# What Is Mathematics? Pauli, Jung, and Contemporary Cognitive Science

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## 1 Introduction

The brilliant physicist Wolfgang Pauli was seriously involved with investigations in natural philosophy. Many (often unpublished) manuscripts and an abundant correspondence with prominent scholars of his time reveal his thoughts on causality, consciousness, the relationship between physics and psyche, and the complementarity of mind and matter, among others. His writings show that he was genuinely interested in the history of human ideas, from Western scientific thought to Eastern philosophies, to alchemy and occultism. Regarding the human mind, Pauli's philosophical investigations were deeply influenced by the work of the Swiss psychiatrist Carl G. Jung, especially through Jung's notions of "archetype" and "collective unconscious". Pauli's philosophical investigations addressed core and fundamental issues, such as the nature of scientific observation and the ontology of scientific theories. For the cutting-edge physicist that Pauli was, a natural extension to these questions would have been: What is mathematics? What is the nature of such a precise conceptual apparatus that makes modern physics possible?

To our knowledge, Pauli did not address these questions directly. He was more of a user of mathematics rather than a philosopher of mathematics or a pure mathematician. From his writings, however, it is possible to infer some aspects of Pauli's views on the nature of mathematics. Keeping in mind the focus of the academic meeting that the present volume addresses – *Pauli's philosophical ideas and contemporary science* – in this chapter I intend to accomplish three things. First, I want to analyze some of Pauli's views on the nature of mathematics mainly as seen through his analysis of Kepler's scientific theories and in his rich correspondence with Jung between 1932 and 1958. We will see that, inspired by Jung's archetypes, some of Pauli's ideas appear to be idealistic (or Platonic) in the sense that they seem to take mathematical ideas to exist somewhat independently of human beings, while others seem to defend the position that mathematical ideas are man-made. Second, from the perspective of contemporary cognitive science, I want to comment on some methodological and theoretical issues in Pauli's work, focusing on the method of introspection used by most thinkers of Pauli's time, and on the relationships between the notions of Jungian archetype and of image schema as it is used in contemporary cognitive linguistics. Finally, I will briefly describe my own approach to the question of the nature of mathematics, by looking at current work in the cognitive science of mathematics and the embodied cognition of human everyday abstraction. I will defend the argument that bodily-grounded human cognitive mechanisms underlying everyday abstraction, such as image-schemas and conceptual metaphor, play a crucial role in making mathematics possible. Mathematics, then, from elementary geometry to transfinite numbers is a biologically-grounded wonderful human creation.

## 2 Pauli, a Mathematical Platonist?

Mathematics is a very peculiar body of knowledge. On the one hand, it is an extraordinary conceptual system characterized by the fact that the very entities that constitute it are imaginary, idealized mental abstractions. These entities cannot be perceived directly through the senses. A Euclidean point, for instance, has only location but no extension(!), and, as such, it cannot be found anywhere in the entire universe. A Euclidean point cannot be actually *perceived* or observed through any scientific empirical method. Yet, the truth of many facts in Euclidean geometry depends on this essential imaginary property and cannot be demonstrated empirically (e.g., "only one line passes through two points"). And on the other hand, mathematics provides extremely stable inferential patterns (i.e., theorems) that, once proved, stayed proved forever. What is then the nature of such a unique body of knowledge?

Two main schools of thought in the philosophy of mathematics stand out: Platonism and formalism. The former, following Plato's doctrine, sees mathematical entities, their truths and properties, as atemporal and immutable, transcending the existence of human beings. The latter views these entities as reducible to pure formal properties and rule-driven manipulations of meaningless symbols. Perhaps because mathematics appears to be so pristine, precise, objective, and transcendental, many mathematicians and physicists (even today) endorse a Platonic view of mathematics.

The famous logician Kurt Gödel, for instance, was a hard-core Platonist. Even the shocking results of his "incompleteness theorems" did not change his views of mathematics. Gödel had formally proved that given an axiomatic system for arithmetic, there are true arithmetical statements that cannot be proved within that system. He took this result to solidify his philosophical position, that the ultimate truth in mathematics lies beyond mundane axiomatic systems and human mathematical practices.[1]

[1] For a summarized and non-technical analysis of Gödel's seminal work see Hintikka (2000).

One of the presenters in the meeting on which this volume is based – the French mathematician Alain Connes – holds a very similar position of mathematical Platonism. Connes believes that, at least in what concerns basic arithmetic, there is a *réalité archaïque* (archaic reality) where certain facts about numbers are true, independently of how humans do mathematics, carry formal proofs and concoct axiom systems. As Connes puts it (Connes *et al.*, 2000, p. 14–15, my translation, emphases in the original):

> "What logic brings to us, is, above all, a means of showing the limitations of the formalized axiomatic method, that is, of logical deductions within a formal system ... This intrinsic limitation leads to the separation of what is provable within a given logico-deductive system from what is true, and that I will call 'the archaic mathematical reality'. With this term, voluntarily imprecise but whose intuitive sense must be clear, I mean to encompass at least the vast continent of arithmetical truths ... In other words, the formal system that one uses will never exhaust the archaic mathematical reality."

And what about Pauli? Was he a mathematical Platonist? Certain passages of his writings suggest that he was close to a Platonic position. In the opening section of his essay on the influence of archetypal ideas on Kepler's scientific theories he writes (Pauli, 1952; translated in Pauli, 1994, p. 220):

> "What is the nature of the bridge between the sense perceptions and the concepts? All logical thinkers have arrived at the conclusion that pure logic is fundamentally incapable of constructing such a link. It seems most satisfactory to introduce at this point the postulate of a cosmic order independent of our choice and distinct from the world of phenomena."

Although Pauli in these passages does not directly refer to mathematics, he wonders about the nature of concepts and their relation to sense perceptions. And in order to deal with such questions he dismisses pure logic as a candidate and postulates a "cosmic order" that is independent of human beings and, most importantly, distinct from world facts. This is a kind of reality that has an ontology separate from the "world of phenomena" and transcends the existence of human beings. He then further explicates this view by referring to Plato himself, and by citing Kepler as an important figure who endorsed such a view (Pauli, 1952; translated in Pauli, 1994, p. 221):

> "The process of understanding nature as well as the happiness that man feels in understanding, that is, in the conscious realization of new knowledge, seems thus to be based on a correspondence, a 'matching' of inner images *pre-existent* in the human psyche with external objects and their behavior. This interpretation of scientific knowledge, of course, goes back to Plato, and is, as we shall see, very clearly advocated by Kepler."

