9 Determination of periodic orbits

Periodic orbits play a very important role in many problems of Celestial Mechanics; for example, their study provides interesting information on spin–orbit and orbital resonances (see [96, 136]). From the dynamical point of view periodic orbits can be used to approximate quasi–periodic trajectories; more precisely, a truncation of the continued fraction expansion of an irrational frequency yields a sequence of rational numbers, which correspond to periodic orbits eventually approximating a quasi–periodic torus.

We present some results on the existence of periodic orbits through a constructive version of the implicit function theorem, both in a conservative and in a dissipative setting (Section 9.1). Then we review classical methods for computing periodic orbits, like the Lindstedt–Poincarè (Section 9.2) and the KBM (Section 9.3) techniques. We conclude with a discussion of Lyapunov's theorem on the determination of families of periodic orbits (Section 9.4) and an application to the J_2–problem.

9.1 Existence of periodic orbits

The existence of periodic orbits can be proved through the implementation of an implicit function theorem, which yields a constructive algorithm to find suitable approximations of the solution [29, 149]. We discuss the existence of periodic orbits in the conservative and in the dissipative setting, with concrete reference to the specific sample provided by the spin–orbit problem (see Section 5.5.1).

9.1.1 Existence of periodic orbits (conservative setting)

Let us write the spin–orbit equation of motion (5.16) in the form

$$\ddot{x} - \varepsilon g(x,t) \;=\; 0 \;, \tag{9.1}$$

where $g(x,t) \equiv -(\frac{a}{r})^3 \sin(2x - 2f)$. Equation (9.1) can also be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon g(x,t) \;. \end{aligned} \tag{9.2}$$

Here ε represents the equatorial ellipticity and it can be assumed that $\varepsilon < 1$. A spin–orbit resonance of order $p : q$ is a periodic solution of (9.2) with period

$T = 2\pi q$ $(q \in \mathbf{Z}_+)$, such that

$$\begin{aligned} x(t+2\pi q) &= x(t) + 2\pi p \\ y(t+2\pi q) &= y(t) \ . \end{aligned} \tag{9.3}$$

From (9.2) one obtains

$$\begin{aligned} x(t) &= x(0) + y(0)t + \varepsilon \int_0^t \int_0^\tau g(x(s),s)\ ds\ d\tau = x(0) + \int_0^t y(s)\ ds \\ y(t) &= y(0) + \varepsilon \int_0^t g(x(s),s)\ ds \ . \end{aligned} \tag{9.4}$$

Using the periodicity conditions (9.3) one gets

$$\begin{aligned} \int_0^{2\pi q} y(s)ds - 2\pi p &= 0 \\ \int_0^{2\pi q} g(x(s),s)ds &= 0 \ . \end{aligned} \tag{9.5}$$

Let us expand the solution in powers of ε as

$$\begin{aligned} x(t) &\equiv \overline{x} + \overline{y}t + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots \\ y(t) &\equiv \overline{y} + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots \ , \end{aligned}$$

where $x(0) = \overline{x}$ and $y(0) = \overline{y}$ are suitable initial conditions, while $x_j(t)$, $y_j(t)$, $j \geq 1$, are unknown corrections to higher orders in ε. Let us expand also the initial conditions in powers of ε as

$$\begin{aligned} \overline{x} &= \overline{x}_0 + \varepsilon \overline{x}_1 + \varepsilon^2 \overline{x}_2 + \dots \\ \overline{y} &= \overline{y}_0 + \varepsilon \overline{y}_1 + \varepsilon^2 \overline{y}_2 + \dots \ , \end{aligned} \tag{9.6}$$

for some unknown terms $\overline{x}_0$, $\overline{y}_0$, $\overline{x}_1$, $\overline{y}_1$, ... Equating in (9.2) the same orders in ε and using (9.6), one obtains

$$\begin{aligned} \overline{y} + \varepsilon \dot{x}_1(t) + \dots &= \overline{y} + \varepsilon y_1(t) + \dots \\ \varepsilon \dot{y}_1(t) + \dots &= \varepsilon g(\overline{x} + \overline{y}t, t) + \dots \ , \end{aligned}$$

which yield

$$\begin{aligned} \dot{x}_1(t) &= y_1(t) \\ \dot{y}_1(t) &= g(\overline{x}_0 + \overline{y}_0 t, t) \ , \end{aligned}$$

namely

$$\begin{aligned} x_1(t) &= x_1(t; \overline{x}, \overline{y}) = \int_0^t y_1(s)\ ds \\ y_1(t) &= y_1(t; \overline{x}, \overline{y}) = \int_0^t g(\overline{x}_0 + \overline{y}_0 s, s)\ ds \ . \end{aligned} \tag{9.7}$$

Notice that $x_1(t)$ and $y_1(t)$ can be computed explicitly. Concerning the initial data, using the second of (9.4) and the periodicity conditions (9.5) one obtains

$$\int_0^{2\pi q} \left[\overline{y}_0 + \varepsilon \overline{y}_1 + \varepsilon \int_0^t g(\overline{x}_0 + \overline{y}_0 s, s)\ ds \right]\ dt\ = 2\pi p\ .$$

Therefore, $\overline{y}_0$ and $\overline{y}_1$ are given by

$$\begin{aligned} \overline{y}_0 &= \frac{p}{q} \\ \overline{y}_1 &= -\frac{1}{2\pi q} \int_0^{2\pi q} \int_0^t g(\overline{x}_0 + \overline{y}_0 s, s)\ ds\ dt\ . \end{aligned} \tag{9.8}$$

In a similar way, $\overline{x}_0$ and $\overline{x}_1$ are obtained using

$$\int_0^{2\pi q} g(\overline{x}_0 + \overline{y}_0 s + \varepsilon(\overline{x}_1 + \overline{y}_1 s + x_1(s)), s)\ ds\ =\ 0\ ;$$

expanding in series of ε, the quantity $\overline{x}_0$ is determined as the solution of

$$\int_0^{2\pi q} g(\overline{x}_0 + \overline{y}_0 s, s)\ ds\ =\ 0\ , \tag{9.9}$$

while $\overline{x}_1$ is given by

$$\overline{x}_1 = -\frac{1}{\int_0^{2\pi q} g_x^0 dt} \left[\overline{y}_1 \int_0^{2\pi q} g_x^0\ t\ dt + \int_0^{2\pi q} g_x^0\ x_1(t)\ dt \right]\ , \tag{9.10}$$

where $g_x^0 = g_x(\overline{x}_0 + \overline{y}_0 t, t)$.

