7 Invariant tori

Perturbation theory fails whenever a resonance condition is met; however, even if the non–resonance condition is fulfilled, there could be linear combinations with integer coefficients of the frequency vector which become arbitrarily small. These quantities, which are called the *small divisors,* appear at the denominator of the series defining the canonical transformation needed to implement perturbation theory. Small divisors might prevent the convergence of the series and therefore the application of perturbation theory. To overcome this problem, a breakthrough came with the work of Kolmogorov [105], later proved in different mathematical settings by Arnold [3] and Moser [138]. The overall theory is known with the acronym of KAM theory (Section 7.2) and it allows us to prove the persistence of invariant tori (Section 7.1) under perturbation (compare with [28,31,117,118]). KAM theory was applied to several physical models of interest in Celestial Mechanics (Section 7.3). However, the original versions of the theory gave concrete results very far from the physical measurements of the parameters involved in the proof. The implementation of computer–assisted KAM proofs allowed us to obtain realistic results in simple models of Celestial Mechanics, like the spin–orbit problem or the planar, circular, restricted three–body problem. The validity of such results is also attested by numerical methods for the determination of the breakdown threshold, like the well–known Greene's method (Section 7.4). KAM theory can also be extended to encompass the case of lower–dimensional tori (Section 7.5) as well as of nearly–integrable, dissipative systems (see Section 7.6, [19,32]), like the dissipative spin–orbit problem introduced in Chapter 5. While KAM theory provides a lower bound on the persistence of invariant tori, converse KAM theory gives an upper bound on the non–existence of invariant tori (Section 7.7). Moreover, just above the critical breakdown threshold the invariant tori transform into cantori, which are still invariant sets though being graphs of Cantor sets. Their explicit construction is discussed in a specific example, precisely the sawtooth map where constructive formulae for the cantori can be given (Section 7.8).

7.1 The existence of KAM tori

Let us start by considering the spin–orbit equations (5.15) that we write in the form

$$\begin{aligned} \dot{y} &= -\varepsilon f_x(x,t) \\ \dot{x} &= y \ , \end{aligned} \tag{7.1}$$

where $f_x(x,t) \equiv \frac{\partial f}{\partial x} = (\frac{a}{r})^3 \sin(2x - 2f)$ with $r = r(t)$, $f = f(t)$ being known periodic functions of the time. Equations (7.1) can be viewed as Hamilton's equations associated to the Hamiltonian function

$$\mathcal{H}(y, x, t) = h(y) + \varepsilon f(x, t) \ ,$$

where $h(y) = \frac{y^2}{2}$ is the unperturbed Hamiltonian, ε denotes the perturbing parameter, while the perturbation $f = f(x, t)$ is a continuous periodic function whose explicit expression has been given as in (5.15). The perturbing function can be expanded in Fourier series as

$$f(x, t) = -\frac{1}{2} \sum_{m \neq 0, m = N_1}^{N_2} W\left(\frac{m}{2}, \mathrm{e}\right) \cos(2x - mt) \tag{7.2}$$

for suitable coefficients $W(\frac{m}{2}, \mathrm{e})$ listed in Table 5.1, which depend on the orbital eccentricity. For $\varepsilon = 0$ equations (7.1) can be integrated as

$$\begin{aligned} y(t) &= y(0) \\ x(t) &= x(0) + y(0)t \ ; \end{aligned}$$

henceforth, the motion takes place on a plane in the phase space $\mathbf{T}^2 \times \mathbf{R}$, labeled by the initial condition $y(0)$. The value $y(0)$ coincides with the frequency (or rotation number) $\omega = \omega(y)$ of the motion, which in general is defined as the first derivative of the unperturbed Hamiltonian: $\omega(y) = \frac{dh(y)}{dy}$. Let us fix an irrational frequency $\omega_0 = \omega(y(0))$; the surface $\{y(0)\} \times \mathbf{T}^2$ is invariant for the unperturbed system and we wonder whether for $\varepsilon \neq 0$ there still exists an invariant surface for the perturbed system with the same frequency as the unperturbed case. The answer is provided by KAM theory, which allows us to prove the persistence of invariant tori provided some generic conditions are satisfied.

In a general framework, let us consider a nearly–integrable Hamiltonian function with n degrees of freedom:

$$\mathcal{H}(\underline{y}, \underline{x}) = h(\underline{y}) + \varepsilon f(\underline{y}, \underline{x}) \ , \qquad \underline{y} \in \mathbf{R}^n \ , \quad \underline{x} \in \mathbf{T}^n \ ; \tag{7.3}$$

let $\underline{\omega} \equiv \frac{\partial h(\underline{y})}{\partial \underline{y}} \in \mathbf{R}^n$ be the frequency vector. The first assumption required by KAM theory concerns a non–degeneracy of the unperturbed Hamiltonian. More precisely, let us introduce the following notions.

(i) An n–dimensional Hamiltonian function $h = h(\underline{y})$, $\underline{y} \in V$, being V an open subset of $\mathbf{R}^n$, is said to be *non–degenerate* if

$$\det\left(\frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2}\right) \neq 0 \qquad \text{for any } \ \underline{y} \in V \subset \mathbf{R}^n \ . \tag{7.4}$$

Condition (7.4) is equivalent to require that the frequencies vary with the actions as

$$\det\left(\frac{\partial \underline{\omega}(\underline{y})}{\partial \underline{y}}\right) \neq 0 \qquad \text{for any } \underline{y} \in V \ .$$

The non–degeneracy condition guarantees the persistence of invariant tori with fixed frequency.

(*ii*) An n–dimensional Hamiltonian function $h = h(\underline{y})$, $\underline{y} \in V \subset \mathbf{R}^n$, is said to be *isoenergetically non–degenerate* if

$$\det \begin{pmatrix} \frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} & \frac{\partial h(\underline{y})}{\partial \underline{y}} \\ \frac{\partial h(\underline{y})}{\partial \underline{y}} & \underline{0} \end{pmatrix} \neq 0 \qquad \text{for any } \ \underline{y} \in V \subset \mathbf{R}^n \ . \tag{7.5}$$

This condition can be written as

$$\det \begin{pmatrix} \frac{\partial \underline{\omega}(\underline{y})}{\partial \underline{y}} & \underline{\omega} \\ \underline{\omega} & \underline{0} \end{pmatrix} \neq 0 \qquad \text{for any } \ \underline{y} \in V \subset \mathbf{R}^n \ .$$

The isoenergetic non–degeneracy condition, which is independent of the non–degeneracy condition (7.4), guarantees that the frequency ratio of the invariant tori varies as one crosses the tori on fixed energy surfaces (see [6]).

(*iii*) An n–dimensional Hamiltonian function $\mathcal{H}(\underline{y}, \underline{x}) = h(\underline{y}) + \varepsilon f(\underline{y}, \underline{x})$, $\underline{y} \in \mathbf{R}^n$, $\underline{x} \in \mathbf{T}^n$, is said to be *properly degenerate* if the unperturbed Hamiltonian $h(\underline{y})$ does not depend explicitly on some action variables. In this case, the perturbation $f(\underline{y}, \underline{x})$ is said to remove the degeneracy if it can be split as the sum of two functions, say $f(\underline{y}, \underline{x}) = \bar{f}(\underline{y}) + \varepsilon f_1(\underline{y}, \underline{x})$ with the property that $h(\underline{y}) + \varepsilon \bar{f}(\underline{y})$ is non–degenerate.

In order to apply KAM theory it will be assumed that the unperturbed Hamiltonian satisfies (7.4) or (7.5). Beside non–degeneracy, the second requirement for the applicability of the KAM theorem is that the frequency $\underline{\omega}$ satisfies a strong irrationality assumption, namely the so–called diophantine condition which is defined as follows.

Definition. The frequency vector $\underline{\omega}$ satisfies a diophantine condition of type (C, τ) for some $C \in \mathbf{R}_+$, $\tau \geq 1$, if for any integer vector $\underline{m} \in \mathbf{R}^n \backslash \{\underline{0}\}$:

$$|\underline{\omega} \cdot \underline{m}| \geq \frac{1}{C |\underline{m}|^\tau} \ . \tag{7.6}$$

Under the non–degeneracy condition, the KAM theorem guarantees the persistence of invariant tori with diophantine frequency, provided the perturbing parameter is sufficiently small. More precisely, Kolmogorov [105] stated the following

Theorem (Kolmogorov). *Given the Hamiltonian system* (7.3) *satisfying the non–degeneracy condition* (7.4)*, having fixed a diophantine frequency* $\underline{\omega}$ *for the unperturbed system, if* ε *is sufficiently small there still exists an invariant torus on which the motion is quasi–periodic with frequency* $\underline{\omega}$.