Pauli was well aware that the question of the nature of the "matching" between human pre-existing inner images with objects in the external world had been at the core of philosophy of mind and scientific psychology for more than a century. In this passage he presents the issue through Kepler's eyes, explaining how he straightforwardly solved the question by invoking God and creation dogmas of Christianity (Pauli, 1952; translated in Pauli, 1994, p. 221, emphasis in the original):

> "He [Kepler] speaks in fact of ideas that are pre-existent in the mind of God and were implanted in the soul, the image of God, at the time of creation. These primary images which the soul can perceive with the aid of an innate 'instinct' are called by Kepler archetypal ('archetypalis'). Their agreement with their 'primordial images' or *archetypes* introduced into modern psychology by C.G. Jung and functioning as 'instincts of imagination' is very extensive."

This passage is quite telling. Although Pauli refers to views that Kepler had expressed more than three centuries earlier, he manages to introduce the crucial Keplerian notion of "archetypes", but this time cautiously backed-up with Jung's work in "modern psychology", which at the time of Pauli was considered to be an expression of cutting-edge empirical investigation of the mind. Now, supported by Jung's empirical psychology, and getting away from dogmatic theological arguments, Pauli moves on to close his opening section by explaining how human ideas – including scientific ones – evolve (Pauli, 1952; translated in Pauli, 1994, p. 221):

> "As ordering operators and image-formers in this world of symbolical images, the archetypes thus function as the sought-for-bridge between the sense perceptions and the ideas and are, accordingly, a necessary presupposition even for evolving a scientific theory of nature."

In this opening section, Pauli says nothing about mathematics proper, but he provides the guidelines for the essential building blocks underlying scientific discovery, namely, a process that builds on archetypes and serves as the bridge between sense perceptions and the world of ideas. The semantic content of archetypes, thus, is seen as somewhat independent of human psychological activity, that is, they reside outside of the mind. Therefore they seem to be in line with platonic thought. Such a view shows up in other places in Pauli's writings. For example, in a letter of January 7, 1948, to Fierz, Pauli writes (Meyenn, 1993, pp. 496–497):[2]

> "*The ordering and regulating factors must be placed beyond the distinction of 'physical' and 'psychic'* – as Plato's 'ideas' share the notion of a concept and of a force of nature (they create actions out of themselves). I am very much in favor of referring to the 'ordering' and 'regulating' factors in terms of 'archetypes'; but then it would be inadmissible to *define* them as contents of the *psyche*. The mentioned inner images ('dominant features of the collective unconscious' after Jung) are rather *psychic* manifestations of the archetypes which, *however*, would *also* have to put forth, create, condition anything lawlike in the behavior of the corporeal world. The laws of this world would then be the *physical manifestations of the archetypes*. ... *Each* law of nature should then have an inner correspondence and vice versa, even though this is not always directly visible today."

[2] I want to thank Harald Atmanspacher for pointing me to this quote.

Here Pauli states, in strong terms, that archetypes are not contents of the psyche but, rather, the inner images are psychic manifestations of them. A particular and fairly simple area of mathematics, where the relationship between the physical and the psychical can be studied under the concept of archetypes, is that of numbers. In a letter of October 24, 1953, to Pauli, Jung states that the natural numbers are the simplest of all archetypes (translation in Meier, 2001, p. 127, emphasis in the original):

> "These [the natural numbers] seem to be the simplest and most elementary of all archetypes. That they are archetypes emerges from the psychological fact that simple whole *numbers*, given the chance, amplify themselves immediately and freely through *mythological statements*; e.g. 1 = the One, absolute, nondivisible ... and thus the unconscious, the beginning, God, etc. 2 = the division of the One, the pair, the connection, the difference (agens-patiens, masculine-feminine, etc.), counting, etc. 3 = the renaissance of the One from the Two, the son, the first masculine number, etc."

Pauli, who in is early education had been in touch with the Pythagorean view that man is able to contemplate the numerical proportions of nature thanks to the inherent sense of harmony and beauty of the soul (Gieser, 2005), seems to have accepted this view. He saw mathematics as based on the archetype of numbers, and as a genuine symbolic description of reality, to the point that it can also express mental processes – including dreams – in detail (Gieser, 2005, pp. 309–310). In his letters to Jung, Pauli often describes and analyzes his own dreams in terms of archetypes and numbers, sometimes expressed in quaternarian and trinitarian structures.[3] And in other texts, such as in his "background physics" (Meier, 2001, p. 179–196) he analyzes his dreams by invoking the well-defined imaginary unit $i = \sqrt{-1}$ as a symbol not contained in the real numbers. Pauli interprets it as having the function to unite a pair of opposites and thus produce wholeness. For Pauli, mathematical representations were indeed symbolic descriptions *par excellence* (Meier, 2001, p. 195), but the mathematical entities themselves existed outside the human mind.

But there is more. Pauli was also aware that, beyond the realm of numbers, many areas of mathematics seem to be humanly developed. In his essay on Kepler's work, for instance, he is very cautious in not blindly embarking in a fully timeless, idealistic, and absolute view of mathematics. He writes (Pauli, 1952; translated in Pauli, 1994, p. 229):

> "When *Kepler* says, however, that in the Mind of God it has been eternally true that, for example, the square of the side of a square equals half the square of its diagonal, we do not, to be sure, begrudge one of the first joyful discoverers of quantitative, mathematically formulated natural laws his elation but must, as modern men, remark in criticism that the axioms of Euclidean geometry are not the only possible ones. ... I entirely share the

[3] This is a rich topic whose proper treatment goes beyond the scope of this chapter.

> opinion that man has an instinctive tendency, not rooted merely in external experience, to interpret his sensory perceptions in terms of Euclidean geometry. It took a special intellectual effort to recognize the fact that the assumptions of Euclidean geometry are not the only possible ones."

