
6237 onto the second track.

Pull both into the order 3415266237 (that's two pulls); then back them onto the two tracks in order

1223 onto the first track,
345667 onto the second track.

Then pull the first track's content onto the second, and pull out the whole train in the right order. Thus, it takes 4 pulls on 2 tracks to get the train together.

Define the *index* $m(\sigma)$ of the sequence $\sigma = 6324135726$ as follows: Start at the leftmost (in this case the only) 1, put down all 1's, all 2's to the right of the last 1, 3's to the right of the last 2 if you have covered *all* the 2's, etc. In this case, the first subset defined in this way is 12 (positions 5 and 9). The next subset takes the other 2 and the second 3 (positions 3 and 6); it can't get to the first 3. The next subset takes the first 3, the 4, the 5, and the second 6; the last subset is 67. Thus 6324135726 has been decomposed into 4 nondecreasing, non-overlapping, non-descending sequences

12, 23, 3456, 67. The “4” is the index $m(\sigma)$ of the given sequence σ; the general definition is analogous; it is the number of times the ordering comes to the left end of the sequence.

Now Ma showed that the minimum number of pulls in which a sequence σ can be ordered on k tracks is the integer j such that

$$F_{j-1}^{(k)} < m(\sigma) \leq F_j^{(k)}.$$

Fibonacci would be delighted!

5.9 Egyptian Fractions, Inheritance and Some Unsolved Problems

A rich sheik, shortly before his death (in one of his limousines; he probably wasn't buckled up) bought 11 identical cars, half of which he willed to his eldest daughter, one quarter to his middle daughter, and one sixth to his youngest daughter. But the problem arose how to divide the 11 cars in strict accordance with the will of the (literally) departed, without smashing any more cars. A new-car dealer offered help by lending the heirs a brand-new identical vehicle so that each daughter could now receive a whole car: the eldest 6, the middle 3 and the youngest 2. And lo and behold, after the girls (and their retinues) had driven off, one car remained for the dealer to reclaim!

The problem really solved here was to express $n/(n+1)$ as a sum of 3 *Egyptian fractions*, also called *unit fractions*:

$$\frac{n}{n+1} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}. \tag{5.58}$$

In the above story, $n = 11$ and $a = 2$, $b = 4$, $c = 6$.

Interestingly, for $n = 11$, there is another solution (and potential story) with $a = 2$, $b = 3$, $c = 12$ because *two* subsets of the divisors of 12 $(1,2,3,4,6,12)$ add to 11. Check: $2+3+6 = 1+4+6 = 11$. Check! The inheritance problem is related to *pseudoperfect* numbers, defined as numbers equal to a sum of a subset of their divisors [5.17].

For 3 heirs and 1 borrowed car there are only 5 more possible puzzles, the number of cars being $n = 7$, 17, 19, 23 and 41 [5.18].

As opposed to continued fractions, unit fractions are of relatively little use (other than in tall tales of inheritance, perhaps). In fact, they probably set back the development of Egyptian mathematics incalculably. However, they do provide fertile ground for numerous unsolved problems in Diophantine analysis [5.17].

Of special interest are sums of unit fractions that add up to 1. Thus, for example, it is not known what is the smallest possible value of x_n, called $m(n)$, in

$$\sum_{k=1}^{n} \frac{1}{x_k} = 1, \qquad x_1 < x_2 < \cdots < x_n. \tag{5.59}$$

It is "easy" to check that $m(3) = 6$, $m(4) = 12$ and $m(12) = 30$. But what is the general law? Is $m(n) < cn$ for some constant c? Unknown!

Is $x_{k+1} - x_k \leq 2$ ever possible for all k? Erdös in [5.17] thinks not and offers ten (1971?) dollars for the solution.

Graham [5.19] was able to show that for $n > 77$, a partition of n into distinct positive integers x_k can always be found so that $\sum 1/x_k = 1$.

5.10 Farey Fractions

Another kind of fraction, the *Farey Fractions* have recently shown great usefulness in number theory [5.20].

For a fixed $n > 0$, let all the reduced fractions with nonnegative denominator $\leq n$ be arranged in increasing order of magnitude. The resulting sequence is called the Farey sequence of order n or belonging to n.