9.1.2 Computation of the libration in longitude

Applying the results of Section 9.1.1, we can implement the above formulae to compute the *libration in longitude* of the Moon, which measures the displacement from the synchronous resonance corresponding to $p = q = 1$. The initial data and the first–order corrections are computed through (9.7), (9.8), (9.9), (9.10):

$$\begin{aligned} \overline{x}_0 &= 0 \\ \overline{y}_0 &= 1 \\ x_1(t) &= 0.232086\ t - 0.218318 \sin(t) - 6.36124 \cdot 10^{-3} \sin(2t) \\ &- 3.21314 \cdot 10^{-4} \sin(3t) - 1.89137 \cdot 10^{-5} \sin(4t) \\ &- 1.18628 \cdot 10^{-6} \sin(5t) \\ y_1(t) &= 0.232086 - 0.218318 \cos(t) - 0.0127225 \cos(2t) \\ &- 9.63942 \cdot 10^{-4} \cos(3t) - 7.56548 \cdot 10^{-5} \cos(4t) \\ &- 5.93138 \cdot 10^{-6} \cos(5t) \\ \overline{x}_1 &= 0 \\ \overline{y}_1 &= -0.232086\ , \end{aligned}$$

where for the Moon we used e $= 0.0549$, $\varepsilon = 3.45 \cdot 10^{-4}$. To the first order, the solution corresponding to the synchronous periodic orbit is given by

$$\begin{aligned}
x(t) &= \overline{x}_0 + \overline{y}_0 t + \varepsilon x_1(t) = (1 + 8.00697 \cdot 10^{-5})t \\
&- 7.53196 \cdot 10^{-5} \sin(t) - 2.19463 \cdot 10^{-6} \sin(2t) \\
&- 1.10853 \cdot 10^{-7} \sin(3t) - 6.52523 \cdot 10^{-9} \sin(4t) \\
&- 4.09265 \cdot 10^{-10} \sin(5t) \\
y(t) &= \overline{y}_0 + \varepsilon y_1(t) = 1 - 7.53196 \cdot 10^{-5} \cos(t) - 4.38926 \cdot 10^{-6} \cos(2t) \\
&- 3.3256 \cdot 10^{-7} \cos(3t) - 2.61009 \cdot 10^{-8} \cos(4t) \\
&- 2.04633 \cdot 10^{-9} \cos(5t) \ . \qquad (9.11)
\end{aligned}$$

We remark that having set to unity the angular velocity of rotation, the time t coincides with the Moon's longitude. For $\varepsilon = 0$, the equations of motion can be solved as

$$\begin{aligned}
x(t) &= \overline{x}_0 + \overline{y}_0 t = \overline{x}_0 + t \\
y(t) &= \overline{y}_0 = 1 \ ;
\end{aligned}$$

since $\overline{x}_0 = 0$, the difference between $x(t)$ and t is zero and therefore the direction on the equatorial plane joining the barycenter of the Moon with the Earth does not vary with time. When adding the perturbation due to the non–spherical structure of the Moon, the function $x(t)$ varies by a quantity of order ε, which provides a measure of the *libration in longitude.* The computation to the first order as in (9.11) gives a displacement of the quantity $x(t) - t$ of the order of $8 \cdot 10^{-5}$ in agreement with the astronomical data.

9.1.3 Existence of periodic orbits (dissipative setting)

We consider the dissipative spin–orbit problem described in Section 5.5.3, whose equation of motion (5.21) can be written in compact form as

$$\dot{\underline{z}} = \underline{G}(\underline{z}, t; \mu) \ ,$$

where $\underline{z} = (x, y)$, while $\underline{G}$ is a periodic two–dimensional vector function, depending parametrically on the dissipative constant μ. Assume that for $\mu = 0$ (conservative case) we know a T–periodic solution of the form

$$\underline{z}(t) = \underline{\varphi}(t)$$

with $\underline{\varphi}(T) = \underline{\varphi}(0)$. For μ sufficiently small, there still exists a periodic solution of the dissipative problem with period T [149]; this result is based on the implicit function theorem under quite general hypotheses as we are going to describe. For the dissipative spin–orbit problem we assume for simplicity that the dissipative constant and the perturbing parameter are related by $\mu = \mu_0 \varepsilon$ for a suitable quantity

$\mu_0 < 1$. Then equation (5.21) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon g(x,y,t) \ , \end{aligned} \tag{9.12}$$

with $g(x,y,t) = -(\frac{a}{r})^3 \sin(2x - 2f) - \mu_0(y - \eta)$. We denote by $\bar{x}$ and $\bar{y}$ the initial conditions and by $(x(t;\bar{x},\bar{y}), y(t;\bar{x},\bar{y}))$ the solution at time t with initial conditions $(\bar{x},\bar{y})$. By (9.12) we obtain

$$\begin{aligned} x(t) &\equiv x(t;\bar{x},\bar{y}) = \bar{x} + \bar{y}t + \varepsilon \int_0^t \int_0^\tau g(x(s;\bar{x},\bar{y}), y(s;\bar{x},\bar{y}), s) ds d\tau \\ y(t) &\equiv y(t;\bar{x},\bar{y}) = \bar{y} + \varepsilon \int_0^t g(x(s;\bar{x},\bar{y}), y(s;\bar{x},\bar{y}), s) ds \ . \end{aligned} \tag{9.13}$$