Here we see that Pauli does not ascribe the same "Platonic" status to Euclidean geometry (with its Platonic solids and so on) and to other forms of geometry. He clearly states that other geometries are indeed possible, and specifically points out that they are made possible by human intellectual effort. From this perspective then, according to Pauli, not *all* of mathematics would pre-exist human beings. Some domains of mathematics would be the result of the activity of the human mind. A view along these lines can also be seen in the letter of December, 12, 1950, of Pauli to Jung (translated in Meier, 2001, p. 64) in which he mentions issues regarding the foundations of mathematics, a domain of basic research in mathematics that was very active throughout Pauli's life:

> "It should be noted that the specialized field 'Fundamentals of Mathematics' is in a state of great confusion at the moment as a result of a large-scale undertaking to deal with these questions, an endeavor that failed because it was one-sided and divorced from nature. In this field of research into the fundamentals of mathematics, the 'basis of mathematical probability calculus' marks a particular low point. ... A psychological approach would be both appropriate and very useful here."

In this passage Pauli is most likely criticizing the excessive meaningless formalisms ("divorced from nature") that drove most efforts for settling the foundations of mathematics during the 20th century, and calls for an approach that brings in the richness of the human mind. Pauli's view is certainly far from the mainstream set-theoretical approaches that were *à la mode* at that time. It was much closer to Poincaré's views that saw – unlike the analytical philosophy of Frege or Russell – a strong connection between epistemology and psychology.

So, was Pauli a mathematical Platonist ? Based on the documents we have, it appears that there is no straightforward answer to the question. Or at least, no simple answer that would apply to all of mathematics. Perhaps, Pauli had a view along the lines of Gödel or Connes, that sees most mathematical practices as human – creating axiomatic systems and formal definitions, conceiving symbols and formal proofs – but in what concerns the domain of natural numbers and simple arithmetic seeing an ultimate realm of mathematical truths transcending the human mind. Perhaps Pauli, following Jung, did see something unique in whole numbers. In a letter to Pauli of October 24, 1953 (Meier, 2001, p. 127), Jung wrote that they

> "possess that characteristic of the psychoid archetype in classical form – namely, that *they are as much inside as outside*. Thus, one can never make out whether they have been *devised* or *discovered*; as numbers they are *inside* and as quantity they are *outside*."

## 3 Introspection and Archetypes: A Cognitive Science Perspective

In this section I would like to briefly comment – from the perspective of contemporary cognitive science – on two aspects of Pauli's ideas about mathematics: (1) the method of introspection that he used, and (2) the notion of archetype and its relation to image schemas.

### 3.1 Introspection as a Method of Investigation

In their investigations about the nature of ideas and the properties of the mind, scholars of the time of Pauli and Jung approached these issues heavily relying on the method of *introspection.* They gained insight into the functioning of the mind through the *conscious* examination of their own thoughts, perception, and intuition. Since the time of Greek philosophers, introspection has played a major role in the study of the human mind. Introspection, after all, is a readily available method of investigation, practical and instantaneous, that does not require sophisticated equipment or training. In the philosophy of mathematics, various influential mathematicians of the late 19th and early 20th centuries, such as Richard Dedekind, Georg Cantor, David Hilbert, Henri Poincaré, and Hermann Weyl, developed their philosophical work mainly using introspection as a method of inquiry. They all considered, in one way or another, human intuition as a fundamental starting point for their philosophical investigations: intuitions of small integers, intuitions of collections, intuitions of movement in space, and so on (see Dedekind, 1888; Dauben, 1979, on Cantor; Kitcher, 1976, on Hilbert; Poincaré, 1913; Weyl, 1918). They regarded these fundamental intuitions of the human mind as stable and profound enough to serve as a basis for mathematics.[4] Pauli was aware of their work, and he was especially tuned into Weyl's and Poincaré's philosophical viewpoints (Gieser, 2005).

Pauli's philosophical insights, as well as those from these mathematicians, give us many important elements regarding the personal impressions these scholars had about the nature of mathematics – from the qualitative impressions of having a mathematical insight, to the description of the structure of basic intuitions, to the organization of dreams. But beyond the philosophical and historical interest that these insights may have, they present important limitations when seen from the perspective of nowadays' scientific standards.

[4] However, they did not think of these intuitions and basic ideas as being "rigorous" enough. This was a major reason why, later, formalism would explicitly eliminate ideas, and go on to dominate the foundational debates. Unfortunately, at that time philosophers and mathematicians did not have the scientific and theoretical tools we have today to see that human intuitions and ideas are indeed very precise and rigorous, and that therefore the problems they were facing did not have to do with a lack of rigor of ideas and intuitions. For details see Núñez and Lakoff (1998) and Lakoff and Núñez (2000).

- First, mathematicians of the time of Pauli were professionally trained to do mathematics, not to study ideas and intuitions. And their discipline, mathematics (as such), does not study ideas or intuitions. Today, the study of ideas (concepts and intuitions) in itself is a scientific subject matter, and it is not the vague and elusive philosophical object that it was at the time of Pauli. We will come back to this point in the next section.
- Second, as pointed out above, the methodology they used was mainly *introspection* – the subjective investigation of one's own impressions, feelings, and thoughts. Now we know from substantial evidence in the scientific study of intuition and cognition, that there are fundamental aspects of mental activity that are unconscious in nature and therefore inaccessible to introspection.

Indeed, thanks to the scientific investigation of the human mind, today we know that the method of introspection not only is highly unreliable but also extremely limited in scope. In terms of time scales, introspection as such requires the integration of many cognitive functions at once – attention, perception, memory, and even language – which occur at the time scale of several hundred milliseconds, seconds and minutes. This means that introspection is unable to see anything that occurs below that time scale (e.g., in a few tens of milliseconds), thus missing essential mental processes that take place within those short time frames.

Then there is the neural dynamics underlying attention, perception, memory and so on. These neural dynamics cannot be perceived directly via introspection. We simply cannot say anything about the underlying neural dynamics involved in, say, the recognition of the face of an old friend. We may have impressions and thoughts about it, but via introspection we are completely blind to the properties of the neural dynamics that make face recognition possible. Regarding memory, study after study shows that what we remember is highly unreliable, and therefore introspection applied to memories is likely to be biased by the unreliable nature of memory (see Schacter, 1996).