Example: for $n = 5$, in the interval [0,1] we have:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}. \tag{5.60}$$

For other intervals, the Farey fractions are congruent modulo 1 to the Farey fractions in (5.60). In the interval $(c, c+1]$ there are exactly $\sum_{b=1}^{n} \phi(b) \approx 3n^2/\pi^2$ Farey fractions [see Chap. 8 for the definition of $\phi(n)$].

Calling two successive Farey fractions a/b and c/d, then

$$b + d \geq n + 1, \quad \text{and} \tag{5.61}$$

$$cb - ad = 1, \quad \textit{for} \quad \frac{a}{b} < \frac{c}{d}. \tag{5.62}$$

One of the outstanding properties of the Farey fractions is that given any real number x, there is always a "nearby" Farey fraction a/b belonging to n such that

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b(n+1)}. \tag{5.63}$$

Thus, if $b > n/2$ the approximating error (5.63) is bounded by $2/n^2$. This compares well with the approximate approximating error $\pi^2/12n^2$ which would result if the Farey fractions were completely uniformly distributed.

What is the spectrum (Fourier transform) of the process defined by (5.63) when x goes uniformly from 0 to 1?

The following recursion provides a convenient method of generating the Farey fractions x_i/y_i of order n: Set $x_0 = 0$, $y_0 = x_1 = 1$ and $y_1 = n$. Then

$$\begin{aligned} x_{k+2} &= \left\lfloor \frac{y_k+n}{y_{k+1}} \right\rfloor x_{k+1} - x_k, \\ y_{k+2} &= \left\lfloor \frac{y_k+n}{y_{k+1}} \right\rfloor y_{k+1} - y_k. \end{aligned} \tag{5.64}$$

The *mediant* of two fractions $\frac{a}{b}$ and $\frac{e}{f}$ is defined by

$$\textit{mediant}\left(\frac{a}{b}, \frac{e}{f}\right) := \frac{a+e}{b+f}, \tag{5.65}$$

which lies in the interval $\left(\frac{a}{b}, \frac{e}{f}\right)$. Each term in a Farey series $\dots \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \dots$ is the mediant of its two neighbours:

$$\frac{c}{d} = \frac{a+e}{b+f}. \tag{5.66}$$

In fact, the mediant of any two terms is contained in the Farey series, unless the sum of their (reduced) denominators exceeds the order n of the series.

There is also an interesting geometrical interpretation of Farey fractions in terms of point lattices, especially the fundamental point lattice consisting of all integer pairs (x, y). The Farey fractions a/b belonging to n are precisely all those lattice points $(x = a,\ y = b)$ in the triangle defined by $y = 0$, $y = x$, $y = n$ which can be "seen" from the origin $x = y = 0$, or, equivalently, which can "see" the origin with no other "Farey points" lying on the line of sight (see also Fig. 4.8).

Farey fractions are useful in rational approximations. Continued fractions give the excellent approximation

$$\frac{1}{\pi} \approx \frac{113}{355}.$$

But suppose we want to construct mechanical gears in the approximate ratio $\pi : 1$ using fewer than 100 teeth on the smaller of the two gears. Continued fractions would then give us

$$\frac{1}{\pi} \approx \frac{7}{22},$$

but we can do better with Farey fractions. In a table published by the London Royal Society [5.21] of the Farey series of order 1025 we find near 113/355 the entries

$$\frac{99}{311}, \frac{92}{289}, \frac{85}{267}, \frac{78}{245}, \frac{71}{223} \quad \text{and} \quad \frac{64}{201},$$

any one of which is a better approximation than $7/22$.

Or suppose we want one of the gears to have 2^n teeth. We find in the table

$$\frac{1}{\pi} \approx \frac{163}{512},$$

with an error of $1.5 \cdot 10^{-4}$. This table is of course quite voluminous, having a total of 319765 entries (and a guide to locate the fraction nearest to any given number in the interval (0,1) quickly). (With $\sum_{b=1}^{n} \phi(b) \approx (3/\pi^2)n^2$, (see Chap. 8) we expect about 320000 Farey fractions of order 1025.)

Another important practical application of Farey fractions implied by (5.62) is the solution of Diophantine equations (see Chap. 7). Suppose we are looking for a solution of

$$243b - 256a = 1 \tag{5.67}$$

in integer a and b. By locating the Farey fraction just below $243/256$, namely $785/827$, we find $a = 785$ and $b = 827$. Check: $243 \cdot 827 = 200961$ and $256 \cdot 785 = 200960$. Check!