A spin–orbit resonance of order $p:q$ satisfies the periodicity conditions (9.3) which, together with (9.13), are equivalent to find solutions of the equations

$$\begin{aligned} F_1(\bar{x},\bar{y}) &= 0 \\ F_2(\bar{x},\bar{y}) &= 0 \ , \end{aligned} \tag{9.14}$$

where

$$\begin{aligned} F_1(\bar{x},\bar{y}) &\equiv 2\pi(q\bar{y} - p) + \varepsilon \int_0^{2\pi q} \int_0^\tau g(x(s;\bar{x},\bar{y}), y(s;\bar{x},\bar{y}), s) ds d\tau \\ F_2(\bar{x},\bar{y}) &\equiv \int_0^{2\pi q} g(y(s;\bar{x},\bar{y}), x(s;\bar{x},\bar{y}), s) ds \ . \end{aligned} \tag{9.15}$$

Expanding (9.15) to the first order in ε, one obtains

$$\begin{aligned} F_1(\bar{x},\bar{y}) &= 2\pi(q\bar{y} - p) + \varepsilon\Phi_1(\bar{x},\bar{y}) \\ F_2(\bar{x},\bar{y}) &= \int_0^{2\pi q} g(\bar{x} + \bar{y}s, \bar{y}, s) ds + \varepsilon\Phi_2(\bar{x},\bar{y}) \end{aligned} \tag{9.16}$$

for suitable functions $\Phi_1(\bar{x},\bar{y})$, $\Phi_2(\bar{x},\bar{y})$. Let us expand the initial conditions as $\bar{x} = \bar{x}_0 + \varepsilon\bar{x}_1 + \varepsilon^2\bar{x}_2 + \ldots$, $\bar{y} = \bar{y}_0 + \varepsilon\bar{y}_1 + \varepsilon^2\bar{y}_2 + \ldots$ Then we find $\bar{y}_0 = \frac{p}{q}$, while $\bar{x}_0$ is determined as a non–degenerate critical point of the function

$$\Psi_{p,q}(x) \equiv \frac{1}{2} \int_0^{2\pi q} \left(\frac{a}{r(t)}\right)^3 \cos(2x + 2\frac{p}{q}t - 2f(t)) \ dt \ ,$$

so that using (9.16) one obtains

$$\begin{aligned} F_1(\bar{x}_0,\bar{y}_0) &= \varepsilon\Phi_1(\bar{x}_0,\bar{y}_0) \\ F_2(\bar{x}_0,\bar{y}_0) &= -2\pi q\mu_0\left(\frac{p}{q} - \eta\right) + \varepsilon\Phi_2(\bar{x}_0,\bar{y}_0) \ . \end{aligned}$$

Let us evaluate the Jacobian J of (9.16) at $(\bar{x}_0,\bar{y}_0)$ and let us denote the result by $J_0 + \varepsilon J_1$; then J_0 is non–degenerate, since

$$J_0 = \begin{pmatrix} 2\pi q & 0 \\ \Phi_{p,q}(\bar{x}_0, \bar{y}_0; \mu_0) & \frac{d^2}{dx^2}\Psi_{p,q}(\bar{x}_0) \end{pmatrix}$$

for a suitable function $\Phi_{p,q} = \Phi_{p,q}(\bar{x}_0, \bar{y}_0; \mu_0)$. Let M be the inverse of the Jacobian J evaluated at $(\bar{x}_0, \bar{y}_0)$; let $\rho > 0$ and denote by $\bar{B}_\rho(\bar{x}_0, \bar{y}_0)$ the closed ball of radius ρ around $(\bar{x}_0, \bar{y}_0)$. Let A be a compact subset of $\mathbf{R}$ and let $0 < \alpha < 1$, $R > 0$ be real parameters. The implicit function theorem can be applied provided the following conditions are satisfied (I_2 is the 2×2 identity matrix):

$$\begin{aligned} \sup_{\bar{B}_\rho(\bar{x}_0,\bar{y}_0)\times A} \|I_2 - M\ J\| &\leq \alpha \\ \sup_A |\underline{F}(\bar{x}_0, \bar{y}_0)| \cdot \sup_A \|M\| &\leq (1-\alpha)\ R\ ; \end{aligned}$$

the above inequalities turn out to be smallness conditions on the parameters. Under these conditions the implicit function theorem guarantees that for ε sufficiently small there exists a solution $(x(\varepsilon), y(\varepsilon)) \in \bar{B}_\rho(\bar{x}_0, \bar{y}_0)$ of the system

$$\begin{aligned} F_1(x(\varepsilon), y(\varepsilon)) &= 0 \\ F_2(x(\varepsilon), y(\varepsilon)) &= 0\ , \end{aligned}$$

providing a fixed point of (9.14) with the required periodicity conditions.

9.1.4 Normal form around a periodic orbit

The dynamics in a neighborhood of the periodic orbits determined as in Section 9.1.1 can be studied through the development of a suitable normal form, which turns out to be useful in a number of samples in Celestial Mechanics. We briefly sketch the procedure referring to equations (9.2), whose associated Hamiltonian function takes the form

$$\mathcal{H}_1(y, x, t) = \frac{y^2}{2} - \varepsilon V(x, t)\ , \qquad y \in \mathbf{R}\ , \quad (x, t) \in \mathbf{T}\ ,$$

where $y = \dot{x}$ is the variable conjugated to x and $V_x(x,t) = g(x,t)$. Let $(\tilde{x}(t), \tilde{y}(t))$ be a periodic orbit of order $p : q$ with periodicity conditions (9.3). We assume to know the periodic orbit for example through its series expansion as explained in Section 9.1.1. In the proximity of the periodic orbit, let γ be a positive, small parameter, measuring the distance from the periodic orbit and let $(\gamma\xi(t), \gamma\eta(t))$ be a small displacement such that we can write the solution in the form

$$\begin{aligned} x(t) &= \tilde{x}(t) + \gamma\xi(t) \\ y(t) &= \tilde{y}(t) + \gamma\eta(t)\ . \end{aligned} \tag{9.17}$$