And then, there is the huge amount of phenomena that co-occurs with mental activity but that is outside of conscious awareness (required for introspection to take place), such as eye saccades and speech-gesture coordination, which modern cognitive science recognizes as important indicators of human thinking in real-time (McNeill, 1992; Núñez, 2006).

In sum, philosophical inquiry based mainly on introspection – although very important – gives, at best, a very limited and often biased picture of the conceptual structure that makes mathematics possible. If we want to address the question of the nature of mathematics, introspection is not the right method to do so. This applies to Pauli's (and Jung's) philosophical work.

### 3.2 Archetypes and Image Schemas

Pauli often cited Jung's archetypes in his writings. The following are examples of Jung's views from the 1920s and 1930s:

- "The primordeal image, elsewhere also termed *archetype*, is always collective, i.e., it is at least common to entire peoples or epochs ... [It] is the precursor of the *idea* and its matrix." (Original 1921, translation from Jung, 1971, §747 and §750.)
- "Archetypes are typical modes of apprehension." (Original 1919, translation from Jung, 1969, §280.)
- "The archetypal motifs presumably derive from patterns of the human mind that are transmitted not only by tradition and migration but also by heredity. The latter hypothesis is indispensable, since even complicated archetypal images can be reproduced spontaneously without there being any possibility of direct tradition." (Original 1937, translation from Jung, 1958, §88.)
- "I suppose ... the inherited quality to be something like the formal possibility of producing the same ideas over and over again. I have called this the 'archetype'. Accordingly, the archetype would be a structural quality or condition peculiar to a psyche that is somehow connected with the brain." (Original 1937, translation from Jung, 1958, §165.)

The notion of archetype is deep. It could have had a much greater impact in the study of the human mind of the 20th century if it had not been so difficult to investigate it empirically. The psychology of the 1950s and 1960s, dominated by behaviorism, and that of the 1970s and 1980s, dominated by the information-processing paradigm, simply did not have room for archetypes – a notion too difficult to operationalize and to encompass within strictly individualistic rule-driven views of the human mind. Interestingly, however, certain important aspects of the notion of archetypes as described by the above citations resonate in contemporary cognitive semantics, especially in what concerns the notion of *image schemas*.

Image schemas constitute an important finding in contemporary cognitive linguistics, showing that human conceptual systems can be ultimately decomposed into primitive concepts of spatial relations. Image schemas are basic dynamic topological and orientation structures that characterize spatial inferences and link language to visual-motor experience (Johnson, 1987; Lakoff and Johnson, 1999). Image schemas, like archetypes, have a specific "structural quality" that, as Jung put it, is "somehow connected with the brain" since they appear to be realized neurally, using brain mechanisms such as topographic maps of the visual field, center-surround receptive fields, gating circuitry, and so on (Regier, 1996). Moreover, they are quite close to Jung's original idea of a "primordeal image", the "precursor" of an idea and its "matrix", with a "collective" nature.

Image schemas can be studied empirically through language (and spontaneous gestures), in particular through the linguistic manifestation of spatial relations. Every language has a system of spatial relations, though they differ radically from language to language. In English, for instance, there are prepositions like in, on, through, above, and so on. Other languages have systems that often differ radically from the English system. However, the spatial relations in a given language decompose into conceptual primitives (image schemas) that appear to be universal, that is, they are "typical modes of apprehension" that are "common to entire peoples or epochs", very much like Jung's archetypes. For example, the English word "on", in the sense used in "the book is on the desk" is a composite of three primitive image schemas:

- the *Above Schema* (the book is *above* the desk),
- the *Contact Schema* (the book is *in contact* with the desk),
- the *Support Schema* (the book is *supported* by the desk).

The *Above Schema* is orientational: It specifies an orientation in space relative to the gravitational pull one feels on one's body. The *Contact Schema* is one of a number of topological schemas: It indicates an absence of a gap. The *Support Schema* is force-dynamic in nature: It indicates the direction and nature of a force. In general, static image schemas fall into one of these categories: orientational, topological, and force-dynamic. In other languages, the primitives may combine in very different ways. Not all languages have a single concept like *on* in English. For instance, even in a language as close as German, the *on* in *on* the table is rendered as *auf*, while the *on* in *on the wall* (which does not contain the Above Schema) is translated as *an*.

A common image schema that is of great importance in mathematics is the *Container Schema* (Lakoff and Núñez, 2000), which in everyday cognition occurs as the central part of the meaning of words like *in* and *out*. The Container Schema has three parts: an Interior, a Boundary, and an Exterior. This structure forms a Gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside. This structure is topological in the sense that the boundary can be made larger, smaller, or distorted and still remains the boundary of a Container Schema.

Image schemas have a special cognitive function: they are both *per*ceptual and *con*ceptual in nature. As such, they provide a bridge between language and reasoning on the one hand and vision on the other. Image schemas can fit visual perception, as when we see the milk as being *in* the glass. They can also be imposed on visual scenes, as when we see the bees swarming *in* the garden, where there is no physical container that the bees are in. Because terms of spatial relations in a given language name complex image schemas, image schemas are the link between language and spatial perception, forming, like Jung's archetypes, "patterns of the human mind that are transmitted not only by tradition and migration but also by heredity".

As we will see in the next section, an extremely important feature of image schemas is that their *inferential structure* is *preserved* under metaphorical mappings. This feature will turn out to be a crucial component that helps bringing mathematical ideas into being.

## 4 Mathematics as a Product of the Embodied Human Mind

As we saw earlier, mathematics is a peculiar body of knowledge, whose objects are idealized imaginary entities. Beyond the Euclidean point, we can see the imaginary (but precise nature) of mathematics even clearer if we look at infinity where, because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself. Yet, infinity is at the core of mathematics. It lies at the very basis of many fundamental concepts such as limits, least upper bounds, point-set topology, mathematical induction, infinite sets, points at infinity in projective geometry, to mention only a few.