Of course, we can reduce the above solution for a modulo 243 (Chap. 6) giving the smallest positive solution $a = 56$ and $b = 59$. Thus, a table of Farey fractions of a given order n contains *all* integer solutions to equations like (5.67) with coefficients smaller than n.

Another, and quite recent, application of Farey series is the recovery of undersampled periodic (or nearly periodic) waveforms [5.22]. If we think of "nearly periodic waveforms" as a line-scanned television film, for example, then for most pictorial scenes there are similarities between adjacent picture elements ("pixels"), between adjacent scan lines, and between successive image frames. In other words, the images and their temporal sequence carry redundant information (exception: the proverbial "snowstorm").

Because of this redundancy, such images can, in general, be reconstructed even if the image is severely "undersampled", i. e., if only every nth pixel $(n \gg 1)$ is preserved and the others are discarded. The main problem in the reconstruction is the close approximation of the ratio of the sampling period to one of the quasi periods in the sampled information by a rational number with a given maximal denominator – precisely the problem for which Farey fractions were invented!

5.10.1 Farey Trees

While Farey sequences have many useful applications, such as classifying the rational numbers according to the magnitudes of their denominators, they suffer from a great irregularity: the number of additional fractions in going from Farey sequences of order $n-1$ to those of order n equals the highly fluctuating Euler function $\phi(n)$. A much more regular order is infused into the rational numbers by *Farey trees*, in which the number of fractions added with each generation is simply a power of 2.

Starting with two fractions, we can construct a Farey tree by repeatedly taking the mediants of all numerically adjacent fractions. For the interval $[0, 1]$, we start with $0/1$ and $1/1$ as the initial fractions, or "seeds". The first five generations of the Farey tree then appear as follows:

$$\frac{0}{1} \qquad \frac{1}{1}$$

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad \frac{2}{3}$$

$$\frac{1}{4} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{3}{4}$$

$$\frac{1}{5} \quad \frac{2}{7} \quad \frac{3}{8} \quad \frac{3}{7} \quad \frac{4}{7} \quad \frac{5}{8} \quad \frac{5}{7} \quad \frac{4}{5}$$

Each rational number between 0 and 1 occurs exactly once somewhere in the infinite Farey tree.

The location of each fraction within the tree can be specified by a binary address, in which 0 stands for moving to the left in going from level n to level $n+1$ and 1 stands for moving to the right. Thus, starting at $1/2$, the rational number $3/7$ has the binary address 011. The complement of $3/7$ with respect to 1 (i. e., $4/7$) has the complementary binary address: 100. This binary code for the rational numbers is useful in describing frequency locking in coupled oscillators.

Note that any two *numerically* adjacent fractions of the tree are unimodular. For example, for $4/7$ and $1/2$, we get $2 \cdot 4 - 1 \cdot 7 = 1$.

Some properties of the Farey tree are particularly easy to comprehend in terms of continued fractions w in the interval $[0, 1]$:

$$w = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3} \cdots}}$$

or more conveniently $w = [a_1, a_2, a_3, \ldots]$, where the "partial quotients" a_k are positive integers. Irrational w have nonterminating continued fractions. For quadratic irrational numbers the a_k will (eventually) repeat periodically. For example, $1/\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \ldots] = [1, \overline{1, 2}]$ is preperiodic and has a period of length 2; $1/\sqrt{17} = [\overline{8}]$ has period length 1 and $1/\sqrt{61}$ has period length 11. (It is tantalizing that no simple rule is known that predicts period lengths in general.) Interestingly, for any fraction on level n of the Farey tree, the sum over all its a_k equals n:

$$\sum_k a_k = n \qquad n = 2, 3, 4, \ldots.$$

We leave it to the reader to prove this equation (by a simple combinatorial argument, for example).

There is also a direct way of calculating, from each fraction on level $n-1$, its two neighbours or direct descendants on level n. First write the original fraction as a continued fraction in two different ways, which is always possible by splitting off a 1 from the final a_k. Thus, for example, $2/5 = [2, 2] = [2, 1, 1]$. Then add 1 to the last term of each continued fraction; this yields $[2, 3] = 3/7$ and $[2, 1, 2] = 3/8$, which are indeed the two descendants of $2/5$.

