Chapter 11
The Divisor Functions

Some numbers have few divisors, such as primes, and some numbers have many divisors, such as powers of 2. The number of divisors, although it fluctuates wildly from one integer to the next, obeys some interesting rules, and averages are quite predictable, as we shall see below.

11.1 The Number of Divisors

By definition, $\phi(n)$ is the number of integers in the range from 1 to n which have 1 as greatest common divisor (GCD) with n (see Chap. 8). To codify this statement we introduce the following notation, making use of the number sign #:

$$\phi(n) := \#\{m : 1 \leq m < n,\ (m,n) = 1\}. \tag{11.1}$$

This notation is, of course, equivalent to the more frequently encountered notation using the sum sign:

$$\phi(n) = \sum_{(m,n)=1, m<n} 1.$$

Let (cf. Chap. 2)

$$n = \prod_{p_i|n} p_i^{e_i}. \tag{11.2}$$

Then all divisors of n are of the form

$$d_k = \prod_{p_i|n} p_i^{f_i} \quad \text{with } 0 \leq f_i \leq e_i. \tag{11.3}$$

Here each exponent f_i can take on $e_i + 1$ different values. Thus the number of distinct divisors of n is given by the *divisor function* defined as

$$d(n) := \#\{d_k : d_k|n\},$$

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and equal to

$$d(n) = \prod_i (e_i + 1). \tag{11.4}$$

Example: $n = 12 = 2^2 \cdot 3^1$, $d(12) = 3 \cdot 2 = 6$.

Check: $d_k = 1, 2, 3, 4, 6, 12$, i. e., there are indeed 6 distinct divisors of 12. Check!

In some applications (11.3) is used more than once. Consider a small (elite) university with $N = 3174$ (or was it only 1734) students, subdivided into groups of equal size, each group being cared for by one of x tutors. The tutors in turn are supervised by y professors, each professor looking after the same number of tutors. It is clear that $y|x|N$. Repeated applications of (11.4) show that there are a total of 54 solutions for x and y. Even requiring that there will be more than one professor and more tutors than professors, leaves 31 possibilities. The university president, unhappy with this extravagant freedom of choice, fixed the number z of students in each tutor's group in such a (unique) way that the solution became unique. Not surprisingly, the president, a former mathematician, called this the perfect solution. (But what *is* his unique "perfect" z?)

Now take some m, $1 \le m \le n$. Its GCD with n must be one of the divisors of n:

$$(m,n) = d_k \tag{11.5}$$

for some k. How many numbers are there that share the same GCD? We shall denote the size of this family by N_k:

$$N_k := \#\{m : 1 \le m \le n,\ (m,n) = d_k\}. \tag{11.6}$$

Of course, by definition of Euler's function, for $d_k = 1$, $N_k = \phi(n)$. We can rewrite the above definition of N_k as follows:

$$N_k = \#\left\{m : 1 \le m \le n, \left(\frac{m}{d_k}, \frac{n}{d_k}\right) = 1\right\}, \tag{11.7}$$

and now we see that

$$N_k = \phi\left(\frac{n}{d_k}\right). \tag{11.8}$$

Example: for $n = 12$ and $d_k = 4$, $N_k = \phi(3) = 2$, and there are indeed precisely 2 integers in the range 1 to 12 that share the GCD 2 with 12, namely 2 and 10. Check!

Now each m, $1 \le m \le n$, must have one of the $d(n)$ distinct divisors of n as the GCD with n. Hence,

$$\sum_{k=1}^{d(n)} \phi\left(\frac{n}{d_k}\right) = n. \tag{11.9}$$

Reverting to our old notation of summing over all divisors, we may write instead

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} \phi(d) = n, \tag{11.10}$$

an interesting and important result.

The sum over all divisors of the argument of a number-theoretic function is called its *summatory* function. Thus, the summatory function of Euler's ϕ function is its argument!

Example: $n = 18$:

	Integers m for which $(m,n) = d$						Number of such integers
Divisors $d = 1$	1	5	7	11	13	17	$6 = \phi(18)$
2	2	4	8	10	14	16	$6 = \phi(18/2)$
3	3	15					$2 = \phi(18/3)$
6	6	12					$2 = \phi(18/6)$
9	9						$1 = \phi(18/9)$
18	18						$1 = \phi(18/18)$

The divisor function $d(n)$ is *multiplicative*, i. e., for coprime n and m:

$$d(nm) = d(n) \cdot d(m) \quad \text{for} \quad (n,m) = 1, \tag{11.11}$$

which follows immediately from the formula (11.2) for $d(n)$ in terms of prime exponents.