If mathematics is the product of human imagination, how can we explain the nature of mathematics with its unique features such as precision, objectivity, rigor, generalizability, stability, and, of course, applicability to the real world ? How can we give a cognitive account of what mathematics is, with all the precision and complexities of its theorems, axioms, formal definitions, and proofs ? And how can we do this when the subject matter is truly abstract and apparently detached from anything concrete, as in topics as transfinite numbers, abstract algebra, and hyperset theory ?

In the realm of Platonically oriented philosophies, like Gödel's or Connes', the question of the nature of mathematics does not pose a real problem, since the existence of mathematical ideas transcends the world of human ideas. This view, of course, cannot be tested scientifically and does not provide any link to current empirical work on human ideas and conceptual systems. In such Platonic views issues and questions are a matter of *faith*, not of empirical investigation. The question of the nature of mathematics does not pose major problems to purely formalist philosophies either, because in that worldview mathematics is seen as a manipulation of meaningless symbols. The question of the origin of the meaning of mathematical ideas does not even emerge in the formalist world.

In any case, any precise explanatory proposal of the nature of mathematics should give an account of the unique collection of features that make mathematics so special: precision, objectivity, rigor, generalizability, stability, and, applicability to the real world. This is what makes the scientific study of the nature of mathematics so challenging: Mathematical entities (organized ideas and stable concepts) are abstract and imaginary, yet they are realized through the biological and social peculiarities of the human animal. The challenge then is: How can a bodily-grounded view of the mind give an account of an abstract, idealized, precise, sophisticated and powerful domain of ideas if direct bodily experience with the subject matter is not possible?

In our book *Where Mathematics Comes From*, George Lakoff and I propose some preliminary answers to such questions (Lakoff and Núñez, 2000). Building on findings in mathematical cognition and the neuroscience of numerical cognition, and using mainly methods from cognitive linguistics, a branch of cognitive science, we asked: Which cognitive mechanisms are used in structuring mathematical ideas? And more specifically, which cognitive mechanisms can characterize the inferential organization observed in mathematical ideas themselves?

We suggested that most of the idealized abstract technical entities in mathematics are created via everyday human cognitive mechanisms that extend the structure of bodily experience while preserving inferential organization. Such "natural" mechanisms are, among others, image schemas and conceptual metaphors (Lakoff and Johnson, 1980; Sweetser, 1990; Lakoff, 1993; Lakoff and Núñez, 1997; Núñez and Lakoff, 2005), conceptual blends (Fauconnier and Turner, 1998, 2002; Núñez, 2005), conceptual metonymy (Lakoff and Johnson, 1980), and fictive motion (Talmy, 1988, 2003). Using a technique we called *mathematical idea analysis* we studied in detail many mathematical concepts in several areas of mathematics, from set theory to infinitesimal calculus to transfinite arithmetic. We showed how, via everyday human embodied mechanisms such as image schemas, conceptual metaphor and conceptual blending, the inferential patterns drawn from direct bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains. In order to see how this works, let us now take a closer look into the study of everyday conceptual mappings and inferential organization.

### 4.1 Conceptual Mappings and Inferential Organization

Consider the following two everyday linguistic expressions: "The spring is *ahead* of us" and "the presidential election is now *behind* us". Taken literally, these expressions do not make any sense. "The spring" is not something that can physically be "ahead" of us in any measurable or observable way, and an "election" is not something that can be physically "behind" us. Hundreds of thousands of these expressions, whose meaning is not literal but *metaphorical*, can be observed in human everyday language. They are the product of the human imagination, they convey precise meanings, and allow speakers to make precise inferences about them.

A branch of cognitive science, cognitive linguistics (and more specifically, cognitive semantics), has studied this phenomenon in detail and has shown that the semantics of these hundreds of thousands metaphorical linguistic expressions can be modeled by a relatively small number of *conceptual metaphors* (Lakoff and Johnson, 1980; Lakoff, 1993). These conceptual metaphors, which are inference-preserving cross-domain mappings, are cognitive mechanisms that allow us to project the inferential structure from a *source domain*, which usually is grounded in some form of basic bodily experience, into another one, the *target domain*, usually more abstract. A crucial component of what is modeled is inferential organization, the network of inferences that is generated via the mappings.

The above examples use quite different lexical items (i.e., one refers to a location *ahead of us*, and the other to a location *behind us*), but they are both linguistic manifestations of a single general conceptual metaphor, namely, TIME EVENTS ARE THINGS IN UNIDIMENSIONAL SPACE.[5] As in any conceptual metaphor, the inferential structure of concepts in the target domain (time, in this case) is created via a precise mapping drawn from the source domain (unidimensional space, in this case). In what concerns time expressions, for instance, cognitive linguists have identified two main forms of this general conceptual metaphor, namely, TIME PASSING IS MOTION OF AN OBJECT (which models the inferential organization of expressions such as "Chiristmas is *coming*") and TIME PASSING IS MOTION OVER A LANDSCAPE (which models the inferential organization of expressions such as "we are *approaching* the end of the month") (Lakoff, 1993).[6] The former model has a fixed canonical observer where times are seen as entities moving with respect to the observer, while the latter has times as fixed objects where the observer moves with respect to events in time.

These two forms share some fundamental features: both map (preserving transitivity) spatial locations in front of ego onto temporal events in the future, co-locations with ego onto events in the present, and locations behind ego (also preserving transitivity) onto events in the past. Spatial construals of time are, of course, much more complex, but this is basically all what we need to know here. For the purposes of this chapter, there are two very important morals to keep in mind:

a) *Truth*, when imaginary entities are concerned, is always relative to the inferential organization of the mappings involved in the underlying conceptual metaphors. For instance, "last summer" can be conceptualized as being *behind us* as long as we operate with the general conceptual metaphor TIME EVENTS ARE THINGS IN UNIDIMENSIONAL SPACE, which determines a specific bodily orientation with respect to metaphorically conceived events in time, namely, the future as being "in front of" us, and the past as being "behind" us. Núñez and Sweetser (2006), however, have shown that the details of that mapping are not universal. Through ethnographic field work, as well as cross-linguistic gestural and lexical analysis of the Aymara language of the

[5] Following a convention in cognitive linguistics, capitals here serve to denote the name of the conceptual mapping as such. Particular instances of these mappings, called metaphorical expressions (e.g., "she has a great future in front of her"), are not written with capitals.