The theorem was later proved in different settings by V.I. Arnold [2] and J. Moser [138] and it is nowadays known by the acronym: the KAM theorem. Qualitatively, we can state that for low values of the perturbing parameter there exists an invariant surface with diophantine frequency $\underline{\omega}$; as the perturbing parameter increases the invariant torus with frequency $\underline{\omega}$ is more and more distorted and displaced, until the parameter reaches a critical value at which the torus breaks down (compare with Figure 7.1). The KAM theorem provides a lower bound on the breakdown threshold; effective KAM estimates, together with a *computer–assisted* implementation,

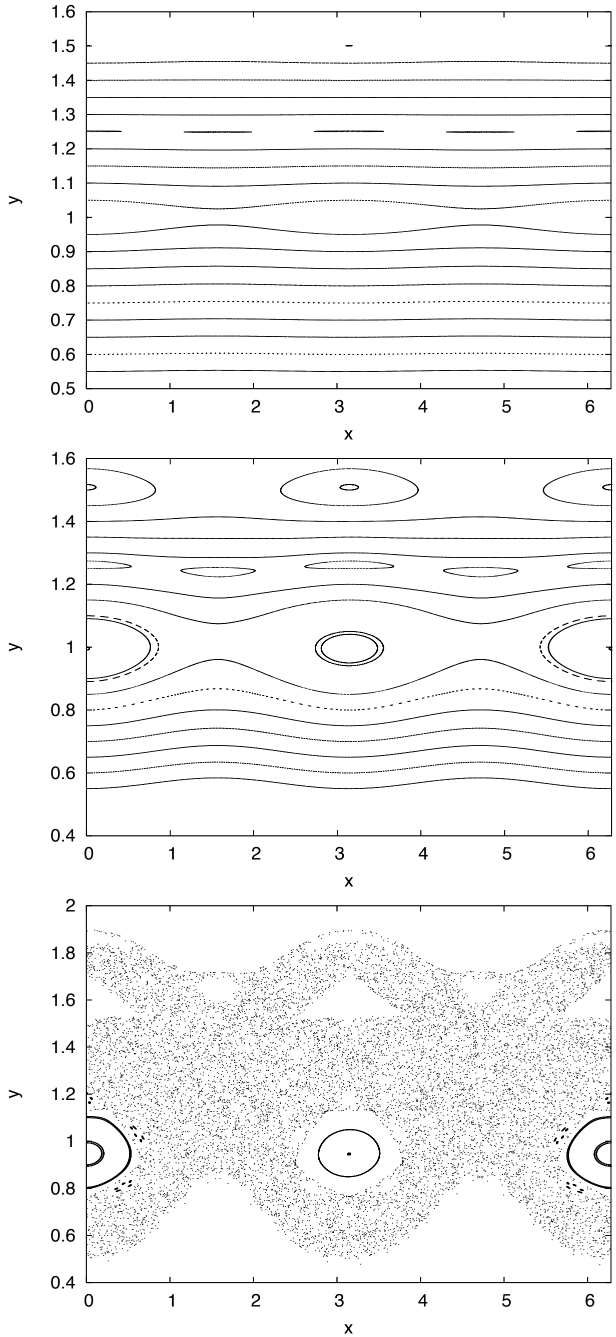

Fig. 7.1. The Poincarè section of the spin–orbit problem (5.19) for 20 different initial conditions and for e = 0.1. Top: $\varepsilon = 10^{-3}$, middle: $\varepsilon = 10^{-2}$, bottom: $\varepsilon = 10^{-1}$.

can provide, in simple examples, results on the parameters which are consistent with the physical values.

Section 7.2 will be devoted to the development of explicit estimates for the specific example of the spin–orbit model given by (7.1) with the perturbing function as in (7.2). Its unperturbed Hamiltonian satisfies the non–degeneracy condition (7.4), being $\frac{\partial^2 h(y)}{\partial y^2} = 1$. To apply the KAM theorem it is required that the frequency of the motion, say ω, satisfies a diophantine condition of type (C, τ) for some $C \in \mathbf{R}_+$, $\tau \geq 1$; therefore we assume that for any integers p and q, relatively coprime, with $q \neq 0$, the following inequality is satisfied:

$$\left| \omega - \frac{p}{q} \right| \geq \frac{1}{C|q|^{\tau+1}} \ . \tag{7.7}$$

An example of a diophantine number satisfying (7.7) with $\tau = 1$ is provided by the golden ratio $\gamma = \frac{\sqrt{5}-1}{2}$, for which (7.7) is fulfilled with the best diophantine constant given by $C = \frac{3+\sqrt{5}}{2}$. Being the system described by a one–dimensional, time–dependent Hamiltonian function, the existence of two invariant tori obtained through the KAM theorem provides a strong stability property, since the motion remains confined between such surfaces. We remark that this property is still valid for a two–dimensional system, since the phase space is four–dimensional and the two–dimensional KAM tori separate the constant energy surfaces into invariant regions. On the other hand, the confinement property is no longer valid whenever the Hamiltonian system has more than two degrees of freedom.

7.2 KAM theory

We present a version of the celebrated KAM theory by providing concrete estimates in the specific case of the spin–orbit model, following the KAM proof given in [31] to which we refer for further details (see also [28]). The goodness of the method strongly depends on the choice of the initial approximation which can be explicitly computed as a suitable truncation of the Taylor series expansion in the perturbing parameter. We also discuss how to choose the (irrational) rotation number, among those satisfying the diophantine condition. In order to obtain optimal results, it is convenient to use a computer to determine the initial approximation as well as to check the estimates provided by the theorem. The so–called interval arithmetic technique allows us to keep control of the numerical errors introduced by the machine. We also review classical and computer–assisted results of KAM applications in Celestial Mechanics.

7.2.1 The KAM theorem

The spin–orbit Hamiltonian associated to (7.1) can be written as the nearly–integrable Hamiltonian function

$$\mathcal{H}(y, x, t) = \frac{y^2}{2} + \varepsilon f(x, t) \ , \tag{7.8}$$

where $y \in \mathbf{R}$, $(x,t) \in \mathbf{T}^2$, the perturbing function $f = f(x,t)$ is assumed to be a periodic analytic function and the positive real number ε represents the perturbing parameter. Hamilton's equations associated to (7.8) can be written as the second–order differential equation

$$\ddot{x} + \varepsilon f_x(x,t) = 0 \ . \tag{7.9}$$

Definition. A KAM torus for (7.9) with rotation number ω is a two–dimensional invariant surface, described parametrically by

$$x = \vartheta + u(\vartheta,t) \ , \quad (\vartheta,t) \in \mathbf{T}^2 \ , \tag{7.10}$$

where $u = u(\vartheta,t)$ is a suitable analytic periodic function such that

$$1 + u_\vartheta(\vartheta,t) \neq 0 \quad \text{for all } (\vartheta,t) \in \mathbf{T}^2 \tag{7.11}$$

and where the flow in the parametric coordinate is linear, namely $\dot{\vartheta} = \omega$.

Notice that the requirement (7.11) ensures that (7.10) is a diffeomorphism. In this Section we want to prove the following KAM result.

Theorem. *Given the spin–orbit Hamiltonian (7.8) and having fixed for the unperturbed system a diophantine frequency ω satisfying (7.7), if ε is sufficiently small there still exists a KAM torus with frequency ω.*

Let us introduce the partial derivative operator D as

$$D \equiv \omega \frac{\partial}{\partial \vartheta} + \frac{\partial}{\partial t} \ . \tag{7.12}$$

We remark that for any function $g = g(\vartheta,t)$ the inversion of the operator D provides

$$(D^{-1}g)(\vartheta,t) = \sum_{(n,m)\in \mathbf{Z}^2 \setminus \{\underline{0}\}} \frac{\hat{g}_{nm}}{i(\omega n + m)} \ e^{i(n\vartheta + mt)} \ ,$$

which provokes the appearance of the small divisors $\omega n + m$. Notice that from the second equation in (7.1) we obtain that

$$y = \omega + Du(\vartheta,t) \ .$$

Inserting the parametrization (7.10) in (7.9) and using the definition (7.12), one obtains that the function u must satisfy the differential equation

$$D^2 u(\vartheta,t) + \varepsilon f_x(\vartheta + u(\vartheta,t), t) = 0 \ . \tag{7.13}$$

To prove the existence of an invariant surface with rotation number ω is equivalent to find a solution of equation (7.13). This goal is achieved by implementing a Newton's method as follows. Let $v = v(\vartheta,t)$ be an approximate solution of (7.13) with an error term $\eta = \eta(\vartheta,t)$:

$$D^2 v(\vartheta,t) + \varepsilon f_x(\vartheta + v(\vartheta,t), t) = \eta(\vartheta,t) \ . \tag{7.14}$$

We assume that $\mathcal{M} \equiv 1 + v_\vartheta(\vartheta, t) \neq 0$ for all $(\vartheta, t) \in \mathbf{T}^2$. We want to determine a new approximate solution $v' = v'(\vartheta, t)$ which satisfies (7.13) with an error $\eta' = \eta'(\vartheta, t)$ quadratically smaller, namely

$$D^2 v'(\vartheta, t) + \varepsilon f_x(\vartheta + v'(\vartheta, t), t) = \eta'(\vartheta, t) \ , \tag{7.15}$$

where $|\eta'| = O(|\eta|^2)$. This task can be accomplished through the following Lemma (see [26]).

Lemma (New approximation). *Let z be a solution of the equation*

$$D(\mathcal{M}^2 Dz) = -\mathcal{M}\eta \ . \tag{7.16}$$

Let

$$w \equiv \mathcal{M}z \ , \qquad v' \equiv v + w \ ;$$

then v' satisfies (7.15) *with*

$$\eta' = \eta_\vartheta z + q_1 \tag{7.17}$$

and

$$q_1 = \varepsilon f_x(\vartheta + v + w, t) - \varepsilon f_x(\vartheta + v, t) - \varepsilon f_{xx}(\vartheta + v, t) w \ . \tag{7.18}$$

Proof. We first remark that taking the derivative of (7.14) with respect to ϑ one has

$$D^2\mathcal{M} + \varepsilon f_{xx}(\vartheta + v, t)\mathcal{M} = \eta_\vartheta \ . \tag{7.19}$$

By (7.15) and (7.17) one has

$$D^2 v + D^2(\mathcal{M}z) + \varepsilon f_x(\vartheta + v, t) + \varepsilon f_{xx}(\vartheta + v, t)\mathcal{M}z = \eta_\vartheta z \ ;$$

using (7.19) and (7.14), one obtains

$$D^2(\mathcal{M}z) - (D^2\mathcal{M})\ z = -\eta \ . \tag{7.20}$$

Multiplying (7.20) by $\mathcal{M}$ one can easily recognize that the function z must solve (7.16). □