For the special case that n is the product of k distinct primes, none of which is repeated,

$$d(n = p_1 p_2 \ldots p_k) = 2^k. \tag{11.12}$$

Such n are also called *squarefree*, for obvious reasons. For example, 18 is not squarefree, but 30 is, being the product of 3 distinct primes. Thus, 30 has $2^3 = 8$ divisors.

As we saw in Sect. 4.4, the probability of a large integer being squarefree is about $6/\pi^2 \approx 0.61$. Thus, a (narrow) majority of integers are squarefree. In fact, of the 100 integers from 2 to 101, exactly 61 are squarefree. And even among the first 20 integers above 1, the proportion (0.65) is already very close to the asymptotic value. Thus, in this particular area of number theory, 20 is already a large number. [But the reader should be reminded that in other areas (cf. Chap. 4) even $10^{10\,000}$, for example, is not so terribly large.]

Using the notation for the sum over all divisors, we could have introduced $d(n)$ in the following way:

$$d(n) := \sum_{d|n} 1. \tag{11.13}$$

Thus, $d(n)$ is the summary function of n^0.

11.2 The Average of the Divisor Function

Using the Gauss bracket, there is still another way of expressing $d(n)$ which is especially suited for estimating an asymptotic average:

$$d(n) = \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right). \tag{11.14}$$

Here, if n is divisible by k, then the difference in the parentheses will be 1; otherwise it will be zero. The important point in the above expression is that it is extended over *all* k, not just the divisors of n, as in the definition of $d(n)$.

Now if we sum, we obtain

$$\sum_{n=1}^{N} d(n) = \sum_{k=1}^{N} \left\lfloor \frac{N}{k} \right\rfloor \approx N \sum_{k=1}^{N} \frac{1}{k}. \tag{11.15}$$

Of course, the estimate on the right is an upper limit, because by dropping the Gauss bracket we have increased (by less than 1) all summands for which N is not divisible by k. However, for large N, this increase should be relatively small. Hence we expect the average value to go with the sum of the reciprocal integers, i. e., the logarithm

$$\frac{1}{N} \sum_{n=1}^{N} d(n) \approx \ln N. \tag{11.16}$$

The exact results is [10.1]

$$\frac{1}{N} \sum_{n=1}^{N} d(n) = \ln N + 2\gamma - 1 + 0\left(\frac{1}{\sqrt{N}}\right). \tag{11.17}$$

Here γ is Euler's constant:

$$\gamma := \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.57721\ \ldots, \tag{11.18}$$

which makes the additive constant in (11.17) about 0.15442. The notation $0(1/\sqrt{N})$ means that the absolute error in (11.17) is smaller than $c/\sqrt{N}$, where c is some constant.

11.3 The Geometric Mean of the Divisors

There is also a nice formula for the *product* of all divisors of a given integer n. With

$$n = \prod_{p_i|n} p_i^{e_i}, \tag{11.19}$$

we have

$$\prod_{d|n} d = n^{\frac{1}{2}d(n)}. \tag{11.20}$$

Example: $n = 12$. The product of divisors equals $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 12 = 1728$. The number of divisors $d(12) = 6$ and $1728 = 12^3$. Check!

To obtain the geometric mean of the divisors of n, we have to take the $d(n)$-th root of their product, giving $\sqrt{n}$ according to (11.20).

A curious result? Not really! Divisors come in *pairs*: if d divides n, so does its distinct "mate" n/d (exception: for $n = d^2$, the mate is not distinct). And the geometric mean of each of these pairs equals $\sqrt{n}$, and so does the overall geometric mean.

11.4 The Summatory Function of the Divisor Function

The summary function

$$\sigma(n) := \sum_{d|n} d, \tag{11.21}$$

like all summatory functions of multiplicative functions, is multiplicative. Thus, it suffices to consider the problem first only for n that are powers of a single prime and then to multiply the individual results. This yields with (11.3):

$$\sigma(n) = \prod_{p_i|n} \frac{p_i^{1+e_i} - 1}{p_i - 1}. \tag{11.22}$$

The asymptotic behaviour of $\sigma(n)$ is given by [11.1]:

$$\frac{1}{N^2} \sum_{n=1}^{N} \sigma(n) = \frac{\pi^2}{12} + 0\left(\frac{\ln N}{N}\right), \tag{11.23}$$

a result that can be understood by dropping the -1 in the denominator in each term of the product (11.23) and converting it into a product over *all* primes with the proper probabilistic factors, and proceeding as in the case of $\phi(n)/n$ (see Sect. 11.6).