Inserting (9.17) in (9.2), one obtains

$$\begin{aligned} \dot{x}(t) &= \dot{\tilde{x}}(t) + \gamma\dot{\xi}(t) = \tilde{y}(t) + \gamma\eta(t) \\ \dot{y}(t) &= \dot{\tilde{y}}(t) + \gamma\dot{\eta}(t) = \varepsilon g(\tilde{x}(t) + \gamma\xi(t), t)\ , \end{aligned}$$

where we can expand g in Taylor series around $\gamma = 0$ as

$$g(\tilde{x}(t) + \gamma\xi(t), t) = g(\tilde{x}(t), t) + \gamma g_x(\tilde{x}(t), t)\xi + \frac{1}{2}\gamma^2 g_{xx}(\tilde{x}(t), t)\xi^2 + \ldots$$

Since $(\tilde{x}(t), \tilde{y}(t))$ is a solution of the equations of motion, one gets

$$\begin{aligned} \dot{\xi} &= \eta \\ \dot{\eta} &= \varepsilon g_x(\tilde{x}(t), t)\xi + \frac{\varepsilon}{2}\gamma g_{xx}(\tilde{x}(t), t)\xi^2 + \ldots\ , \end{aligned} \tag{9.18}$$

whose associated Hamiltonian is

$$\mathcal{H}_2(\eta, \xi, t) = \frac{\eta^2}{2} - \frac{\varepsilon}{2} g(\tilde{x}(t), t)\xi^2 - \frac{\varepsilon}{6} g_x(\tilde{x}(t), t)\gamma\xi^3 + \ldots$$

Defining

$$\underline{u} \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}\ , \quad Q(t) \equiv \begin{pmatrix} 0 & 1 \\ \varepsilon g_x(\tilde{x}(t), t) & 0 \end{pmatrix}\ , \quad R_2(\underline{u}, t) \equiv \begin{pmatrix} 0 \\ \frac{\varepsilon}{2} g_{xx}(\tilde{x}(t), t)\xi^2 + \ldots \end{pmatrix}\ ,$$

we can write (9.18) in the form

$$\dot{\underline{u}} = Q(t)\underline{u} + \gamma R_2(\underline{u}, t)\ . \tag{9.19}$$

Floquet theory (see Appendix D) can be implemented to eliminate the time–dependence in the linear part. Through a symplectic, periodic change of variables one can reduce (9.19) to the form

$$\dot{\underline{v}} = A\underline{v} + \gamma S_2(\underline{v}(t), t)\ , \tag{9.20}$$

where $\underline{v} \equiv (v_1, v_2) \in \mathbf{R}^2$, A is a constant matrix and S_2 is a suitable function. We can assume that A takes the form

$$A \equiv \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\ ,$$

so that the linear part reduces to

$$\begin{aligned} \dot{v}_1 &= \omega v_2 \\ \dot{v}_2 &= -\omega v_1\ , \end{aligned}$$

whose associated Hamiltonian corresponds to that of a harmonic oscillator, namely $\mathcal{H}_3(v_1, v_2) = \frac{\omega}{2}(v_1^2 + v_2^2) + \ldots$. Using action–angle variables (I, φ) for the harmonic oscillator, we can write the Hamiltonian function corresponding to (9.20) in the form

$$\mathcal{H}_4(I, \varphi, t) = \omega I + \gamma F(I, \varphi, t)\ ,$$

for a suitable function $F = F(I, \varphi, t)$. A Birkhoff normal form can now be implemented in the style of Section 6.5 to reduce the perturbation and to get a better approximation in the neighborhood of the periodic orbit.

9.2 The Lindstedt–Poincaré technique

Convergent series approximations of periodic solutions can be found through the *Lindstedt–Poincaré technique*, also known as the *continuation method.* Consider a dynamical system described by the second–order differential equation

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}) \ , \qquad x \in \mathbf{R} \ , \tag{9.21}$$

where $\varepsilon \geq 0$ is a small real parameter and $f : \mathbf{R}^2 \to \mathbf{R}$ is a regular function. For $\varepsilon = 0$ the system reduces to a harmonic oscillator, which has periodic solutions with period $T_0 = \frac{2\pi}{\omega_0}$. The Lindstedt–Poincaré technique allows us to find periodic solutions for ε different from zero by taking into account that the frequency of the motion can change due to the non–linear terms. In fact, when ε is different from zero the period T is equal to T_0 only up to terms of order ε. Basically one expands the solution $x(t)$ and the (unknown) frequency ω of the periodic orbit as a function of ε:

$$\begin{aligned} x(t) &= x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots \ , \end{aligned} \tag{9.22}$$

where we impose that $x_j(T) = x_j(0)$, being the quantities $x_j(t)$, ω_j, $j \geq 0$, unknown. Under the change of variables $s = \omega t$, the equation (9.21) becomes

$$\omega^2 x'' + \omega_0^2 x = \varepsilon f(x, \omega x') \ , \tag{9.23}$$

where x' and x'' denote the first and second derivatives with respect to s. Let us expand the perturbation in powers of ε as

$$\begin{aligned} f(x, \omega x') &= f(x_0, \omega_0 x_0') + \varepsilon \Big[x_1 \frac{\partial f(x_0, \omega_0 x_0')}{\partial x} + x_1' \frac{\partial f(x_0, \omega_0 x_0')}{\partial x'} \\ &+ \ \omega_1 \frac{\partial f(x_0, \omega_0 x_0')}{\partial \omega} \Big] + O(\varepsilon^2) \ . \end{aligned}$$

Inserting the series expansion (9.22) in (9.23) and equating terms of the same order in ε, one obtains

$$\begin{aligned} \omega_0^2 x_0'' + \omega_0^2 x_0 &= 0 \\ \omega_0^2 x_1'' + \omega_0^2 x_1 &= f(x_0, \omega_0 x_0') - 2\omega_0\omega_1 x_0'' \\ &\ldots \end{aligned}$$