[6] For a different and more recent taxonomy based on linguistic data, as well as on gestural and psychological experimental evidence, see Núñez and Sweetser (2006) and Núñez *et al.* (2006).

Andes' highlands, they provided the first well-documented case violating the postulated universality of the metaphorical orientation future-in-front-of-ego and past-behind-ego. In Aymara, for instance, "last summer" is conceptualized as being *in front of* ego, not *behind* of ego, and "next year" is not conceptualized as being *in front of* ego, but *behind* ego. Moreover, Aymara speakers not only utter these words when referring to time, but also produce co-timed corresponding gestures, strongly suggesting that these metaphorical spatial construals of time are not merely about words, but about deeper conceptual phenomena. The moral is that there is no *ultimate truth* regarding these imaginative structures. In this case, there is no ultimate truth about where, really, is the ultimate metaphorical location of the future (or the past). Truth will depend on the details of the mappings of the underlying conceptual metaphor. As we will see, this is of paramount importance when mathematical concepts are concerned: Their ultimate truth is not hidden in the structure of the universe, but it will be relative to the underlying conceptual mappings (e.g., metaphors) used to create them.

b) It is crucial to keep in mind that the abstract conceptual systems we develop are possible *because* we are biological beings with specific morphological and anatomical features. In this sense, human abstraction is *embodied* in nature. It is because we are living creatures with a salient and unambiguous front and a back, that we can build on these properties and the related bodily experiences to bring forth stable and solid concepts such as "the future in front of us". This would be impossible if we had the body of a jellyfish or of an amoeba. Moreover, abstract conceptual systems are not "simply" socially constructed, as a matter of convention. Biological properties and specificities of human bodily-grounded experience impose very strong constraints on what concepts can be created. While social conventions usually have a huge number of degrees of freedom, many human abstract concepts do not. For example, the color pattern of the Euro bills was socially constructed via convention (and so were the design patterns they have). But virtually any color ordering would have done the job. Metaphorical construals of time, on the contrary, are *only* based on a spatial source domain. This is an *empirical* observation, not an arbitrary or speculative statement: As far as we know, there is no language or culture on earth where time is conceived in terms of thermic or chromatic source domains. And there is more: not just any spatial domain does the job. Spatial construals of time are, as far as we know, always based on unidimensional space.[7] Human abstraction is thus not merely "socially constructed". It is constructed through strong non-arbitrary biological and cognitive constraints that play an essential role in constituting what human abstraction is. Human cognition is *embodied*, shaped by species-specific non-arbitrary con-

[7] Although they can, of course, be more complicated, e.g. in the case of cyclic or helix-like conceptions. But even in those cases the building blocks – a segment of a circle or a helix – preserve the topological properties of the uni-dimensional segment.

straints. This property is of key importance when mathematical concepts are concerned.

We are now in a position to analyze how the inferential structure of image schemas (for example, the Container Schema) is preserved under metaphorical mappings like the ones just described to generate more abstract concepts (such as the concept of Boolean class). We shall see exactly how image schemas provide the inferential structure to the source domain of the conceptual metaphor which, via the mapping, is projected onto the target domain of the metaphor to generate sophisticated mathematical concepts, in this case, Boolean-class inferences.

### 4.2 Structure of Image Schemas and Metaphorical Projections

When we draw illustrations of Container Schemas, we find that they look like Venn diagrams for Boolean classes. This is by no means an accident. The reason is that classes are normally conceptualized in terms of Container Schemas. For instance, we think (and speak) of elements as being *in* or *out* of a class. Venn diagrams are visual instantiations of Container Schemas. The reason that Venn diagrams work as symbolizations of classes is that classes are usually metaphorically conceptualized as containers – that is, as bounded regions in space.

Container Schemas have a logic that appears to arise from the structure of our visual and imaging system, adapted for more general use. More specifically, Container Schemas appear to be realized neurally using such brain mechanisms as topographic maps of the visual field, center-surround receptive fields, and gating circuitry (Regier, 1996). The inferential structure of these schemas can be used both for structuring space and for more abstract reason, and is projected onto our everyday conceptual system by a particular conceptual metaphor, the Classes Are Containers metaphor. This accounts for part (by no means all!) of our reasoning about conceptual categories. Boolean logic also arises from our capacity to perceive the world in terms of Container Schemas and to form mental images using them.

So, how do we normally conceptualize the intuitive pre-mathematical notion of classes ? From the perspective of mathematical idea analysis the answer is in terms of Container Schemas. In other words, we normally conceptualize a class of entities in terms of a bounded region of space, with members of the class all *inside* the bounded region and non-members outside of the bounded region. From a cognitive perspective, intuitive classes are thus metaphorical conceptual containers, characterized cognitively by a metaphorical mapping – the Classes Are Containers metaphor. Table 1 shows the corresponding mappings. This is our natural, everyday unconscious conceptual metaphor for what a class is. It grounds our concept of a class in our concept of a bounded region in space, via the conceptual apparatus of the image schema for containment. This is the way we conceptualize classes in everyday life.

| Source Domain<br>Container Schemas | | Target Domain<br>Classes |
|---|---|---|
| interiors of container schemas | → | classes |
| objects in interiors | → | class members |
| being an object in an interior | → | the membership relation |
| an interior of one container schema within a larger one | → | a subclass in a larger class |
| the overlap of the interiors of two container schemas | → | the intersection of two classes |
| the totality of the interiors of two container schemas | → | the union of two classes |
| the exterior of a container schemas | → | the complement of a class |

**Tab. 1.** The metaphor CLASSES ARE CONTAINERS

We can now analyze how conceptual image schemas (in this case, Container Schemas) are the source of four fundamental inferential laws of logic. The structural constraints on Container Schemas mentioned earlier (i.e., brain mechanisms such as topographic maps of the visual field, center-surround receptive fields, gating circuitry, etc.) give them an inferential structure, which Lakoff and I called "Laws of Container Schemas" (Lakoff and Núñez, 2000). These so-called "laws" are conceptual in nature and are reflections at the cognitive level of brain structures at the neural level (see Figure 1). The four inferential laws are Container Schema versions of classical logical laws:

- *Excluded Middle.* Every object $X$ is either *in* Container Schema $A$ or *outside of* Container Schema $A$.
- *Modus Ponens*: Given two Container Schemas $A$ and $B$ and an object $X$, if $A$ is *in* $B$ and $X$ is *in* $A$, then $X$ is *in* $B$.
- *Hypothetical Syllogism*: Given three Container Schemas $A$, $B$ and $C$, if $A$ is *in* $B$ and $B$ is *in* $C$, then $A$ is *in* $C$.
- *Modus Tollens*: Given two Container Schemas $A$ and $B$ and an object $Y$, if $A$ is *in* $B$ and $Y$ is *outside of* $B$, then $Y$ is *outside of* $A$.