The solution z is obtained from (7.16) in the form

$$z \equiv D^{-1}\Big(\mathcal{M}^{-2}[c_0 - D^{-1}(\mathcal{M}\eta)]\Big) + c_1 \ , \tag{7.21}$$

where c_0 and c_1 are suitable constants which take the following expressions:

$$\begin{aligned} c_0 &\equiv \langle \mathcal{M}^{-2}\rangle^{-1}\ \langle \mathcal{M}^{-2}D^{-1}(\mathcal{M}\eta)\rangle \\ c_1 &\equiv -\langle \mathcal{M}^{-1}\rangle \langle \mathcal{M}D^{-1}\Big(\mathcal{M}^{-2}[c_0 - D^{-1}(\mathcal{M}\eta)]\Big)\rangle \ , \end{aligned} \tag{7.22}$$

so that w has zero average. Let us introduce the complex domain

$$\Delta_{\xi,\rho} \equiv \{(\vartheta, t, \varepsilon) \in \mathbf{C}^3 \ : \ |\mathrm{Im}(\vartheta)| \leq \xi \ , \ \ |\mathrm{Im}(t)| \leq \xi \ , \ \ |\varepsilon| \leq \rho\} \ ;$$

then, for a function $g = g(\vartheta, t; \varepsilon)$ we define the norm

$$\|g\|_{\xi,\rho} \equiv \sup_{\Delta_{\xi,\rho}} |g(\vartheta, t; \varepsilon)| \ .$$

Now we need a technical lemma which provides bounds on the derivatives of a function $g = g(\vartheta, t; \varepsilon)$, whose Fourier series expansion is given by $g(\vartheta, t; \varepsilon) = \sum_{(n,m)\in\mathbf{Z}^2} \hat{g}_{nm} e^{i(n\vartheta + mt)}$.

Lemma (Bounds on derivatives). *Let $g = g(\vartheta, t; \varepsilon)$ be an analytic function on the domain $\Delta_{\xi,\rho}$. Then, for any $0 < \delta \le \xi$, one has*

$$\|g_\vartheta\|_{\xi-\delta,\rho} \le \|g\|_{\xi,\rho}\ \delta^{-1}\ . \tag{7.23}$$

Moreover, if $\langle g\rangle = 0$ and ∂_ϑ^ℓ denotes the derivative of order ℓ with respect to ϑ, then for $\ell = 0, 1$,

$$\|\partial_\vartheta^\ell D^{-1} g\|_{\xi-\delta,\rho} \le \sigma_\ell(2\delta)\|g\|_{\xi,\rho}\ ,$$

where

$$\sigma_\ell(\delta) \equiv 2\left[\sum_{(n,m)\in\mathbf{Z}^2\setminus\{\underline{0}\}} \left(\frac{|n|^\ell}{\omega n + m}\right)^2 e^{-\delta(|n|+|m|)}\right]^{1/2}\ . \tag{7.24}$$

Proof. Given a holomorphic function $g = g(\vartheta, t; \varepsilon)$ defined on $\Delta_{\xi,\rho}$, the estimate (7.23) is obtained through Cauchy's integral formula, i.e.

$$\|g_\vartheta\|_{\xi-\delta,\rho} = \|\frac{1}{2\pi i}\oint_{|\vartheta-\gamma|=\delta}\frac{g(\gamma,t;\varepsilon)}{(\vartheta-\gamma)^2}\ d\gamma\|_{\xi-\delta,\rho} \le \|g\|_{\xi,\rho}\ \delta^{-1}\ .$$

Under the condition $\langle g\rangle = 0$, from the maximum principle and Schwarz inequality one obtains

$$\begin{aligned}
\|\partial_\vartheta^\ell D^{-1} g\|_{\xi-\delta,\rho} &= \left\|\sum_{(n,m)\in\mathbf{Z}^2\setminus\{\underline{0}\}} \hat g_{nm}\frac{n^\ell}{\omega n+m}\ e^{i(n\vartheta+mt)}\right\|_{\xi-\delta,\rho}\\
&\le \sup_{|\varepsilon|\le\rho}\sum_{k_1,k_2\in\{-1,1\}}\left|\sum_{(n,m)\in\mathbf{Z}^2\setminus\{\underline{0}\}}\hat g_{nm}\frac{n^\ell}{\omega n+m}e^{(k_1 n+k_2 m)(\xi-\delta)}\right|\\
&\le \sup_{|\varepsilon|\le\rho}\sum_{(n,m)\in\mathbf{Z}^2\setminus\{\underline{0}\}}|\hat g_{nm}|\left(\sum_{k_1,k_2\in\{-1,1\}} e^{2(k_1 n+k_2 m)\xi}\right)^{\frac12} e^{-\delta(|n|+|m|)}\frac{|n|^\ell}{|\omega n+m|}\\
&\le \sigma_\ell(2\delta)\|g\|_{\xi,\rho}\ ,
\end{aligned}$$

with $\sigma_\ell(2\delta)$ defined according to (7.24). □

We introduce the quantities V, V_1, M, $\tilde M$, E, $s_\ell(\delta)$ as the following upper bounds:

$$\begin{aligned}
&\|v\|_{\xi,\rho} \le V\ , \qquad \|v_\vartheta\|_{\xi,\rho} \le V_1\ , \qquad \|\mathcal{M}\|_{\xi,\rho} \le M\ ,\\
&\|\mathcal{M}^{-1}\|_{\xi,\rho} \le \tilde M\ , \qquad \|\eta\|_{\xi,\rho} \le E\ , \qquad \|\sigma_\ell(\delta)\|_{\xi,\rho} \le s_\ell(\delta)\ .
\end{aligned}$$

One obtains that

$$\tilde M^{-2} \le \|\mathcal{M}^2\|_{\xi,\rho} \le M^2\ , \quad M^{-2} \le \|\mathcal{M}^{-2}\|_{\xi,\rho} \le \tilde M^2\ , \quad \|\langle\mathcal{M}^{-2}\rangle^{-1}\|_{\xi,\rho} \le M^2\ .$$

From (7.22), one finds that c_0, c_1 can be bounded as

$$\begin{aligned}
\|c_0\|_{\xi,\rho} &\le M^3\tilde M^2 s_0(2\xi)E\\
\|c_1\|_{\xi,\rho} &\le M\tilde M^3 s_0(\xi)\Big[M^3\tilde M^2 s_0(2\xi)E + M s_0(\xi)E\Big]\ .
\end{aligned}$$

Having introduced the quantities

$$\begin{aligned} a &\equiv (M\tilde{M}s_0(\delta))^2 \Bigg[1 + (M\tilde{M})^2 \frac{s_0(2\xi)}{s_0(\delta)} \\ &+ M\tilde{M}\left(\frac{s_0(\xi)}{s_0(\delta)}\right)^2 \left(1 + (M\tilde{M})^2 \frac{s_0(2\xi)}{s_0(\xi)}\right)\Bigg] \\ b &\equiv \frac{aV_1}{M}\delta^{-1} + a\frac{s_1(\delta)}{s_0(\delta)} \ , \end{aligned}$$

from the definition of z in (7.21) one finds the following bounds W on w and W_1 on the derivative of w with respect to ϑ:

$$\begin{aligned} \|w\|_{\xi-\delta,\rho} &\leq Ea \equiv W \\ \|w_\vartheta\|_{\xi-\delta,\rho} &\leq Eb \equiv W_1 \ . \end{aligned}$$

The first inequality follows from $\|w\|_{\xi-\delta,\rho} \leq M\|z\|_{\xi-\delta,\rho}$ and from the estimate

$$\|z\|_{\xi-\delta,\rho} \leq \|c_1\|_{\xi,\rho} + s_0(\delta)\tilde{M}^2(\|c_0\|_{\xi,\rho} + s_0(\delta)ME) \ .$$

Similar computations hold for $\|w_\vartheta\|_{\xi-\delta,\rho}$. Finally, from (7.17) and (7.18) one obtains a bound E_1 on the new error term as

$$\|\eta'\|_{\xi-\delta,\rho} \leq E^2\left(\frac{a\delta^{-1}}{M} + \frac{a^2F}{2}\right) \equiv E_1 \ ,$$

where $F \equiv \|\varepsilon f_{xxx}\|_{\xi-\delta+V+W,\rho}$.

Let us assume that we start from a given initial approximation $v^{(0)}$ satisfying (7.14) with an error term $\eta^{(0)}$; we construct the solution at the jth step, say $v^{(j)}$, by an iterative application of the *New approximation Lemma* starting from the initial solution $v^{(0)}$. Let $M^{(j)}$, $\tilde{M}^{(j)}$, $E^{(j)}$, $W^{(j)}$, $W_1^{(j)}$ be the bounds corresponding to the solution $v^{(j)}$. From the previous estimates and definitions, the bounds for the solution $v^{(j+1)}$ are obtained through the following Lemma which provides the KAM algorithm needed to construct bounds on the new approximate solution.

Lemma (KAM algorithm). *Let $\xi_0 > 0$, $\xi_j \equiv \frac{\xi_0}{2^j}$ and let $\delta_j \equiv \frac{\xi_0}{2^{j+1}}$. Given the following quantities referring to the solution $v^{(j)}$ on the domain with parameters ξ_j, δ_j: $M^{(j)}$, $\tilde{M}^{(j)}$, $E^{(j)}$, $W^{(j)}$, $W_1^{(j)}$, we define the bounds corresponding to the solution $v^{(j+1)}$ as follows:*

$$
\begin{aligned}
M^{(j+1)} &\equiv M^{(j)} + W_1^{(j)} \\
\tilde{M}^{(j+1)} &\equiv \tilde{M}\left(1 - \tilde{M}\sum_{i=0}^{j} W_1^{(i)}\right)^{-1} && \text{if } \sum_{i=0}^{j} W_1^{(i)} < 1 \\
\tilde{M}^{(j+1)} &\equiv \infty && \text{if } \sum_{i=0}^{j} W_1^{(i)} \geq 1 \\
E^{(j+1)} &\equiv (E^{(j)})^2 \left(\frac{a^{(j)}\delta_j^{-1}}{M^{(j)}} + \frac{(a^{(j)})^2 F}{2}\right) \\
W^{(j+1)} &\equiv E^{(j+1)} a^{(j+1)} \\
W_1^{(j+1)} &\equiv E^{(j+1)} b^{(j+1)} \ .
\end{aligned}
$$

One can iterate the above algorithm for a finite number of steps; the convergence to the true solution of equation (7.13) is obtained once a suitable *KAM condition* is satisfied. To this end, let us premise the following Lemma which provides a bound on the quantity $\sigma_\ell(\delta)$ introduced in (7.24).