These equations can be solved recursively and the quantities ω_j can be found by imposing the periodicity conditions $x_j(s + 2\pi) = x_j(s)$, $j = 0, 1, 2, \ldots$

As a concrete example we consider the Duffing equation [146]

$$\ddot{x} + \omega_0^2 x = -\varepsilon\omega_0^2 x^3 \ .$$

Changing time as $s = \omega t$, one gets

$$\omega^2 x''(s) + \omega_0^2 x(s) = -\varepsilon\omega_0^2 x(s)^3 \ .$$

Let us expand the solution $x(s)$ and the unknown frequency ω as in (9.22) and assume that $x'_j(0) = 0$ for any $j \geq 0$. To the zeroth order in ε one obtains the equation

$$x''_0 + x_0 = 0$$

and, taking into account the initial conditions, one finds $x_0(s) = A\cos s$ for some real constant A. To the first order in ε one obtains the equation

$$x''_1 + x_1 = A\left(2\frac{\omega_1}{\omega_0} - \frac{3}{4}A^2\right)\cos s - \frac{1}{4}A^3\cos 3s \ ;$$

secular terms are avoided provided $\frac{\omega_1}{\omega_0} = \frac{3}{8}A^2$, thus yielding the first–order solution $x_1(s) = \frac{1}{32}A^3\cos 3s$. The solution at all subsequent orders can be obtained implementing iteratively the above procedure.

9.3 The KBM method

The Krylov–Bogoliubov–Mitropolsky (KBM) method allows us to find periodic solutions for systems of the form (9.21); for $\varepsilon = 0$ such systems admit the solution

$$x(t) = A\cos\xi(t) \qquad \text{with} \ \ \xi(t) \equiv \omega_0 t + \varphi \ ,$$

for some constants A, φ depending on the initial conditions. For ε different from zero, one can write the solution as

$$x(t) = A\cos\xi + \varepsilon x_1(A,\xi) + \varepsilon^2 x_2(A,\xi) + \dots \ , \tag{9.24}$$

where $x_j(A,\xi)$ are 2π–periodic functions. The quantities A, ξ satisfy the equations

$$\begin{aligned} \dot{A} &= \varepsilon\alpha_1(A) + \varepsilon^2\alpha_2(A) + \dots \\ \dot{\xi} &= \omega_0 + \varepsilon\beta_1(A) + \varepsilon^2\beta_2(A) + \dots \end{aligned} \tag{9.25}$$

for some unknown functions $\alpha_j(A)$, $\beta_j(A)$. Inserting (9.24) and (9.25) in the left hand side of (9.21), one obtains

$$\ddot{x} + \omega_0^2 x = \varepsilon\left[-2\omega_0\alpha_1\sin\xi - 2\omega_0 A\beta_1\cos\xi + \omega_0^2\left(\frac{\partial^2 x_1}{\partial\xi^2} + x_1\right)\right] + O(\varepsilon^2) \ .$$

Concerning the right–hand side of (9.21) one has

$$\varepsilon f(x,\dot{x}) = \varepsilon f(x_0,\dot{x}_0) + O(\varepsilon^2) \ ,$$

where $x_0 = A\cos\xi$, $\dot{x}_0 = -A\omega_0\sin\xi$, being x_0 the lowest–order approximation in which A and $\dot{\xi}$ are constant. Equating same powers of ε, the first order is given by

$$\omega_0^2\left(\frac{\partial^2 x_1}{\partial\xi^2} + x_1\right) = f(x_0,\dot{x}_0) + 2\omega_0\alpha_1\sin\xi + 2\omega_0 A\beta_1\cos\xi \tag{9.26}$$

and similarly for higher orders which can be solved recursively. For the first order, let us expand x_1 and f in Fourier series as

$$\begin{aligned} f(A,\xi) &= f_0(A) + \sum_{j=1}^{\infty} \Big[f_j^{(c)}(A) \cos j\xi + f_j^{(s)}(A) \sin j\xi \Big] \\ x_1(A,\xi) &= x_0^{(1)}(A) + \sum_{j=2}^{\infty} \Big[x_j^{(1c)}(A) \cos j\xi + x_j^{(1s)}(A) \sin j\xi \Big] \ , \end{aligned} \tag{9.27}$$

for suitable functions $f_0(A)$, $f_j^{(c)}(A)$, $f_j^{(s)}(A)$, $x_0^{(1)}(A)$, $x_j^{(1c)}(A)$, $x_j^{(1s)}(A)$. To avoid secular terms we impose that

$$\int_0^{2\pi} x_1(A,\xi) \cos\xi d\xi = 0 \ , \qquad \int_0^{2\pi} x_1(A,\xi) \sin\xi d\xi = 0 \ ,$$

which yield $x_1^{(1c)}(A) = x_1^{(1s)}(A) = 0$. Inserting (9.27) in (9.26) and equating Fourier coefficients of the same order, one obtains

$$f_1^{(c)}(A) + 2\omega_0 A\beta_1(A) = 0 \ , \qquad f_1^{(s)}(A) + 2\omega_0 \alpha_1(A) = 0 \ ,$$

which provide explicit expressions for $\alpha_1(A)$ and $\beta_1(A)$. Moreover, one has

$$x_0^{(1)}(A) = \frac{f_0(A)}{\omega_0^2} \ , \qquad x_j^{(1c)}(A) = \frac{f_j^{(c)}(A)}{\omega_0^2(1-j^2)} \ , \qquad x_j^{(1s)}(A) = \frac{f_j^{(s)}(A)}{\omega_0^2(1-j^2)} \ , \quad j \geq 2 \ ,$$

which provide the Fourier coefficients appearing in (9.27). Similar computations can be performed to determine iteratively the solution to higher orders.