Now, recall that conceptual metaphors allow the inferential structure of the source domain to be used to structure the target domain. So, the CLASSES ARE CONTAINERS metaphor maps the inferential laws given above for embodied Container Schemas (source domain) onto conceptual classes (target domain). These include both everyday classes and Boolean classes, which are metaphorical extensions of everyday classes. The entailment of such conceptual mapping is the following:

- *Excluded Middle.* Every element $X$ is either a *member of* class $A$ or *not a member of* class $A$.
- *Modus Ponens:* Given two classes $A$ and $B$ and an element $X$, if $A$ is a *subclass* of $B$ and $X$ is a *member* of $A$, then $X$ is a *member* of $B$.
- *Hypothetical Syllogism:* Given three classes $A$, $B$, and $C$, if $A$ is a *subclass of* $B$ and $B$ is a *subclass of* $C$, then $A$ is a *subclass of* $C$.
- *Modus Tollens:* Given two classes $A$ and $B$ and an element $Y$, if $A$ is a *subclass of* $B$ and $Y$ is *not a member* of $B$, then $Y$ is *not a member of* $A$.

The moral is that these traditional laws of logic are in fact cognitive entities and, as such, grounded in the neural structures that characterize Container Schemas. In other words, these laws are part of our bodies. Since they do not transcend our bodies, they are not laws of any transcendent reason. The truths of these traditional laws of logic are thus not dogmatic. They are true by virtue of what they mean.

## 4.3 Are Hypersets Sets ?

Let us close this chapter by asking the following question in modern mathematics: Are hypersets sets ? If not, what are they ? We will see that the answer to these questions shows that mathematics is made possible by the embodied mechanisms of human imagination, such as image schemas and conceptual metaphor. Let us begin with the question: What are sets ? On the formalist


**Fig. 1.** The "laws" of cognitive Container Schemas. The figure shows one cognitive Container Schema, $A$, occurring inside another, $B$. By inspection, one can see that, if $X$ is in $A$, then $X$ is in $B$, and that, if $Y$ is outside of $B$, then $Y$ is outside of $A$. We conceptualize physical containers in terms of cognitive containers. Cognitive Container Schemas are used not only in perception and imagination but also in conceptualization, as when we conceptualize bees as swarming *in* the garden. Container Schemas are the cognitive structures that allow us to make sense of familiar Venn diagrams.

view of the axiomatic method, a "set" is any mathematical structure that "satisfies" the axioms of set theory as written in symbols. The traditional axioms for set theory (the Zermelo-Fraenkel axioms) are often taught as being about sets conceptualized as containers. Many writers speak of sets as "containing" their members, and most students think of them that way. Even the choice of the word "member" suggests such a reading, as do the Venn diagrams used to introduce the subject. But if you look carefully through those axioms, you will find nothing in them that characterizes a container. The terms "set" and "member of" are both taken as undefined primitives. In formal mathematics, that means that they can be anything that fits the axioms. Here are the classic Zermelo-Fraenkel axioms including the axiom of choice, commonly called the ZFC axioms.

- *The axiom of extension:* Two sets are equal if and only if they have the same members. In other words, a set is uniquely determined by its members.
- *The axiom of specification:* Given a set $A$ and a one-place predicate $P(x)$ that is either true or false for each member of $A$, there exists a subset of $A$ whose members are exactly those members of $A$ for which $P(x)$ is true.
- *The axiom of pairing:* For any two sets, there exists a set that they are both members of.
- *The axiom of union:* For every collection of sets, there is a set whose members are exactly the members of the sets of that collection.
- *The axiom of powers:* For each set $A$, there is a set $P(A)$ whose members are exactly the subsets of set $A$.
- *The axiom of infinity:* There exists a set $A$ such that (i) the empty set is a member of $A$, and (ii) if $x$ is a member of $A$, then the successor of $x$ is a member of $A$.
- *The axiom of choice:* Given a disjointed set $S$ whose members are nonempty sets, there exists a set $C$ which has as its members one and only one element from each member of $S$.

We can see that there is absolutely nothing in these axioms that explicitly requires sets to be containers. What these axioms do, collectively, is to *create* entities called "sets", first from elements and then from previously created sets. The axioms do not say explicitly how sets are to be conceptualized.

The point here is that, within formal mathematics, where all mathematical concepts are mapped onto set-theoretical structures, the "sets" used in these structures are not technically conceptualized as the Container Schemas we described above. They do not have container-schema structure with an interior, boundary, and exterior at all. Indeed, within formal mathematics, there are no concepts at all, and hence sets are not conceptualized as anything in particular. They are undefined entities whose only constraints are that they must "fit" the axioms. For formal logicians and model theorists, sets are those entities that fit the axioms and are used in the modeling of other branches of mathematics.

Of course, most of us do conceptualize sets in terms of Container Schemas, and that is perfectly consistent with the axioms given above. However, when we conceptualize sets as Container Schemas, a particular entailment follows automatically: *Sets cannot be members of themselves*, since containers cannot be inside themselves. Strictly speaking, this entailment does *not* follow from the axioms, but rather from our metaphorical understanding of sets in terms of containers. The above axioms do not rule out sets that contain themselves. Indeed, an extra axiom was proposed by von Neumann to rule out this possibility:

- *The axiom of foundation:* There are no infinite descending sequences of sets under the membership relation. That is, $\cdots \in S_{i+1} \in S_i \in \cdots \in S$ is ruled out.