Lemma (Bound on $\sigma_\ell(\delta)$). *Let $0 < \delta \leq \frac{1}{2}$; for $\ell = 0, 1$, if $k_\ell \equiv \tau + \ell + 1$, then*

$$\sigma_\ell(\delta) < K_\ell C \delta^{-k_\ell} \ , \tag{7.25}$$

where $K_0 \equiv \frac{25}{2}(\frac{\Gamma(2\tau+1)}{\pi})^{1/2}$, $K_1 \equiv K_0\sqrt{(2\tau+2)(2\tau+1)}$, with Γ being the Euler's gamma function.

Proof. For $t \geq 1$ and $0 < \delta \leq \frac{1}{2}$, one has

$$\sum_{n \in \mathbf{Z}} |n|^t e^{-\delta|n|} < 2e^{\frac{1}{2}}\Gamma(t+1)\delta^{-(t+1)} \ .$$

Being[1] $C > 2$ and $\tau \geq 1$, one finds

$$
\begin{aligned}
\sigma_\ell(\delta) &< 2\left(\sum_{m\neq 0}\frac{e^{-\delta|m|}}{m^2} + C^2\sum_{n\neq 0}|n|^{2\tau+2\ell}e^{-\delta|n|}\sum_m e^{-\delta|m|}\right) \\
&< 2\left(\frac{2}{\delta} + 2C^2(1+\sqrt{e})\sqrt{e}\ \Gamma(2(\tau+\ell)+1)\ \delta^{-2(\tau+\ell+1)}\right)^{\frac{1}{2}} \\
&< 2C(1+2(1+\sqrt{e})\sqrt{e})^{\frac{1}{2}}(\Gamma(2(\tau+\ell)+1))^{\frac{1}{2}}\delta^{-(\tau+\ell+1)} \ ,
\end{aligned}
$$

which gives (7.25). □

Finally, let v, η satisfy (7.14); for some $\xi_* > 0$, $\rho > 0$, let $E \equiv \|\eta\|_{\xi_*,\rho}$, $M \equiv \|\mathcal{M}\|_{\xi_*,\rho}$, $\tilde{M} \equiv \|\mathcal{M}^{-1}\|_{\xi_*,\rho}$, $F \equiv \|f_{xxx}\|_{\xi_*+V,\rho}$. The convergence of the sequence of approximate solutions to the solution of (7.13) is obtained through the following result, which gives the persistence of the invariant torus with diophantine frequency ω, provided ε is sufficiently small (compare with (7.28) below).

[1] The smallest value of the diophantine constant corresponds to the golden ratio $\frac{\sqrt{5}-1}{2}$ and it amounts to $C \equiv \frac{3+\sqrt{5}}{2} \simeq 2.618$.

Proposition (KAM condition). *Let* $\xi_* > 0$, $\rho > 0$ *and let* β_0, β_1, β_2, η_0, η_1, η_2 *be positive constants defined as follows:*

$$\begin{aligned}
\beta_0 &\equiv \left(M\tilde{M}\ K_0 C\left(\frac{4}{\xi_*}\right)^{k_0}\right)^2 \left[1+(M\tilde{M})^2\frac{1}{8^{k_0}}+M\tilde{M}\left(\frac{1}{4}\right)^{2k_0}\left(1+(M\tilde{M})^2\frac{1}{2^{k_0}}\right)\right] \\
\beta_1 &= (M\tilde{M}C)^2 2^{4k_0+3}\xi_*^{-2k_0-1}K_1K_0 \\
&\quad \cdot \left[1+(M\tilde{M})^2\frac{1}{8^{k_0}}+M\tilde{M}\left(\frac{1}{4}\right)^{2k_0}\left(1+(M\tilde{M})^2\frac{1}{2^{k_0}}\right)\right] \\
\beta_2 &= \frac{4\beta_0}{\xi_*}+\frac{\beta_0^2 F}{2} \\
\eta_0 &= 2^{2k_0}\ , \qquad \eta_1 = 2^{2k_0+1}\ , \qquad \eta_2 = \max(2\eta_0,\eta_0^2)\ .
\end{aligned} \tag{7.26}$$

Defining

$$\mathcal{K} \equiv 2\tilde{M}\beta_1(1+2\eta_1\beta_2\eta_2)\ , \tag{7.27}$$

if

$$\mathcal{K}\ E\ <\ 1\ , \tag{7.28}$$

then (7.13) *has a unique solution* u*, with* $\langle u\rangle = \langle v\rangle$ *and*

$$\begin{aligned}
\|u-v\|_{\frac{\xi_*}{2},\rho} &< \mathcal{K}E\frac{\xi_*}{4} \\
\|u_\vartheta - v_\vartheta\|_{\frac{\xi_*}{2},\rho} &< \frac{\mathcal{K}E}{2\tilde{M}}\ .
\end{aligned} \tag{7.29}$$

Proof. Define the sequences $\{\xi_*^{(j)}\}$, $\{\delta_j\}$, $j \in \mathbf{Z}_+$, as $\xi_*^{(j)} = \frac{\xi_*}{2}+\frac{\xi_*}{2^{j+1}}$, $\delta_j = \frac{\xi_*}{2^{j+2}}$. Under the assumption (7.28), for a suitable $\mathcal{K}_0 < \mathcal{K}$ one has the following relations, valid for any $j \geq 0$:

$$\begin{aligned}
E^{(j)} &< (\mathcal{K}_0 E)^{2^j} \\
\xi_*^{(j)} + V^{(j)} &\leq \xi_* + V \\
\tilde{M}^{(j)} &\leq 2\tilde{M}\ ,
\end{aligned} \tag{7.30}$$

where $V^{(j)}$ is an upper bound on $v^{(j)}$. The first of (7.30) implies that the sequence of the error terms $\{E^{(j)}\}_{j\in\mathbf{Z}_+}$ converges to zero. Moreover, from the second of (7.30) we get that the sequence of approximate solutions $\{v^{(j)}\}_{j\in\mathbf{Z}_+}$ tends to a unique solution u. The third equation in (7.30) is equivalent to

$$\tilde{M}\sum_{i=0}^{j-1} W_1^{(i)} \leq 1\ . \tag{7.31}$$

The proof of the validity of (7.30) and (7.31) can be done by induction on j. It is readily seen that these relations are valid for $j = 0$. Assume they are true for $1,..,j$; we want to prove that (7.30) and (7.31) are valid for $j+1$. We first show that the following inequalities hold:

$$\begin{aligned} E^{(i+1)} &\le (E^{(i)})^2 \beta_2 \eta_2^i \\ W^{(i)} &\le E^{(i)} \beta_0 \eta_0^i \\ W_1^{(i)} &\le E^{(i)} \beta_1 \eta_1^i \ , \end{aligned} \tag{7.32}$$

where the real constants β_0, β_1, β_2, η_0, η_1, η_2 are defined as in (7.26). Let $A^{(i)} \equiv \beta_0 \eta_0^i$; we prove the first in (7.32) through the following chain of inequalities:

$$\begin{aligned} E^{(i+1)} &\le (E^{(i)})^2 \left(A^{(i)} \frac{2^{i+2}}{\xi_*} + \frac{(A^{(i)})^2 F}{2} \right) \\ &\le (E^{(i)})^2 \left(\frac{4\beta_0 (2\eta_0)^i}{\xi_*} + \frac{\beta_0^2 \eta_0^{2i} F}{2} \right) \\ &\le (E^{(i)})^2 \beta_2 \eta_2^i \ . \end{aligned}$$

Concerning the second relation in (7.32) one has

$$W^{(i)} \le E^{(i)} A^{(i)} = E^{(i)} \beta_0 \eta_0^i \ .$$

Finally, the third inequality in (7.32) is obtained as follows:

$$\begin{aligned} W_1^{(i)} &\le E^{(i)} A^{(i)} \frac{2^{i+2}}{\xi_*} \left(1 + \frac{K_1}{K_0} \right) \\ &\le E^{(i)} \beta_1 \eta_1^i \ . \end{aligned}$$

The first relation in (7.32) yields the first in (7.30): setting

$$\mathcal{K}_0 \equiv \beta_2 \eta_2 \ ,$$

one has

$$E^{(j+1)} \le E^{2^{j+1}} \prod_{i=0}^{j} (\beta_2 \eta_2^{j-i})^{2^i} = E^{2^{j+1}} \left[\beta_2^{\sum_{i=1}^{j+1} \frac{1}{2^i}} \ \eta_2^{\sum_{i=1}^{j+1} \frac{i-1}{2^i}} \right]^{2^{j+1}} < (\mathcal{K}_0 E)^{2^{j+1}} \ .$$