Conversely, the close parent of any fraction (the one on the adjacent level) is found by subtracting 1 from its last term (in the form where the last term exceeds 1, because $a_k = 0$ is an illegal entry in a continued fraction). The other (distant) parent is found by simply *omitting* the last term. Thus, the two parents of $3/7 = [2,3]$ are the close parent $[2,2] = 2/5$ and the distant parent $[2] = 1/2$. (But which parent is greater, in general – the close or the distant one? And how are mediants calculated using only continued fraction?)

Interestingly, if we zigzag down the Farey tree from its upper right ($1/1 \to 1/2 \to 2/3 \to 3/5 \to 5/8$, and so on), we land on fractions whose numerators and denominators are given by the Fibonacci numbers F_n, defined by $F_n = F_{n-1} + F_{n-2}$; $F_0 = 0$, $F_1 = 1$. In fact, on the nth zig or zag, starting at $1/1$, we reach the fraction F_{n+1}/F_{n+2}, which approaches the golden mean $\gamma = (\sqrt{5}-1)/2 = 0.618\ldots$ as $n \to \infty$. (Starting with $0/1$ we land on the fractions F_n/F_{n+2}, which converge on $\gamma^2 = 1-\gamma$.) The binary address of γ in the Farey tree is $101010\ldots$.

As already noted, the continued fraction expansions of the ratios F_n/F_{n+1} have a particularly simple form:

$$\frac{F_n}{F_{n+1}} = [1,1,\ldots,1] \quad (\text{with } n\ 1'\text{s}).$$

Obviously, continued fractions with small a_k converge relatively slowly to their final values, and continued fractions with only 1's are the most slowly converging of all. Since

$$\gamma = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = [1,1,1,\ldots] = [\overline{1}],$$

where the bar over the 1 indicates infinitely many 1's, the golden mean γ has the most slowly converging continued fraction expansion of all irrational numbers. The golden mean γ is therefore sometimes called (by physicists and their ilk) "the most irrational of all irrational numbers" – a property of γ with momentous consequences in a wide selection of problems in nonlinear physics, from the double swing to the three-body problem.

Roughly speaking, if the frequency ratio of two coupled oscillators is a rational number P/Q, then the coupling between the driving force and the "slaved" oscillator is particularly effective because of a kind of a resonance: every Q cycles of the driver, the same physical situation prevails so that energy transfer effects have a chance to build up in a resonancelike manner. This resonance effect is particularly strong if Q is a *small* integer. This is precisely what happened with our moon: resonant energy transfer between the Moon and the Earth by tidal forces slowed the Moon's spinning motion until the spin period around its own axis locked into the 28-day cycle of its revolution around the Earth. As a consequence the Moon always shows us the same face, although it wiggles ("librates") a little.

Similarly, the frequency of Mercury's spin has locked into its orbital frequency at the rational number 3/2. As a consequence, one day on Mercury lasts two Mercury *years*. (And one day – in the distant future, one hopes – something strange like that may happen to Mother Earth!)

The rings of Saturn, or rather the gaps between them, are another consequence of this resonance mechanism. The orbital periods of any material (flocks of ice and rocks) in these gaps would be in a rational resonance with some periodic force (such as the gravitational pull from one of Saturn's "shepherding" moons). As a consequence, even relatively weak forces have a cumulatively significant effect over long time intervals, accelerating any material out of the gaps.

For rational frequency ratios with large denominators Q, such a resonance effect would, of course, be relatively weak, and for *irrational* frequency ratios, resonance would be weaker still or absent.

For strong enough coupling, however, even irrational frequency ratios might be affected. But there is always one irrational frequency ratio that would be least disturbed: the golden mean, because, in a rational approximation to within a certain accuracy, it requires the largest denominators Q. This property is also reflected in the Farey tree: on each level n the two fractions with the largest denominators are the ones that equal F_{n-1}/F_{n+1} and F_n/F_{n+1}, which for $n \to \infty$ approach $\gamma^2 = 0.382\ldots$ and $\gamma = 0.618\ldots$, respectively. (Conversely, the fractions with the smallest Q on a given level of the Farey tree are from the harmonic series $1/Q$ and $1 - 1/Q$.)