9.4 Lyapunov's theorem

A remarkable result due to Lyapunov allows us to determine a family of periodic solutions around an equilibrium position. We sketch the proof of the theorem, referring to [162] for complete details. As an illustrative example, we consider the J_2–problem introduced in Section 5.6.

9.4.1 Families of periodic orbits

We consider an n–dimensional Hamiltonian system described by the Hamiltonian function $\mathcal{H} = \mathcal{H}(\underline{w})$, $\underline{w} \in \mathbf{R}^{2n}$, which is assumed to be regular in a suitable neighborhood of the origin. We assume that the origin is an equilibrium position; let $\pm\lambda_1, \dots, \pm\lambda_n$ be distinct eigenvalues of the linearized matrix L associated to the Hamiltonian $\mathcal{H}$ around the origin.

Lyapunov's Theorem. *Let λ_1 be purely imaginary, not identically zero, and assume that the ratios $\frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_n}{\lambda_1}$ are not integers; then, there exists a family of periodic solutions around the equilibrium position, depending analytically on a real* *parameter ρ, such that $\rho = 0$ corresponds to the equilibrium solution and that the period $T(\rho)$ is analytic in ρ with $T(0) = \frac{2\pi}{|\lambda_1|}$.*

Proof. We look for a solution $\underline{w} = \underline{w}(\xi, \eta)$ as a power series of some unknown functions $\xi = \xi(t)$, $\eta = \eta(t)$. Then, Hamilton's equations $\dot{\underline{w}} = J\mathcal{H}_{\underline{w}}(\underline{w})$ become

$$\underline{w}_\xi \dot{\xi} + \underline{w}_\eta \dot{\eta} \ = \ J\mathcal{H}_{\underline{w}}(\underline{w}) \ . \tag{9.28}$$

We assume that ξ, η satisfy the relations

$$\dot{\xi} = \alpha\xi \ , \qquad \dot{\eta} = \beta\eta \tag{9.29}$$

with α, β being suitable power series in ξ, η. We next perform a linear canonical transformation, say $\underline{w} = C\underline{z}$, for some constant matrix C, such that the linearized matrix is transformed into $C^T JLC = \Lambda$, where Λ is the diagonal matrix with non–zero elements $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$. Moreover, the matrix C is chosen to be symplectic and such that its components are suitably normalized according to [162]. With this transformation, equation (9.28) takes the form

$$\underline{z}_\xi \xi\alpha + \underline{z}_\eta \eta\beta - \Lambda\underline{z} = \underline{g}(\underline{z}) \ , \tag{9.30}$$

where

$$\underline{g}(\underline{z}) \equiv C^{-1}J(C^{-1})^T \ \mathcal{H}_{\underline{z}}(C\underline{z}) - \Lambda\underline{z} \ .$$

In order to determine uniquely the power series $z_k(\xi, \eta)$ $(k = 1, \ldots, 2n)$, $\alpha(\xi, \eta)$, $\beta(\xi, \eta)$, by comparison of the coefficients in (9.30), one needs to impose the following compatibility conditions:

$(C1)$ $z_1 - \xi$, $z_2 - \eta$, $z_3, \ldots, z_{2n}$ start with quadratic terms;
$(C2)$ there are no terms of the form $\xi(\xi\eta)^\ell$ in $z_1 - \xi$ and no terms of the form $\eta(\xi\eta)^\ell$ in $z_2 - \eta$;
$(C3)$ the series for α and β depend only on the quantity $\omega \equiv \xi\eta$.

By induction, one easily proves that equation (9.29) can be effectively solved and that the coefficients are uniquely determined. The constant terms of α and β are, respectively, λ_1 and $-\lambda_1$. Moreover, it can be shown (see [162]) that α and β satisfy the relation

$$\alpha + \beta = 0 \tag{9.31}$$

and that the Hamiltonian $\mathcal{H}$ becomes a series of $\omega = \xi\eta$. Referring to [162] for the proof of the convergence of the series $z_k(\xi, \eta)$ $(k = 1, \ldots, 2n)$, α, β for sufficiently small values of $|\xi|$, $|\eta|$, by (9.31) one finds that

$$\frac{d\omega}{dt} = \dot{\xi}\eta + \xi\dot{\eta} = (\alpha + \beta)\xi\eta = (\alpha + \beta)\omega = 0 \ ;$$

therefore ω, α, β do not depend on the time and consequently from (9.29) one obtains

$$\xi = \xi_0 e^{\alpha t} \ , \qquad \eta = \eta_0 e^{\beta t} \ , \tag{9.32}$$

where ξ_0, η_0 are the initial conditions. The value $|\xi_0|$ should be taken sufficiently small, say $|\xi_0| \leq \rho$ for some positive real parameter ρ, to ensure the convergence. By (9.32) one obtains a family of periodic orbits with periods $T(\rho) = \frac{2\pi}{|\alpha|}$; since to the lowest order α coincides with λ_1, the period of the equilibrium position is $T(0) = \frac{2\pi}{|\lambda_1|}$. □

9.4.2 An example: the J_2–problem

As an application of Lyapunov's theorem, we consider the motion of a homogeneous rigid body $\mathcal{S}$ moving around an oblate planet $\mathcal{P}$. Assuming that the central planet is axially symmetric, using spherical coordinates the potential function governing the motion of the satellite is provided in Section 5.6. The J_2–problem consists in retaining only the lowest–order term in the series expansion of the potential as a series of the Legendre's polynomials (see equation (5.27)):

$$U(r, \varphi) = \frac{\mu}{r} + \frac{\mu J_2 R_e^2}{r^3} \left(\frac{1}{2} - \frac{3}{2} \sin^2 \theta \right) ,$$

where J_2 is constant, $\mu = \mathcal{G}M$, M being the mass of $\mathcal{P}$, R_e is the equatorial radius of $\mathcal{P}$, while r and θ are, respectively, the radius and the latitude of the satellite $\mathcal{S}$ with respect to the central body $\mathcal{P}$. The Hamiltonian function describing the J_2–problem is derived as follows. In a reference frame with the origin coinciding with the barycenter $\mathcal{O}$ of $\mathcal{P}$, the spherical coordinates of $\mathcal{S}$ are:

$$\begin{aligned} x_S &= r \cos \phi \cos \theta \\ y_S &= r \sin \phi \cos \theta \\ z_S &= r \sin \theta \ , \end{aligned}$$

where $r \geq 0$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$. Assuming that the mass of $\mathcal{S}$ is normalized to one, the Lagrangian function is given by:

$$\mathcal{L}(\dot{r}, \dot{\phi}, \dot{\theta}, r, \phi, \theta,) = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\phi}^2 \cos^2\theta + r^2 \dot{\theta}^2) + U(r, \theta) \ . \tag{9.33}$$