Let $\mathcal{K}$ satisfy the inequality

$$\sqrt{2^5 \beta_0 \eta_0 \xi_*^{-1}} \ \mathcal{K}_0 \le \mathcal{K} \ , \tag{7.33}$$

from the second relation in (7.32) and from (7.28) we obtain

$$\begin{aligned} \sum_{i=0}^{j} W^{(i)} &< \beta_0 E + \beta_0 \sum_{i=1}^{\infty} \eta_0^i (\mathcal{K}_0 E)^{2^i} \\ &< \beta_0 E + \beta_0 (\mathcal{K}_0 E)^2 \eta_0 \left(1 + \frac{1}{\log \frac{1}{\mathcal{K}_0 E \sqrt{\eta_0}}} \right) \\ &< \mathcal{K} E \frac{\xi_*}{4} < \xi_* \left(\frac{1}{2} - \frac{1}{2^{j+2}} \right) \ , \end{aligned} \tag{7.34}$$

due to the following estimates:

$$\beta_0 E < \frac{\xi_*}{2^5}\mathcal{K}_0 E \ , \qquad \beta_0(\mathcal{K}_0 E)^2\eta_0 \le \frac{(\mathcal{K}E)^2\xi_*}{2^5} \ , \qquad \mathcal{K}_0 E\sqrt{\eta_0} < \frac{\mathcal{K}E}{2^{10}} \ .$$

Since

$$V^{(j+1)} \equiv V + \sum_{i=0}^{j} W^{(i)} \ ,$$

one obtains the second of (7.30). From the third in (7.32) and from $\mathcal{K}_0 E\sqrt{\eta_1} < \frac{\mathcal{K}E}{2^9}$ we get

$$\begin{aligned} 2\tilde{M}\sum_{i=0}^{j} W_1^{(i)} &= 2\tilde{M}\sum_{i=0}^{j} E^{(i)}\beta_1\eta_1^i \\ &< 2\tilde{M}\beta_1 E + 2\tilde{M}\beta_1(\mathcal{K}_0E)^2\eta_1\left(1+\frac{1}{\log\frac{1}{\mathcal{K}_0E\sqrt{\eta_1}}}\right) \\ &< 2\tilde{M}\beta_1 E + 4\tilde{M}\beta_1(\mathcal{K}_0E)^2\eta_1 \ ; \end{aligned} \qquad (7.35)$$

if

$$2\tilde{M}\beta_1 + 4\tilde{M}\beta_1\eta_1\mathcal{K}_0 \le \mathcal{K} \ , \qquad (7.36)$$

one obtains (7.31). Notice that $\mathcal{K}$ is determined by the inequalities (7.33) and (7.36); these inequalities are satisfied provided

$$\mathcal{K} \equiv \max\{\sqrt{2^5\beta_0\eta_0\xi_*^{-1}}\beta_2\eta_2, 2\tilde{M}\beta_1(1+2\eta_1\beta_2\eta_2)\} \ ,$$

which is equivalent to (7.27). Finally, (7.34) and (7.35) imply (7.29). □

Remark. Let us consider the general case of a Hamiltonian function with n degrees of freedom:

$$\mathcal{H}(\underline{y},\underline{x}) = h(\underline{y}) + \varepsilon f(\underline{y},\underline{x}) \ , \qquad \underline{y}\in\mathbf{R}^n \ , \ \underline{x}\in\mathbf{T}^n \ .$$

The equations of motion are

$$\begin{aligned} \dot{\underline{x}} &= h_{\underline{y}}(\underline{y}) + \varepsilon f_{\underline{y}}(\underline{y},\underline{x}) \\ \dot{\underline{y}} &= -\varepsilon f_{\underline{x}}(\underline{y},\underline{x}) \ . \end{aligned} \qquad (7.37)$$

A KAM torus with rotation vector $\underline{\omega}$ is defined by the parametric equations

$$\begin{aligned} \underline{x}(\underline{\vartheta}) &= \underline{\vartheta} + \underline{u}(\underline{\vartheta}) \\ \underline{y}(\underline{\vartheta}) &= \underline{v}(\underline{\vartheta}) \ , \end{aligned} \qquad (7.38)$$

where $\underline{\vartheta}\in\mathbf{T}^n$ with $\dot{\underline{\vartheta}} = \underline{\omega}$ and $\underline{u}$, $\underline{v}$ are suitable vector functions. Let us introduce the operator $D \equiv \underline{\omega}\ \frac{\partial}{\partial\underline{\vartheta}}$. Inserting (7.38) in (7.37), one finds that $\underline{u}$ and $\underline{v}$ must satisfy the following quasi–linear partial differential equations on $\mathbf{T}^n$:

$$\begin{aligned} \underline{\omega} + D\underline{u} - h_{\underline{y}}(\underline{v}) - \varepsilon f_{\underline{y}}(\underline{v},\underline{\vartheta}+\underline{u}) &= \underline{0} \\ D\underline{v} + \varepsilon f_{\underline{x}}(\underline{v},\underline{\vartheta}+\underline{u}) &= \underline{0} \ . \end{aligned} \qquad (7.39)$$

The KAM proof is obtained by solving (7.39) through a Newton iteration method, extending the procedure as it was described for finding the solution of (7.13).

7.2.2 The initial approximation and the estimate of the error term

The initial approximation $v \equiv v^{(0)}$ (see (7.14)) of the KAM theorem can be obtained taking advantage of the analyticity of the KAM surfaces with respect to the perturbing parameter in a neighborhood of the origin [139–141]). We consider the parametrization (7.10), where the function $u = u(\vartheta, t)$ depends parametrically on ε and therefore we denote it as $u = u(\vartheta, t; \varepsilon)$. Let us expand u in power series as

$$u(\vartheta, t; \varepsilon) \;=\; \sum_{k=1}^{\infty} u_k(\vartheta, t)\ \varepsilon^k \ . \tag{7.40}$$

In the case of the spin–orbit problem the coefficients u_k can be recursively computed as follows. Write equation (7.13) with the perturbation given by (7.2) as

$$D^2 u \ + \ \varepsilon \sum_{m\neq 0, m=N_1}^{N_2} W\left(\frac{m}{2}, \mathrm{e}\right)\ \sin(2\vartheta + 2u - mt) \;=\; 0 \ . \tag{7.41}$$

For u expanded as in (7.40), define the power series

$$e^{i(2\vartheta+2u)} \;\equiv\; \sum_{n=0}^{\infty} c_n(\vartheta, t)\ \varepsilon^n \ , \tag{7.42}$$

for some unknown complex coefficients c_n which can be determined as follows. Differentiating (7.42) with respect to ε and using the series expansion (7.40), one obtains

$$2i \sum_{k=1}^{\infty} k u_k \varepsilon^{k-1} \ \cdot \sum_{j=0}^{\infty} c_j \varepsilon^j \;=\; \sum_{n=1}^{\infty} n c_n \varepsilon^{n-1} \ .$$

Equating same powers of ε one obtains:

$$\begin{aligned} c_0(\vartheta, t) &\;\equiv\; e^{2i\vartheta} \\ c_n(\vartheta, t) &\;\equiv\; \frac{2i}{n} \sum_{k=1}^{n} k u_k c_{n-k} \ . \end{aligned} \tag{7.43}$$

Finally, (7.41) can be written as

$$D^2 u = -\frac{1}{2i} \sum_{n=1}^{\infty} \varepsilon^n \left[\sum_{m\neq 0, m=N_1}^{N_2} W\left(\frac{m}{2}, \mathrm{e}\right)\ (e^{-imt} c_{n-1} \ - \ e^{imt} \overline{c}_{n-1}) \right] \ ,$$

where the bar denotes complex conjugacy. A recursive relation defining the functions u_n is obtained comparing the terms of the same order in ε:

$$u_n(\vartheta, t) \;\equiv\; -\frac{1}{2i}\, D^{-2} \left[\sum_{m\neq 0, m=N_1}^{N_2} W\left(\frac{m}{2}, \mathrm{e}\right)\ (e^{-imt} c_{n-1} \ - \ e^{imt} \overline{c}_{n-1}) \right] \ . \tag{7.44}$$

Notice that u_n depends on the previous functions $u_1, \ldots, u_{n-1}$. The initial approximation can be obtained as the finite truncation up to a suitable order k_0 (for some positive integer k_0) of the series expansion (7.40):

$$v^{(0)}(\vartheta, t; \varepsilon) \equiv \sum_{k=1}^{k_0} u_k(\vartheta, t)\, \varepsilon^k . \tag{7.45}$$

To give a concrete example, let us assume that the perturbing function in (7.2) is given by

$$\begin{aligned} f(x,t) \equiv\ & \frac{\mathrm{e}}{4}\cos(2x-t) - \left(\frac{1}{2} - \frac{5}{4}\mathrm{e}^2\right)\cos(2x-2t) \\ & - \frac{7}{4}\mathrm{e}\cos(2x-3t) - \frac{17}{4}\mathrm{e}^2\cos(2x-4t) . \end{aligned}$$