Another way to demonstrate the unique position of the golden mean among all the irrational numbers is based on the theory of rational approximation, an important part of number theory. For a good rational approximation, one expands an irrational number w into a continued fraction and terminates it after n terms to yield a rational number $[a_1, a_2, \ldots, a_n] = p_n/q_n$. This rational approximation to w is in fact the best for a given maximum denominator q_n. For example, for $w = 1/\pi = [3, 7, 15, 1, 293, \ldots]$ and $n = 2$, we get $p_n/q_n = 7/22$, and there is no closer approximation to $1/\pi$ with a denominator smaller than 22.

Now, even with such an optimal approximation as afforded by continued fractions, the differences for the golden mean γ

$$\left|\gamma - \frac{p_n}{q_n}\right|$$

exceed c/q_n^2 (where c is a constant that is smaller than but arbitrarily close to $1/\sqrt{5}$) for *all* values of n above some n_0. And this is true only for the golden mean γ and the "noble numbers" (defined as irrational numbers whose continued fractions end in all 1's). Thus, in this precise sense, the golden mean (and the noble numbers) keep a greater distance from the rational numbers than does any other irrational number. Small wonder that the golden mean plays such an important role in synchronization problems.

5.10.2 Locked Pallas

On 5 May, 1812, Gauss communicated to his friend Friedrich Wilhelm Bessel (1784–1846) a strange discovery he had just made: the periods of revolution around the sun of Jupiter and Pallas are exactly in the ratio $18/7$. Gauss asked Bessel not to tell anyone else because he was afraid of being accused of *Zahlen-Mystik* (number mysticism), then, as now, rampant in astronomy. Instead (to preserve priority) he "published" his result in the *Göttingschen Gelehrten Anzeigen* (No. 67, 25 April 1812) as a cryptogram, a string of 16 0's and 1's:

$$1\,1\,1\,1\,0\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1.$$

He would divulge the key to unlock his discovery at an appropriate time, Gauss added. But he never did "divulge" and the great Gauss was rather peeved (although he did not perish) when later other astronomers did publish and claimed priority.

Knowing the encrypted message $(18/7)$ my student Inga Holube showed the way to read Gauss's cryptogram. By appropriate segmentation,

$$1\,1\,1 \quad 1\,0\,0\,0 \quad 1\,0\,0\,1\,0 \quad 1\,0\,0\,1,$$

and by interpreting the resulting snippets as binary numbers she obtained the numbers

$$7 \qquad 8 \qquad 18 \qquad 9.$$

Thus Gauss was probably saying that the period 7 belongs to the 8th planet (as Pallas was considered at the time) and the period 18 belongs to the 9th planet (Jupiter) [5.23].

Incidentally, 7 and 18 are not "any old" numbers; they are close *Lucas* numbers (see Sect. 5.8). The series of Lucas numbers are constructed like the Fibonacci series: each number is the sum of its two predecessors, but the series begins with 1 and 3 and continues 4, 7, 11, 18, ... As mentioned before (Sect. 5.8), the ratios of successive Lucas numbers have simple continued fractions and approach the golden mean. Like the Fibonacci numbers they play an important role in nonlinear dynamics and synchronization problems.

What ratio of periods would the orbits of Pallas and Jupiter lock into if the gravitational coupling between them was increased (for example by increasing the mass of Jupiter). The real experiment cannot of course be done but such questions can conveniently be studied by computer simulation.

Analytically, too, we can venture a guess. For increased coupling strengths, the period ratios are typically represented by fractions lying higher in the Farey tree (see Sect. 5.10.1), such as one of their "parent" fractions. For $7/18 = [2,1,1,3]$, the parent fractions are $[2,1,1,2] = 5/13$ and $[2,1,1] = 2/5$. Thus, the period ratio of Jupiter and Pallas might lock into $5/13$ or $2/5$.

For an even stronger gravitational coupling we have to consult the "grandparents" of $7/18$, i. e., the parents of $5/13$ and $2/5$. With our familiar algorithm for Farey families this yields $3/8$ and $2/5$ and $1/3$ and $1/2$, respectively. Curiously, the ratio $2/5$ is both parent and *grand*parent to $7/18$. Unusual relationships between humans but perhaps par for the course among the Greek gods – especially if Jupiter is part of the party.