Due to the cylindrical symmetry of the problem, the variable ϕ is cyclic; therefore the vertical component of the angular momentum (coinciding with the momentum p_ϕ conjugated to ϕ) is constant, say equal to $\overline{g}$, providing

$$p_\phi = r^2 \dot{\phi} \cos^2\theta \equiv \overline{g} \ .$$

Since the Lagrangian (9.33) does not depend explicitly on the time, another constant of the motion is given by the total energy. Using the first integral $\overline{g}$, the energy E becomes:

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{\overline{g}^2}{2r^2}(1 + \tan^2\theta) - \frac{\mu}{r} + \frac{\mu J_2 R_e^2}{r^3} \left(\frac{3}{2} \sin^2 \theta - \frac{1}{2} \right) .$$

We introduce a new pair of coordinates (ρ, z) defined as

$$\begin{aligned}\rho &= r\cos\theta\\ z &= r\sin\theta\ .\end{aligned}$$

Adopting the units of measure so that $\mu = 1$ and $R_e = 1$, the Lagrangian becomes

$$\mathcal{L}(\dot\rho, \dot z, \rho, z) = \frac{1}{2}(\dot\rho^2 + \dot z^2) - \frac{\overline{g}^2}{2\rho^2} + U(\rho, z) \tag{9.34}$$

with

$$U(\rho, z) = (\rho^2 + z^2)^{-\frac{1}{2}} - \frac{J_2}{2}(\rho^2 + z^2)^{-\frac{5}{2}}(2z^2 - \rho^2)\ .$$

Let $p_\rho = \dot\rho$, $p_z = \dot z$; the Hamiltonian associated to the Lagrangian (9.34) is given by:

$$\mathcal{H}(p_\rho, p_z, \rho, z) = \frac{1}{2}({p_\rho}^2 + {p_z}^2) + \frac{\overline{g}^2}{2\rho^2} - U(\rho, z)\ .$$

The corresponding equations of motion are:

$$\begin{aligned}\dot\rho &= p_\rho\\ \dot z &= p_z\\ \dot{p_\rho} &= \tfrac{\overline{g}^2}{\rho^3} + U_\rho(\rho, z)\\ \dot{p_z} &= U_z(\rho, z)\ ,\end{aligned} \tag{9.35}$$

where $U_\rho(\rho, z)$ and $U_z(\rho, z)$ denote the derivatives of U with respect to ρ and z:

$$U_\rho(\rho, z) = -\rho(\rho^2 + z^2)^{-\frac{3}{2}} + \frac{5}{2}J_2\rho(\rho^2 + z^2)^{-\frac{7}{2}}(2z^2 - \rho^2) + J_2\rho(\rho^2 + z^2)^{-\frac{5}{2}}\ ,$$

$$U_z(\rho, z) = -z(\rho^2 + z^2)^{-\frac{3}{2}} + \frac{5}{2}J_2 z(\rho^2 + z^2)^{-\frac{7}{2}}(2z^2 - \rho^2) - 2J_2 z(\rho^2 + z^2)^{-\frac{5}{2}}\ .$$

In order to compute the equilibrium points, we set equal to zero the right–hand side of (9.35). Selecting the solution with $z = 0$, one easily obtains that ρ must be a root of the equation $2\rho^2 - 2\overline{g}^2\rho + 3J_2 = 0$. Therefore, two equilibrium points of (9.35) are given by $P_0 \equiv (\rho_0, z_0, p_{\rho_0}, p_{z_0}) = (\frac{\overline{g}^2 + \sqrt{\overline{g}^4 - 6J_2}}{2}, 0, 0, 0)$ and $P_1 = (\frac{\overline{g}^2 - \sqrt{\overline{g}^4 - 6J_2}}{2}, 0, 0, 0)$ provided $\overline{g}^4 > 6J_2$ so as to have real positive values of ρ. In the following sections we focus our attention on the equilibrium position P_0.

9.4.3 Linearization of the Hamiltonian around the equilibrium point

We proceed to linearize the equations of motion in a neighborhood of the equilibrium point P_0. First of all, through the transformation $\tilde z = z$, $\tilde\rho = \rho - \rho_0$ (which shifts P_0 to the origin of the reference frame), we get the Hamiltonian function

$$\begin{aligned}\tilde{\mathcal{H}}(p_{\tilde\rho},p_{\tilde z},\tilde\rho,\tilde z) &= \frac{1}{2}({p_{\tilde\rho}}^2+{p_{\tilde z}}^2)+\frac{\overline{g}^2}{2(\tilde\rho+\rho_0)^2}-[(\tilde\rho+\rho_0)^2+\tilde z^2]^{-\frac{1}{2}}\\ &\quad+\frac{J_2}{2}[(\tilde\rho+\rho_0)^2+\tilde z^2]^{-\frac{5}{2}}(2\tilde z^2-(\tilde\rho+\rho_0)^2)\ ,\end{aligned}$$

where $p_{\tilde\rho}$, $p_{\tilde z}$ are the momenta conjugated to $\tilde\rho$, $\tilde z$; the corresponding equations of motion are:

$$\begin{aligned}&\dot{\tilde\rho}=p_{\tilde\rho}\\ &\dot{\tilde z}=p_{\tilde z}\\ &\dot p_{\tilde\rho}=\tfrac{\overline{g}^2}{(\tilde\rho+\rho_0)^3}+U_{\tilde\rho}(\tilde\rho+\rho_0,\tilde z)\\ &\dot p_{\tilde z}=U_{\tilde z}(\tilde\rho+\rho_0,\tilde z)\ .\end{aligned}$$