Since allowing sets to be members of themselves would result in such a sequence, this axiom has the indirect effect of ruling out self-membership.

Within formal mathematics, model theory has nothing to do with everyday understanding. Model theorists do not depend upon our ordinary container-based concept of a set. Indeed, certain model theorists have found that our ordinary grounding metaphor that SETS[8] ARE CONTAINERS gets in the way of modeling kinds of phenomena they want to model, especially recursive phenomena. For example, take expressions like

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}.$$

If we observe carefully, we can see that the denominator of the main fraction has in fact the value defined for $x$ itself. In other words, the above expression is equivalent to

$$x = 1 + \frac{1}{x}.$$

Such recursive expressions are common in mathematics and computer science. The possibilities for modeling such expressions using "sets" are ruled out if the only kind of "sets" used in the modeling cannot have themselves as members. Set theorists have realized that a new non-container metaphor is needed for thinking about sets, and have explicitly constructed one (see Barwise and Moss, 1991).

The idea is to use graphs, not containers, for characterizing sets. The kinds of graphs used are accessible pointed graphs, or APGs. "Pointed" indicates an asymmetric relation between nodes in the graph, indicated visually by an arrow pointing from one node to another – or from one node back to that node itself (see Figure 2). "Accessible" indicates that there is a single node which is linked to all other nodes in the graph, and can therefore be "accessed" from any other node.

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[8] There are technical differences between classes and sets whose analysis goes beyond the scope of this text. For a discussion see Lakoff and Núñez (2000).

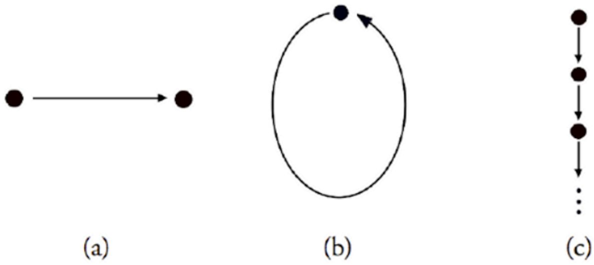


**Fig. 2.** Hypersets: Sets conceptualized as graphs, with the empty set as the graph with no arrows leading from it. The set containing the empty set is a graph whose root has one arrow leading to the empty set (a). Illustration (b) depicts a graph of a set that is a "member" of itself, under the SETS ARE GRAPHS metaphor. Illustration (c) depicts an infinitely long chain of nodes in an infinite graph, which is equivalent to (b).

From the axiomatic perspective, the axiom of foundation has been replaced by another axiom that implies its negation, the "anti-foundation axiom". From the perspective of mathematical idea analysis, the creators of hypersets implicitly used a conceptual metaphor which has the mapping shown in Table 2. The effect of this metaphor is to eliminate the notion of containment from the concept of a "set". The graphs have no notion of containment built into them at all. And containment is not modeled by the graphs.

Graphs that have no loops satisfy the ZFC axioms and the axiom of foundation. They thus work just like sets conceptualized as containers. But graphs that do have loops model sets that can "have themselves as members". They do not work like sets that are conceptualized as containers, and they do not satisfy the axiom of foundation.

| Source Domain<br>Accessible Pointed Graphs | | Target Domain<br>Sets |
|---|---|---|
| an AGP | → | the membership structure of a set |
| an arrow | → | the membership relation |
| nodes that are tails of arrows | → | sets |
| decorations on nodes that are heads of arrows | → | members |
| AGP's with no loops | → | classical sets with the foundation axiom |
| AGP's with or without loops | → | hypersets with the the anti-foundation axiom |

**Tab. 2.** The metaphor SETS ARE GRAPHS

A "hyperset" is an APG that may or may not contain loops. Hypersets thus do not fit the axiom of foundation, but rather another axiom with the opposite intent:

- *The anti-foundation axiom:* Every APG pictures a unique set.

The fact that hypersets satisfy the Zermelo-Fraenkel axioms confirms what we said above: *The Zermelo-Fraenkel axioms for set theory – generally accepted in mathematics – do not define our ordinary concept of a set as a container.* That is, the axioms of "set theory" are not, and were never meant to be, about what we ordinarily call "sets" as conceptualized in terms of Container Schemas.

So what are sets, really ? The answer to this question allows us to see the power of conceptual metaphor in mathematics. Sets, conceptualized in everyday terms as containers, do not have the right properties to model everything needed. So we can now metaphorically reconceptualize "sets" to exclude containment by using certain kinds of graphs. The only confusing thing is that this special case of graph theory is still called "set theory" for historical reasons.

Because of this misleading terminology, it is sometimes said that the theory of hypersets is "a set theory in which sets can contain themselves." From a cognitive point of view this is completely misleading because it is not a theory of "sets" as we ordinarily understand them in terms of containment. The reason that these graph theoretical objects are called "sets" is a functional one: They play the role in modeling axioms that classical sets with the axiom of foundation used to play.

The moral is that mathematics has (at least) two internally consistent, but mutually inconsistent metaphorical conceptions of sets: one in terms of Container Schemas and one in terms of graphs. Is one of these conceptions right and the other wrong ? A Platonist might want to think that there must be only one literally correct notion of a "set" transcending the human mind. But from the perspective of mathematical idea analysis these two distinct notions of a "set" define different and mutually inconsistent subject matters, conceptualized via radically different human conceptual metaphors. Mathematics is full of cases like this one.

As we mentioned at the beginning, Wolfgang Pauli in his essay on Kepler, made very clear, for the case of geometry, that "it took a special intellectual effort to recognize the fact that the assumptions of Euclidean geometry are not the only possible ones". Perhaps we will never know what exactly Pauli meant by "intellectual effort". Was it an effort for discovering other forms of truth in some Platonic realm ? Or was it an effort in the sense of conceiving entirely new ideas thanks to cognitive mechanisms that sustain human imagination ? Our work in the cognitive science of mathematics endorses the latter, which sees mathematics, from Euclidean points to transfinite numbers and hypersets, as a bodily-grounded wonderful human creation.

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