Then, the first two approximating functions $u_1(\vartheta, t)$ and $u_2(\vartheta, t)$ are given by the following expressions:

$$\begin{aligned} u_1(\vartheta,t) =\ & \frac{-\mathrm{e}}{2(2\omega-1)^2}\sin(2\vartheta-t) + \frac{(1-\frac{5}{2}\mathrm{e}^2)}{(2\omega-2)^2}\sin(2\vartheta-2t) + \\ & + \frac{7\mathrm{e}}{2(2\omega-3)^2}\sin(2\vartheta-3t) + \frac{17\mathrm{e}^2}{2(2\omega-4)^2}\sin(2\vartheta-4t) \end{aligned}$$

and

$$\begin{aligned} u_2(\vartheta,t) =\ & \left[-\frac{\mathrm{e}}{2(2\omega-1)^2} + \frac{4\mathrm{e}}{(2\omega-2)^2} - \frac{7\mathrm{e}}{2(2\omega-3)^2}\right]\sin t + \\ & + \frac{1}{4}\left[-\frac{7\mathrm{e}^2}{4(2\omega-1)^2} + \frac{17\mathrm{e}^2}{2(2\omega-2)^2} + \frac{7\mathrm{e}^2}{4(2\omega-3)^2} - \frac{17\mathrm{e}^2}{2(2\omega-4)^2}\right]\sin 2t \\ & + \frac{\mathrm{e}^2}{4(2\omega-1)^2}\,\frac{\sin(4\vartheta-2t)}{(4\omega-2)^2} \\ & + \left[-\frac{\mathrm{e}}{2(2\omega-2)^2} - \frac{\mathrm{e}}{2(2\omega-1)^2}\right]\frac{\sin(4\vartheta-3t)}{(4\omega-3)^2} + \\ & + \left[\frac{1-5\mathrm{e}^2}{(2\omega-2)^2} - \frac{7\mathrm{e}^2}{4}\left(\frac{1}{(2\omega-1)^2} + \frac{1}{(2\omega-3)^2}\right)\right]\frac{\sin(4\vartheta-4t)}{(4\omega-4)^2} + \\ & + \left[\frac{7\mathrm{e}}{2(2\omega-2)^2} + \frac{7\mathrm{e}}{2(2\omega-3)^2} + \right]\frac{\sin(4\vartheta-5t)}{(4\omega-5)^2} + \\ & + \left[\frac{17\mathrm{e}^2}{2(2\omega-2)^2} + \frac{49\mathrm{e}^2}{4(2\omega-3)^2} + \frac{17\mathrm{e}^2}{2(2\omega-4)^2}\right]\frac{\sin(4\vartheta-6t)}{(4\omega-6)^2} . \end{aligned}$$

To implement the KAM algorithm and to check the KAM condition, it is necessary to provide explicit estimates on some quantities, like the initial approximation, its derivative, the error term, etc. The most difficult task is the estimate of the error function $|\eta^{(0)}|_{\xi,\rho}$ (for some positive parameters ξ, ρ) associated to a given initial approximation $v^{(0)}$, which can be constructed by means of the recursive formulae (7.43), (7.44). The estimate of $\eta^{(0)}$ can be obtained through the following Lemma (see also [27]).

Lemma (Estimate of the error term). *Let* $v^{(0)}(\vartheta,t;\varepsilon) \equiv \sum_{k=1}^{k_0} u_k(\vartheta,t)\ \varepsilon^k$ *for some positive integer* k_0 *and let* $\eta \equiv \eta^{(0)}$ *satisfy* (7.14) *with* $v \equiv v^{(0)}$*. For some positive parameters* ξ, ρ, *let* $S^{(0)} \equiv \|v^{(0)}\|_{\xi,\rho}$, $U_k \equiv \|u_k\|_{\xi,\rho}$ *and* $\bar{F} \equiv \|f_x\|_{\xi,\rho}$. *Define recursively the sequences* $\{\alpha_j\}$, $\{\beta_j\}$ *as*

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_j &= \frac{2}{j}\sum_{k=1}^{j} kU_k\alpha_{j-k}\ , \qquad j \geq 1 \end{aligned}$$

and

$$\begin{aligned} \beta_0 &= 1 \\ \beta_j &= -\frac{2}{j}\sum_{k=1}^{j} kU_k\beta_{j-k}\ , \qquad j \geq 1\ . \end{aligned}$$

Then, setting

$$\begin{aligned} a &= e^{2S^{(0)}} - \sum_{j=1}^{k_0-1} \alpha_j\rho^j \\ b &= e^{-2S^{(0)}} - \sum_{j=1}^{k_0-1} \beta_j\rho^j\ , \end{aligned}$$

the error term is estimated as

$$\|\eta^{(0)}\|_{\xi,\rho} = \bar{F}\sqrt{\frac{a^2+b^2}{2}}\ .$$

We remark that in concrete applications the convergence of the KAM algorithm is improved as the order k_0 of the initial approximation (7.45) gets larger. Indeed, let us denote by $\varepsilon_{KAM}^{(k_0)} = \varepsilon_{KAM}^{(k_0)}(\omega)$ the lower bound provided by the KAM theorem on the persistence of the invariant torus with frequency ω, starting from the initial approximation (7.45) truncated at the order k_0. We report in Table 7.1 some results associated to (5.19) for the frequency $\omega = 1+\frac{1}{2+\frac{\sqrt{5}-1}{2}}$; the results concern the values

Table 7.1. The threshold $\varepsilon_{KAM}^{(k_0)}(\omega)$ as a function of the order k_0 of the initial approximation.

k_0	$\varepsilon_{KAM}^{(k_0)}(\omega)$
1	$2 \cdot 10^{-5}$
5	$1.5 \cdot 10^{-3}$
10	$4.1 \cdot 10^{-3}$
15	$6 \cdot 10^{-3}$
20	$6.6 \cdot 10^{-3}$
25	$7.5 \cdot 10^{-3}$
30	$8.2 \cdot 10^{-3}$

$\varepsilon_{KAM}^{(k_0)}(\omega)$ as the order k_0 of the initial approximation increases (here we selected $\xi = 0.05$). We remark that the relative improvement of the threshold $\varepsilon_{KAM}^{(k_0)}(\omega)$ is higher as k_0 is small, while it gets smaller as k_0 increases.

7.2.3 Diophantine rotation numbers

One of the assumptions which is required to apply the KAM theorem is that the frequency of the motion must satisfy the diophantine condition (7.6). Moreover, we recall that the KAM estimates depend on the value of the diophantine constant (see, e.g., (7.26), (7.27), (7.28)) and a proper choice of the frequency certainly improves the performances of the theorem. In this section we review some results from number theory concerning the choice of diophantine numbers and the computation of the corresponding diophantine constants.

We start by introducing the *continued fraction* expansion of a positive real number α defined as the sequence of positive integer numbers $a_0, a_1, a_2, \ldots$, such that

$$\alpha \equiv a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} , \qquad a_j \in \mathbf{Z}_+ . \tag{7.46}$$

Using standard notation, we shall write

$$\alpha \equiv [a_0;\ a_1,\ a_2,\ a_3,\ \ldots] .$$

A rational number has a finite continued fraction expansion, while irrationals have an infinite continued fraction expansion. For any irrational number α there exists an infinite *approximant* sequence of rational numbers, say $\{\frac{p_n}{q_n}\}_{n\in\mathbf{Z}_+}$, such that $\frac{p_n}{q_n}$ converges to α as n goes to infinity. Each $\frac{p_n}{q_n}$ can be obtained as the truncation to the order n of the continued fraction expansion (7.46):

$$\begin{aligned}
\frac{p_0}{q_0} &= a_0 \\
\frac{p_1}{q_1} &= a_0 + \frac{1}{a_1} \\
\frac{p_2}{q_2} &= a_0 + \frac{1}{a_1 + \frac{1}{a_2}} \\
&\ldots
\end{aligned}$$

For the golden number $\gamma = \frac{\sqrt{5}-1}{2}$, the rational approximants are given by the ratio of the Fibonacci's numbers:

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$$

A bound on how close the rational numbers $\frac{p_n}{q_n}$ approximate α is given by the following inequalities:

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} .$$

Definition. An *algebraic* number ω is a solution of a polynomial $P_n(z)$ of degree n with integer coefficients, say $c_0, \ldots, c_n$:

$$P_n(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \ , \tag{7.47}$$

provided ω is not a solution of a polynomial of lower degree with integer coefficients. A *quadratic* number is an algebraic number of degree 2. An irrational number α is called a *noble number* if the terms of its continued fraction expansion (7.46) are definitely one, namely there exists an integer N such that $a_k = 1$ for all $k > N$. In this case we write

$$\alpha \ \equiv \ [a_0; \ a_1, \ \ldots, \ a_N, \ 1^\infty] \ ;$$

the number $[a_0; \ a_1, \ \ldots, \ a_N]$ is called the *head* of the noble number.

Noble numbers are a subset of the quadratic irrationals, which are in turn a subset of the algebraic irrationals. By a theorem due to Liouville one can show that an algebraic number is diophantine [104].

Theorem (Liouville). *Let ω be an algebraic number of degree n; then ω satisfies the diophantine condition* (7.7) *for some positive constant C and for $\tau = n - 1$.*

Proof. Let ω be a root of (7.47) so that we can write

$$P_n(z) = (z - \omega) P_{n-1}(z) \ , \tag{7.48}$$

for a suitable polynomial $P_{n-1}(z)$ of degree $n-1$. It is $P_{n-1}(\omega) \neq 0$, otherwise we could write $P_n(z) = (z-\omega)^2 P_{n-2}(z)$ for some polynomial $P_{n-2}(z)$. In this case $\frac{d}{dz} P_n(\omega) = 0$, in contrast to the assumption that ω is an algebraic number of degree n, being $\frac{d}{dz} P_n(z)$ a polynomial of degree $n-1$ with integer coefficients. Therefore there exists $\delta > 0$ such that $P_{n-1}(z) \neq 0$ for any $|z - \omega| \leq \delta$. If p, q are integer numbers such that $|\omega - \frac{p}{q}| \leq \delta$, from (7.48) we can write

$$\frac{p}{q} - \omega = \frac{P_n(\frac{p}{q})}{P_{n-1}(\frac{p}{q})} = \frac{c_0 q^n + c_1 p q^{n-1} + \cdots + c_n p^n}{q^n P_{n-1}(\frac{p}{q})} \ . \tag{7.49}$$

The numerator of the last expression in (7.49) is an integer greater or equal than one; let

$$M \equiv \sup_{|z-\omega| \leq \delta} |P_{n-1}(z)| \ .$$

Then we obtain

$$\left| \frac{p}{q} - \omega \right| \geq \frac{1}{M q^n} \ .$$

On the other hand, if $|\frac{p}{q} - \omega| > \delta$, then $|\frac{p}{q} - \omega| > \frac{\delta}{q^n}$, so that (7.7) is satisfied by defining

$$C \equiv \left(\min \left(\delta, \frac{1}{M} \right) \right)^{-1} \ . \qquad \square$$

We stress that there exist diophantine numbers which are not algebraic numbers. The set of diophantine numbers with constant C and exponent τ, say $D(C,\tau)$, has measure one as C tends to zero. For example, the measure $\mu(D(C,\tau)^c)$ of the complement $D(C,\tau)^c$ of the set $D(C,\tau)$ in the interval $[0,1]$ can be computed as follows. For any coprime integers m, n, one has

$$\mu(D(C,\tau)^c) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{C}{n^{\tau+1}} = C \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{\tau+1}} = C \frac{\zeta(\tau)}{\zeta(\tau+1)} \ ,$$

where $\phi(n)$ is the Euler function and $\zeta(\tau)$ is the Riemann zeta function. In conclusion, $\mu(D(C,\tau)^c)$ tends to zero as C tends to zero for any $\tau \geq 1$. The set of diophantine numbers is the union of the sets $D(C,\tau)$ for any positive C and τ.