Next, we expand the equations of motion by means of a Taylor power series around the equilibrium point up to the second order. The linearized system becomes

$$\begin{pmatrix}\dot{\tilde\rho}\\ \dot{\tilde z}\\ \dot p_{\tilde\rho}\\ \dot p_{\tilde z}\end{pmatrix}=L\begin{pmatrix}\tilde\rho\\ \tilde z\\ p_{\tilde\rho}\\ p_{\tilde z}\end{pmatrix}\ ,$$

where

$$L=\begin{pmatrix}0&0&1&0\\0&0&0&1\\ \gamma&0&0&0\\0&\delta&0&0\end{pmatrix}$$

is the matrix corresponding to the linearization and

$$\begin{aligned}\gamma &= \frac{1}{{\rho_0}^3}\Big(-\frac{3\overline{g}^2}{\rho_0}+2+\frac{6J_2}{{\rho_0}^2}\Big)\\ \delta &= \frac{1}{2{\rho_0}^5}(-2{\rho_0}^2-9J_2)\ .\end{aligned}$$

Notice that δ is negative for any initial condition. Up to constant terms, the linearized Hamiltonian in a neighborhood of $\tilde P_0=(0,0,0,0)$ is given by

$$\mathcal{H}_L(p_{\tilde\rho},p_{\tilde z},\tilde\rho,\tilde z)\ =\ \frac{1}{2}({p_{\tilde\rho}}^2+{p_{\tilde z}}^2-\gamma\tilde\rho^2-\delta\tilde z^2)\ +\ \mathcal{H}_3(p_{\tilde\rho},p_{\tilde z},\tilde\rho,\tilde z)\ , \qquad (9.36)$$

where $\mathcal{H}_3(p_{\tilde\rho},p_{\tilde z},\tilde\rho,\tilde z)$ denotes terms of order higher than three.

9.4.4 Application of Lyapunov's theorem

In this section we apply Lyapunov's theorem to the existence of families of periodic orbits starting from the Hamiltonian (9.36). To this end the following conditions must be satisfied by the linearized system associated to (9.36):

(*i*) the eigenvalues $\lambda_1,\lambda_2,-\lambda_1,-\lambda_2$ of the matrix L must be distinct;
(*ii*) let λ_1 be purely imaginary; then the ratio $\frac{\lambda_2}{\lambda_1}$ must not be an integer.

The eigenvalues associated to (9.36) are obtained as follows. Let

$$\underline{w} \equiv \begin{pmatrix} \tilde{\rho} \\ \tilde{z} \\ p_{\tilde{\rho}} \\ p_{\tilde{z}} \end{pmatrix} , \qquad J \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ,$$

and let $\underline{w}^T$ be the transposed of $\underline{w}$. Then, (9.36) can be written as $\mathcal{H}_L(\underline{w}) = -\frac{1}{2}\underline{w}^T \cdot JL\underline{w} + \mathcal{H}_3(\underline{w})$ and the eigenvalues of the linearization L are $\lambda_1 = \sqrt{\delta}$, $\lambda_2 = \sqrt{\gamma}$, $\lambda_3 = -\sqrt{\delta}$, $\lambda_4 = -\sqrt{\gamma}$. Excluding degenerate cases (see the remarks below), conditions (i)–(ii) above are satisfied, so that Lyapunov's theorem applies.

Remarks.
(1) Since $\delta < 0$ for each initial condition, λ_1, λ_3 are always purely imaginary. Moreover if $3J_2 < \overline{g}^2\rho_0$, then λ_2, λ_4 are also purely imaginary. Therefore, if we assume that $3J_2 < \overline{g}^2\rho_0$, then the four eigenvalues are equal to $\lambda_{1,3} = \pm i\sqrt{|\delta|}$, $\lambda_{2,4} = \pm i\sqrt{|\gamma|}$; using the relation $2{\rho_0}^2 = 2\overline{g}^2\rho_0 - 3J_2$, we obtain

$$\begin{aligned} \gamma &= \frac{1}{2{\rho_0}^5}(-2\overline{g}^2\rho_0 + 6J_2) , \\ \delta &= \frac{1}{2{\rho_0}^5}(-2\overline{g}^2\rho_0 - 6J_2) . \end{aligned}$$

Their ratio is given by

$$\frac{\gamma}{\delta} = \frac{-2\overline{g}^2\rho_0 + 6J_2}{-2\overline{g}^2\rho_0 - 6J_2}.$$

(2) If $J_2 \neq 0$, $\overline{g} \neq 0$, $\rho_0 \neq 0$ and $3J_2 < \overline{g}^2\rho_0$, then λ_j $(j = 1, \ldots, 4)$ are purely imaginary and $\frac{\gamma}{\delta}$ is not an integer. Therefore, by Lyapunov's theorem there exist two families of periodic orbits with periods $\frac{2\pi}{\sqrt{|\gamma|}}$ and $\frac{2\pi}{\sqrt{|\delta|}}$.

(3) If $\overline{g} = 0$, then $\frac{\gamma}{\delta} = -1$ and Lyapunov's theorem cannot be applied.

(4) If $J_2 = 0$, then $\gamma = \delta$ and Lyapunov's theorem cannot be applied. Notice that in this case the system is integrable.

(5) The main condition for the applicability of Lyapunov's theorem is $6J_2 < \overline{g}^4$, which guarantees that ρ_0 is real. One can easily see that this condition implies the inequality $3J_2 < \overline{g}^2\rho_0$, which ensures that γ is negative.