The above expression converges quite rapidly. For example, for $N = 5$, the result is 0.84 compared to $\pi^n/12 \approx 0.82$.

11.5 The Generalized Divisor Functions

Generalized divisor functions are defined as follows:

$$\sigma_k(n) := \sum_{d|n} d^k. \tag{11.24}$$

Of course, $\sigma_o(n) = d(n)$ and $\sigma_1(n) = \sigma(n)$. These generalized divisor functions obey a simple symmetry with respect to their index:

$$\sigma_k(n) = \sum_{d|n} d^k = \sum_{d|n} \left(\frac{d}{n}\right)^{-k} = n^k \sigma_{-k}(n). \tag{11.25}$$

Example: $n = 6$, $k = 3$. Divisors $d = 1, 2, 3, 6$.

$$1 + 8 + 27 + 216 = 216\left(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{216}\right).$$

The above symmetry relation, for $k = 1$, leads to the following relation between divisor means of n: Arithmetic mean $\overline{d}$ times harmonic mean $\hat{d}$ equals geometric mean $\tilde{d}$ squared equals n:

$$\overline{d}\hat{d} = \tilde{d}^2 = n. \tag{11.26}$$

Example: $n = 6$; $d = 1, 2, 3, 6$. $\overline{d} = 3$, $\hat{d} = 2$, $\tilde{d} = \sqrt{6}$, $3 \cdot 2 = (\sqrt{6})^2 = 6$. Check!

11.6 The Average Value of Euler's Function

Euler's ϕ function is a pretty "wild" function. For example,

$$\phi(29) = 28, \quad \phi(30) = 8, \quad \phi(31) = 30, \quad \phi(32) = 16.$$

If we are interested in the asymptotic behaviour of $\phi(n)$, we had better consider some *average* value. The following probabilistic argument will give such an average automatically, because our probabilities ignore fine-grain fluctuations such as those in the above numerical example.

Consider first

$$\frac{\phi(n)}{n} = \prod_{p_i|n} \left(1 - \frac{1}{p_i}\right). \tag{11.27}$$

Using our by now customary (but of course unproved) probabilistic argument, we convert the above product over primes that divide n into a product over *all* primes. The probability that "any old" prime will divide n equals $1/p_i$, and the probability that it will *not* equals $1 - 1/p_i$. In that case, the prime p_i "contributes" the factor 1 to the product. Thus, we may write

$$\begin{aligned}\prod_{p_i|n} \left(1 - \frac{1}{p_i}\right) &\approx \prod_{p_i} \left[\left(1 - \frac{1}{p_i}\right)\frac{1}{p_i} + 1\left(1 - \frac{1}{p_i}\right)\right] \\ &= \prod_{p_i} \left(1 - \frac{1}{p_i^2}\right),\end{aligned} \tag{11.28}$$

an infinite product that we have encountered before and which we calculated by converting the reciprocal of each factor into an infinite geometric series and then multiplying everything out. This produces every squared integer exactly once. Thus, we find

$$\text{average of } \frac{\phi(n)}{n} \approx \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{-1} = \frac{6}{\pi^2}. \tag{11.29}$$

The result corresponds to the asymptotic probability that two arbitrarily selected integers are coprime – as it should, if we remember the definition of $\phi(n)$. In fact, the number of white dots in Fig. 4.8 in a vertical line up to the 45° diagonal equals Euler's function for that coordinate.

The formal results for the asymptotic behaviour of $\phi(n)$ are as follows [11.1]:

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\phi(k)}{k} = \frac{6}{\pi^2} + 0\left(\frac{\ln n}{n}\right), \quad \text{and} \tag{11.30}$$

$$\frac{1}{n^2}\sum_{k=1}^{n} \phi(k) = \frac{3}{\pi^2} + 0\left(\frac{\ln n}{n^2}\right). \tag{11.31}$$

The result (11.30) corresponds of course to our probabilistic estimate, and the formula (11.31) is likewise unsurprising because the *average* factor inside the sum in (11.31) compared to the sum (11.30) is $n/2$.

Example: $n = 4$

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\phi(k)}{k} = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{2}{3} + \frac{1}{2}\right) = 0.667,$$

as compared to the asymptotic value 0.608. And

$$\frac{1}{n^2}\sum_{k=1}^{n} \phi(k) = \frac{1}{16}(1+1+2+2) = 0.375,$$

which also compares well with the asymptotic value (0.304).

What are the probabilities that *three* integers will not have a common divisor? And what is the probability that each of the three *pairs* that can be formed with three integers will be made up of coprime integers? The reader can find the (simple) answers or look them up in Sect. 4.4.