7.2.4 Trapping diophantine numbers

For Hamiltonian systems like the spin–orbit problem, the KAM tori separate the phase space into invariant regions. One can make use of this property to trap periodic orbits between two KAM tori with suitable rotation numbers bounding the frequency of the periodic orbit from above and below. In this section we address the question concerning the choice of the bounding rotation numbers. In particular, the stability of the resonance of order $p:q$ (for some integers p, q with $q \neq 0$) can be inferred by proving the existence of a pair of invariant tori with frequency bounding the $p:q$ resonance from above and below. Having in mind an application of KAM theorem to the spin–orbit problem, we focus our attention on the 1:1 and 3:2 resonances. Let the golden ratio be $\gamma = \frac{\sqrt{5}-1}{2}$; a possible choice of trapping diophantine numbers for $p = q = 1$ is given by the sequences of noble numbers defined as

$$\begin{aligned} \Gamma_k &\equiv [0;\ 1,\ k-1,\ 1^\infty] \equiv 1 - \frac{1}{k+\gamma} \ , \\ \Delta_k &\equiv [1;\ k,\ 1^\infty] \equiv 1 + \frac{1}{k+\gamma} \ , \qquad k \geq 2 \ . \end{aligned} \tag{7.50}$$

Both Γ_k and Δ_k converge to one from below and above, respectively, and have the property that for all k, $|\Gamma_k - 1| = |\Delta_k - 1|$. Notice that Γ_k and Δ_k are noble algebraic numbers of degree two, since they are roots of the polynomials

$$\begin{aligned} P_{\Gamma_k}(x) \equiv\ & 4(k^4 - 2k^3 - k^2 + 2k + 1)x^2 - 4(2k^4 - 6k^3 + k^2 + 5k + 1)x \ + \\ &+ 4k^4 - 16k^3 + 12k^2 + 8k - 4 \end{aligned}$$

and

$$\begin{aligned} P_{\Delta_k}(x) \equiv\ & 4(k^4 - 2k^3 - k^2 + 2k + 1)x^2 - 4(2k^4 - 2k^3 - 5k^2 + 3k + 3)x \ + \\ &+ 4k^4 - 12k^2 + 4 \ . \end{aligned}$$

We remark that noble tori are conjectured to be the last surfaces to disappear in any interval of rotation numbers ([124, 125, 148], see also [62]). Numerical experiments on the standard and quadratic maps [124] show that noble tori are locally the most robust in the sense that

(i) for any critical (i.e., close to breakdown) noble surface of rotation number ω, there exists an interval around ω containing no other invariant tori;

(ii) let $\mathcal{T}_\varepsilon(\alpha)$ be a critical non–noble torus; then in any interval around α there always exists a non–critical noble.

Concerning the 3:2 resonance we can consider the trapping rotation numbers

$$\begin{aligned} \Gamma'_k &\equiv \frac{3}{2} - \frac{1}{k+\gamma} , \\ \Delta'_k &\equiv \frac{3}{2} + \frac{1}{k+\gamma} , \qquad k \geq 2 , \end{aligned} \tag{7.51}$$

converging to $\frac{3}{2}$ from below and above, respectively, and with $|\Gamma'_k - \frac{3}{2}| = |\Delta'_k - \frac{3}{2}|$ for any k. Notice that Γ'_k and Δ'_k are not necessarily noble numbers, but they are second–order algebraic numbers, since they are roots of the polynomials

$$\begin{aligned} P_{\Gamma'_k}(x) \equiv\ & 4(k^4 - 2k^3 - k^2 + 2k + 1)x^2 - 4(3k^4 - 8k^3 + 7k + 2)x \\ & + 9k^4 - 30k^3 + 13k^2 + 20k - 1 \end{aligned}$$

and

$$\begin{aligned} P_{\Delta'_k}(x) \equiv\ & 4(k^4 - 2k^3 - k^2 + 2k + 1)x^2 - 4(3k^4 - 4k^3 - 6k^2 + 5k + 4)x \\ & + 9k^4 - 6k^3 - 23k^2 + 8k + 11 . \end{aligned}$$

The computation of the diophantine constant C for the numbers Γ_k, Δ_k, Γ'_k, Δ'_k can be performed as follows.

Proposition. *Let Γ_k, Δ_k, Γ'_k, Δ'_k be as in (7.50), (7.51); then for any $k \geq 2$ the corresponding diophantine constants are, respectively,*

$$k+\gamma, \quad k+\gamma, \quad 4(k+\gamma), \quad 4(k+\gamma) .$$

Proof. Let us provide the details for the derivation of the diophantine constant associated to Δ_k; the computations for the other numbers follow easily. We want to show that

$$\left| \left(1 + \frac{1}{k+\gamma}\right) - \frac{p}{q} \right| \geq \frac{1}{(k+\gamma)q^2} \qquad \text{for all } p,\ q \in \mathbf{Z},\quad q \neq 0 ,$$

which is equivalent to require that

$$\left| \frac{1}{k+\gamma} - \frac{p}{q} \right| \geq \frac{1}{(k+\gamma)q^2} \qquad \text{for all } p,\ q \in \mathbf{Z} ,\quad q \neq 0 . \tag{7.52}$$

The rational approximants to $\frac{1}{k+\gamma}$ are given by

$$\left\{ \frac{p_j}{q_j} \right\}_{j \geq 0} \equiv \left\{ \frac{\alpha_j}{\alpha_j k + \alpha_{j-1}} \right\}_{j \geq 0} ,$$

where the α_j's are the Fibonacci's numbers defined via the recursive relation

$$\alpha_0 = 1,\ \alpha_1 = 1,\ \ldots,\ \alpha_{j+1} = \alpha_j + \alpha_{j-1} \qquad \text{for all } j \geq 1 \ ;$$

then, it is sufficient to show (7.52) with $\frac{p}{q}$ replaced by the approximant $\frac{\alpha_j}{\alpha_j k + \alpha_{j-1}}$, namely

$$\left| \frac{1}{k+\gamma} - \frac{\alpha_j}{\alpha_j k + \alpha_{j-1}} \right| \geq \frac{1}{(k+\gamma)(\alpha_j k + \alpha_{j-1})^2} \qquad \text{for all } k \geq 2 \ . \tag{7.53}$$

From (7.53) one gets the inequality

$$\left| 1 - \frac{\alpha_j (k+\gamma)}{\alpha_j k + \alpha_{j-1}} \right| \geq \frac{1}{(\alpha_j k + \alpha_{j-1})^2} ,$$

which is equivalent to

$$\left| \gamma - \frac{\alpha_{j-1}}{\alpha_j} \right| \geq \frac{1}{\alpha_j^2 (k + \frac{\alpha_{j-1}}{\alpha_j})} .$$

Since

$$k + \frac{\alpha_{j-1}}{\alpha_j} \geq 2 + \frac{\alpha_{j-1}}{\alpha_j} ,$$

it is sufficient to show that

$$\left| \gamma - \frac{\alpha_{j-1}}{\alpha_j} \right| \geq \frac{1}{\alpha_j^2 (2 + \frac{\alpha_{j-1}}{\alpha_j})} .$$

Defining A_j by the equality

$$\left| \gamma - \frac{\alpha_{j-1}}{\alpha_j} \right| \equiv \frac{1}{A_j \alpha_j^2} ,$$

it is readily seen that

$$A_j = \gamma + 1 + \frac{\alpha_{j-1}}{\alpha_j} ; \tag{7.54}$$

therefore we get that

$$\left| \gamma - \frac{\alpha_{j-1}}{\alpha_j} \right| = \frac{1}{\alpha_j^2 (\gamma + 1 + \frac{\alpha_{j-1}}{\alpha_j})} \geq \frac{1}{\alpha_j^2 (2 + \frac{\alpha_{j-1}}{\alpha_j})} ,$$

since

$$2 + \frac{\alpha_{j-1}}{\alpha_j} \geq \gamma + 1 + \frac{\alpha_{j-1}}{\alpha_j} .$$

Applying the same procedure one proves that

$$\left|\Gamma_k - \frac{p}{q}\right| \geq \frac{1}{(k+\gamma)q^2} \qquad \text{for all } p,q \in \mathbf{Z}\ , \quad q \neq 0\ , \quad k \geq 2\ ,$$

where the sequence of rational approximants to Γ_k is given by

$$\left\{ \frac{\alpha_j(k-1)+\alpha_{j-1}}{\alpha_j k + \alpha_{j-1}} \right\}\ .$$

Analogous considerations hold for Γ'_k and Δ'_k. □

We remark that for the golden ratio equation (7.54) implies that the diophantine constant is equal to $C = \frac{3+\sqrt{5}}{2}$.