5.11 Fibonacci and the Problem of Bank Deposits

There is an interesting family of problems, appearing in many guises, to which Fibonacci numbers provide a quick solution. Suppose Bill, a wealthy Texan chemist, opens a new bank account with x_1 dollars. The next (business) day he deposits x_2 dollars, both integer dollar amounts. Thereafter his daily deposits are always the sum of the previous two deposits. On the nth day Bill is known to have deposited x_n dollars. What were the original deposits?

A solution of this problem, posed by L. A. Monzert (cf. Martin Gardner [5.24]), argues that, for sufficiently large n, successive deposits should be in the golden ratio. This reasoning permits one to find the $(n-1)$th deposit x_{n-1} and, together with x_n, by *backward* recursion, all prior ones.

However, with the knowledge gained in this chapter, we can find a *direct* answer to this financial problem, one that is valid even for small n. Since the recursion rule for x_n is like that for the Fibonacci numbers, the x_n must be expressible as a linear combination of Fibonacci numbers. In fact, two such terms suffice:

$$x_n = a F_{n+k} + b F_{n+m}. \tag{5.68}$$

With $F_0 = 0$ and $F_{-1} = F_1 = 1$, the initial conditions are satisfied by

$$x_n = x_1 F_{n-2} + x_2 F_{n-1}. \tag{5.69}$$

Now, because of (5.27), an integer solution to (5.69) is given by

$$x_1 = (-1)^n x_n F_{n-3}, \tag{5.70}$$

$$x_2 = -(-1)^n x_n F_{n-4}.$$

Solutions (not only to mathematical problems) become that simple if one knows and *uses* the proper relations!

However, we are not quite done yet. According to (5.70) one or the other initial deposit is negative; but we want all deposits to be positive of course. Looking at (5.69) we notice that we can add to x_1 any multiple of F_{n-1} as long as we subtract the same multiple of F_{n-2} from x_2. Thus, the general solution is

$$\begin{aligned} x_1 &= (-1)^n x_n F_{n-3} + m F_{n-1}, \\ x_2 &= -(-1)^n x_n F_{n-4} - m F_{n-2}. \end{aligned} \tag{5.71}$$

We can now ask for what values (if any!) of m *both* x_1 and x_2 are positive. Or, perhaps, for what value of m x_2 is positive and as small as possible. The answer, which leads to the longest chain of deposits to reach a given x_n, is

$$m = \left\lfloor -(-1)^n x_n \frac{F_{n-4}}{F_{n-2}} \right\rfloor. \tag{5.72}$$

For $x_{20} = 1\,000\,000$ (dollars), (5.72) yields $m = -381966$ and (5.71) gives $x_1 = 154$ and $x_2 = 144$.

If we had asked that the twenty-*first* deposit be one million dollars, (5.72) would have given the same absolute value of m, and with (5.71), $x_2 = 154$ and $x_1 = -10$. In other words, we would have posed an illicit problem.

Here we have, unwittingly, solved a Diophantine equation, of which more in Chaps. 6 and 7.

5.12 Error-Free Computing

One of the overriding problems in contemporary computing is the accumulation of *rounding errors* to such a degree as to make the final result all but useless. This is particularly true if results depend on the input data in a *discontinuous* manner. Think of matrix inversion.

The inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix} \tag{5.73}$$

for $\varepsilon \neq 0$ equals

$$A^{-1} = \begin{pmatrix} 1+1/\varepsilon & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon \end{pmatrix}. \tag{5.74}$$

An important generalization of a matrix inverse, applicable also to singular matrices, is the *Moore-Penrose inverse* A^+ [5.25]. For nonsingular matrices, the Moore-Penrose inverse equals the ordinary inverse:

$$A^+ = A^{-1}. \tag{5.75}$$

As $\varepsilon \to 0$ in (5.73), the matrix A becomes singular and the Moore-Penrose inverse no longer equals A^{-1} but can be shown to be

$$A^+ = \tfrac{1}{4}A. \tag{5.76}$$

In other words, as $\varepsilon \to 0$, the elements of A^+ become larger and larger only to drop discontinuously to $1/4$ for $\varepsilon = 0$.

Examples of this kind of sensitivity to small errors abound in numerical analysis. For many computations the only legal results are integers, for example, the coefficients in chemical reaction equations. If the computation gives noninteger coefficients, their values are rounded to near integers, often suggesting impossible chemical reactions.

In some applications of this kind, double-precision arithmetic is a convenient remedy. (The author once had to invoke double precision in a very early (ca. 1959) digital filter, designed to simulate concert hall reverberation, because the sound would refuse to die away when the music stopped.) In other situations, number-theoretic transforms can be used, of which the Hadamard transform (see Chap. 18) is only one example.

However, quite general methods for error-free computing have become available in the recent past, and it is on these that we shall focus attention in this section.

Specifically, we want to sketch a strategy for computing that will not introduce *any* rounding errors whatsoever, no matter how long or complex the computation. How is this possible? Of the four basic mathematical operations, three (addition, subtraction and multiplication) are harmless: if we start with integers, we stay with integers – no rounding problems there. But *division* is a real bugbear. Computers can never represent the fraction $1/7$, for example, in the binary (or decimal system) without error, no matter how many digits are allowed. If we could only do away with division in our computations! Surprisingly, this is in fact possible, as we shall see.

Of course, computers cannot deal with continuous data – both input and output are, by necessity, rational numbers, and the rational numbers we select here to represent both input data and final results are *Farey fractions* of a given order N (see Sect. 5.10). Once we have chosen a large enough value of N to describe adequately the input data of a problem and all of the answers to that problem, then within this precision, *no* errors will be generated or accumulated.

In this application, we shall generalize our definition of Farey fractions a/b of order N, where a and b are coprime, to include negative and improper fractions:

$$0 \leq |a| \leq N, \quad 0 < |b| \leq N. \tag{5.77}$$

The error-free strategy, in its simplest form, [5.25] then proceeds as follows. A prime modulus m is selected such that

$$m \geq 2N^2 + 1, \tag{5.78}$$

and each Farey fraction a/b, with $(b,m) = 1$, is mapped into an *integer* k modulo m:

$$k = \left\langle a b^{-1} \right\rangle_m, \tag{5.79}$$

where the *integer* b^{-1} is the inverse of b modulo m and the acute brackets signify the smallest nonnegative remainder modulo m (see Sect. 1.5). It is in this manner that we have abolished division! The inverse b^{-1} can be calculated by solving the Diophantine equation

$$bx + my = 1, \tag{5.80}$$

using the Euclidean algorithm (see Sect. 7.2). The desired inverse b^{-1} is then congruent modulo m to a solution x of (5.80).

After this conversion to integers, all calculations are performed in the integers modulo m. For example, for $N = 3$ and $m = 19$, and with $3^{-1} = 13$, the fraction $2/3$ is mapped into $26 \equiv 7$, and the fraction $-1/3$ is mapped into $-13 \equiv 6$. The operation $(2/3)+(-1/3)$ is then performed as $7+6=13$, which is mapped back into $1/3$, the correct answer.

It is essential for the practical application of this method that fast algorithms be available for both the forward and backward mappings. Such algorithms, based on the Euclidean algorithm, were described by *Gregory* and *Krishnamurthy* [5.25], thereby reclaiming error-free computing from the land of pious promise for the real world.

Sometimes *intermediate* results may be in error, but with no consequence for the final result, as long as it is an order-N Farey fraction. For example, for $m=19$, $2^{-1}=10$, so that $1/2$ maps into 10, and $(1/3)-(1/2)$ maps into $13-10=3$, which is the image of 3 – an erroneous result because $-1/6$ is the correct answer! But 3 is still useful as an intermediate result. For example, multiplying 3 by 2 produces 6, which is the image of $-1/3$, the correct result.

For the large values of N that are needed in practical applications, the prime m has to be correspondingly large. Since calculating modulo very large primes is not very convenient, a multiple-modulus residue (or Chinese remainder) system, see Chap. 16, is often adopted. For example, for $N=4$, the smallest prime not smaller than $2N^2+1=33$ is 37. Instead, one can calculate with the residues modulo the *two* primes $m_1=5$ and $m_2=7$, whose product $m=m_1\cdot m_2=35$ exceeds $2N^2+1=33$. Such calculations, described in Chap. 17, are much more efficient than the corresponding operation in single-modulus systems, the savings factor being proportional to $m/\sum m_i$. For decomposition of large m into many small prime factors, the savings can be so large as to make many otherwise impossible calculations feasible.

Another preferred number system for carrying out the calculations is based on the integers modulo a prime *power*: $m=p^r$. For example, for $N=17$, the modulus m must exceed 578 and a convenient choice would be $p=5$ and $r=4$, so that m equals $5^4=625$. There is only one problem with this approach: all fractions whose denominators contain the factor 5 cannot be represented because 5 has no inverse modulo 625. However, an ingenious application of p-adic algebra and finite-length *Hensel codes* has solved the problem and looks like the wave of the future in error-free computing. We shall attempt a brief description; for details and practical applications the reader is referred to [5.25].

Essentially, what the Hensel codes do is to remove bothersome factors p in the denominators, so that the "purified" fractions do have unique inverses.

For *integers*, the p-adic Hensel codes are simply obtained by "mirroring" the p-ary expansion. With the 5-ary expansion of 14, for example,

$$14=2\cdot 5^1+4,$$

the Hensel code for 14 becomes

$$H(5,4,14)=.4200. \tag{5.81}$$

In general, $H(p,r,\alpha)$ is the Hensel code of α to the (prime) base p, having precisely r digits.

A *fraction* a/b whose denominator b does *not* contain the factor p is converted to an integer modulo p^r, which is then expressed as a Hensel code. For example, with $p^r = 5^4 = 625$, we get

$$\tfrac{1}{16} \mathrel{\hat{=}} \langle 16^{-1} \rangle_{625} = 586 = 4 \cdot 5^3 + 3 \cdot 5^2 + 2 \cdot 5 + 1,$$

or in Hensel code:

$$H\left(5, 4, \tfrac{1}{16}\right) = .1234. \tag{5.82}$$

Similarly, with $\langle 3/16 \rangle_{625} = 508 = 4 \cdot 5^3 + 1 \cdot 5 + 3$, becomes

$$H\left(5, 4, \tfrac{3}{16}\right) = .3104. \tag{5.83}$$

Of course, the Hensel code for $3/16$ can be obtained directly by multiplying (5.82) with the code for 3:

$$H(5, 4, 3) = .3000, \tag{5.84}$$

where the multiplication proceeds from *left to right*. (Remember, Hensel codes are based on a *mirrored* p-ray notation.) Thus, $H(5, 4, 1/16) \times H(5, 4, 3)$ equals

$$\begin{array}{rr} & .1234 \\ \times & .3000 \\ \hline & .3142 \\ \text{carries} & 112 \\ \hline = & .3104 \end{array}$$

which agrees with $H(5, 4, 3/16)$, see (5.83). Note that any carries beyond four digits (the digit 2 in the above example) are simply dropped. It is ironic that such a "slipshod" code is the basis of *error-free* computation!

If the numerator contains powers of p, the corresponding Hensel code is simply right-shifted, always maintaining precisely r digits. For example

$$H\left(5, 4, \tfrac{5}{16}\right) = 0.0123. \tag{5.85}$$

Powers of p in the denominator are represented by a *left*-shift. Thus, with $H(5, 4, 1/3) = .2313$,

$$H\left(5, 4, \tfrac{1}{15}\right) = 2.313. \tag{5.86}$$

To expand the range of the Hensel codes to arbitrary powers of p in the denominator or numerator, a floating-point notion, $\hat{H}(p, r, \alpha)$, is introduced. For example,

$$\hat{H}\left(5, 4, \tfrac{1}{15}\right) = (.2313, -1), \quad \text{and} \tag{5.87}$$

$$\hat{H}(5,4,375) = (.3000,3), \tag{5.88}$$

where the first number on the right is the mantissa and the second number the exponent.

When multiplying floating-point Hensel codes, their mantissas are multiplied and their exponents are added. For example,

$$\tfrac{1}{3} \times \tfrac{6}{5} \stackrel{\wedge}{=} (.2313,0) \times (.1100,-1) = (.2000,-1),$$

and with

$$.2313 \times .1100 = .2000,$$

we obtain

$$(.2313,0) \times (.1100,-1) = (.2000,-1),$$

which corresponds to $2/5$, the correct answer. (Remember, all operations proceed from left to right and Hensel code .1000 corresponds to 1 and not $1/10$.)

Strange and artificial as they are, Hensel codes perform numerical stunts and never slip a single digit.