7.2.5 Computer–assisted proofs

The computation of the initial approximation and the control of the KAM algorithm usually require the use of a computer, due to the high number of operations involved. However, the computer introduces rounding–off and propagation errors. In order to leave unaltered the rigorous character of the result, one can keep track of the computer rounding–off errors through the application of the so–called *interval arithmetic* technique [60,106], whose implementation is briefly explained as follows. The computer stores real numbers using a sign–exponent–fraction representation; the number of digits in the fraction and the exponent varies with the machine. The result of any elementary operation, i.e. sum, subtraction, multiplication and division, usually produces an approximation of the true result; other calculations, like exponent, square root, logarithm, etc., can be reduced to a sequence of elementary operations through a Taylor series expansion. The idea of the interval arithmetic technique is to represent any real number as an interval and to perform elementary operations on intervals, rather than on real numbers. For example, suppose we perform the sum of two numbers a and b, which are contained, respectively, within the intervals $[a_1, a_2]$ and $[b_1, b_2]$. Adding these two intervals one obtains $[c_1, c_2] \equiv [a_1 + b_1, a_2 + b_2]$. However, we have to consider that the end–points c_1, c_2 of the new interval are themselves produced by an elementary operation and therefore they are affected by rounding errors. Henceforth one needs to construct a new interval which gets rid of the fact that c_1 and c_2 are rounded. This can be done as follows. Let δ be the limiting precision of the machine (see, e.g., [159]). Then, multiply c_1 by $1 \mp \delta$ according to whether c_1 is positive or negative and let us call the final result $c_- \equiv down(c_1)$. Similarly, to get an upper bound of c_2 multiply it by $1 \pm \delta$ according to whether c_2 is positive or negative; let us call the final result $c_+ \equiv up(c_2)$. We finally get that $a + b \in [c_-, c_+]$. The subtraction can be treated in a similar way.

Concerning the multiplication (as well as the division), one needs to consider different cases according to the signs of the factors. More precisely, suppose we compute the multiplication $a \cdot b$, where a and b are represented by the intervals $[a_1, a_2]$ and $[b_1, b_2]$, while the result will be contained in $[c_-, c_+]$. We must distinguish the following cases:

(1) $a_1 \geq 0$ and $b_1 \geq 0$, then $c_- = down(a_1 b_1)$, $c_+ = up(a_2 b_2)$;
(2) $a_1 \geq 0$ and $b_2 \leq 0$, then $c_- = down(a_2 b_1)$, $c_+ = up(a_1 b_2)$;
(3) $a_1 \geq 0$ and $b_1 < 0$, $b_2 > 0$, then $c_- = down(a_2 b_1)$, $c_+ = up(a_2 b_2)$;
(4) $a_2 \leq 0$ and $b_1 \geq 0$, then $c_- = down(a_1 b_2)$, $c_+ = up(a_2 b_1)$;
(5) $a_2 \leq 0$ and $b_2 \leq 0$, then $c_- = down(a_2 b_2)$, $c_+ = up(a_1 b_1)$;
(6) $a_2 \leq 0$ and $b_1 < 0$, $b_2 > 0$, then $c_- = down(a_1 b_2)$, $c_+ = up(a_1 b_1)$;
(7) $a_1 < 0$, $a_2 > 0$ and $b_1 \geq 0$, then $c_- = down(a_1 b_2)$, $c_+ = up(a_2 b_2)$;
(8) $a_1 < 0$, $a_2 > 0$ and $b_2 \leq 0$, then $c_- = down(a_2 b_1)$, $c_+ = up(a_1 b_1)$;
(9) $a_1 < 0$, $a_2 > 0$ and $b_1 < 0$, $b_2 > 0$, then
(9a) let $\ell_- = down(a_1 b_2)$, $r_- = down(a_2 b_1)$; if $r_- < \ell_-$ then $\ell_- = r_-$;
(9b) let $\ell_+ = up(a_1 b_1)$, $r_+ = up(a_2 b_2)$; if $r_+ > \ell_+$ then $\ell_+ = r_+$;
set $b_1 = \ell_-$, $b_2 = \ell_+$.

A similar approach is used to deal with the division. Casting together the elementary operations on intervals one obtains the implementation of the interval arithmetic technique, where complex operations are reduced to a sequence of elementary operations by using their series expansion.

7.3 A survey of KAM results in Celestial Mechanics

7.3.1 Rotational tori in the spin–orbit problem

We consider the spin–orbit problem widely discussed in the previous sections and we aim to prove the existence of rotational invariant tori, trapping the synchronous resonance from above and below, thus providing a confinement property of the dynamics in the phase space. As a specific example we consider the Earth–Moon system. In writing the model (7.1)–(7.2) we have neglected all perturbations due to other celestial bodies as well as dissipative effects. Among the discarded contributions the most important term is due to the tidal torque generated by the non–rigidity of the satellite. For consistency, we expand the perturbing function in Fourier–Taylor series, neglecting all terms which are of the same order or less than the neglected tidal torque. Taking into account that the eccentricity of the Moon amounts to $\mathrm{e} = 0.0549$, one is led to consider the perturbing function (7.2) with $N_1 = 1$ and $N_2 = 7$. The corresponding Hamiltonian function reads as

$$
\begin{aligned}
\mathcal{H}(y,x,t) \equiv \frac{y^2}{2} - \varepsilon \Bigg[& \left(-\frac{\mathrm{e}}{4} + \frac{\mathrm{e}^3}{32} \right) \cos(2x - t) + \\
& + \left(\frac{1}{2} - \frac{5}{4}\mathrm{e}^2 + \frac{13}{32}\mathrm{e}^4 \right) \cos(2x - 2t) + \left(\frac{7}{4}\mathrm{e} - \frac{123}{32}\mathrm{e}^3 \right) \cos(2x - 3t) + \\
& + \left(\frac{17}{4}\mathrm{e}^2 - \frac{115}{12}\mathrm{e}^4 \right) \cos(2x - 4t) + \left(\frac{845}{96}\mathrm{e}^3 - \frac{32\,525}{1536}\mathrm{e}^5 \right) \cos(2x - 5t) + \\
& + \frac{533}{32}\mathrm{e}^4 \cos(2x - 6t) + \frac{228\,347}{7680}\mathrm{e}^5 \cos(2x - 7t) \Bigg] , \qquad (7.55)
\end{aligned}
$$

where the physical value of the perturbing parameter amounts for the Moon to $\varepsilon \simeq 3.45 \cdot 10^{-4}$. The existence of two bounding tori with frequencies Γ_{40} and Δ_{40} (see (7.50)) has been proven in [23] by performing the following steps. Compute the initial approximation (7.45) up to the order $k_0 = 15$; apply the KAM theorem presented in Section 7.2; implement the interval arithmetic technique. Then, one gets [23] that the synchronous motion of the Moon is trapped in the region enclosed by the tori $\mathcal{T}(\Gamma_{40})$ and $\mathcal{T}(\Delta_{40})$, which is shown to be a subset of $\{(y, x, t) : (x, t) \in \mathbf{T}^2, 0.97 \leq y \leq 1.03\}$.

In a similar way one can prove the stability of the Mercury–Sun system. However, due to the bigger eccentricity of Mercury, being $\mathrm{e} = 0.2056$, the perturbing function contains a larger number of terms, so that the corresponding Hamiltonian is given by

$$\mathcal{H}(y, x, t) \equiv \frac{y^2}{2} - \frac{\varepsilon}{2} \sum_{m \neq 0, m=-11}^{3} W\left(\frac{m}{2}, \mathrm{e}\right) \cos(2x - mt) ,$$

with the coefficients $W(\frac{m}{2}, \mathrm{e})$ truncated to $O(\mathrm{e}^7)$. The stability of the observed 3:2 resonance is obtained for the true value of the perturbing parameter, i.e. $\varepsilon = 1.5 \cdot 10^{-4}$, by proving the existence of the tori with frequencies Γ'_{70} and Δ'_{70} (see (7.51)); the corresponding trapping region is contained in $\{(y, x, t) : (x, t) \in \mathbf{T}^2, 1.48 \leq y \leq 1.52\}$.

7.3.2 Librational invariant surfaces in the spin–orbit problem

The confinement of the motion associated to periodic orbits of the spin–orbit problem can also be obtained by constructing *librational* invariant surfaces. In the following we provide some details of the proof concerning the case of the 1:1 resonance (see [24]), whose outline is the following. The first task is to center the Hamiltonian on the 1:1 periodic orbit and to expand in Taylor series around the new origin. Next, diagonalize the quadratic terms to obtain a harmonic oscillator, perturbed by higher degree (time–dependent) terms. After introducing the action–angle variables associated to the harmonic oscillator, implement a Birkhoff normal form to reduce the size of the perturbation and then apply the KAM theorem to prove the existence of trapping librational tori.

According to the above strategy, we start by writing the Hamiltonian function as

$$\mathcal{H}_0(y, x, t) = \frac{y^2}{2} - \varepsilon a \cos(2x - 2t) - \frac{\varepsilon}{2} \sum_{m \neq 0,2,\ m=N_1}^{N_2} W\left(\frac{m}{2}, \mathrm{e}\right) \cos(2x - mt) , \quad (7.56)$$

where $a \equiv \frac{1}{2} W(1, \mathrm{e})$. Perform the coordinate change $x' = 2x - 2t$, $y' = \frac{1}{2}(y - 1)$, expand in Taylor series around the origin and diagonalize the time–independent quadratic terms by means of the symplectic transformation

$$\begin{aligned} p &= \alpha\, y' \\ q &= \beta\, x' , \end{aligned}$$

with $\alpha = \frac{\sqrt{2}}{(\varepsilon a)^{1/4}}$, $\beta = \frac{(\varepsilon a)^{1/4}}{\sqrt{2}}$. After these steps the Hamiltonian function becomes

$$\begin{aligned}
\mathcal{H}_1(p,q,t) &= \frac{\omega}{2}(p^2+q^2) \ -\varepsilon a \left(\frac{q^4}{4!\beta^4} - \frac{q^6}{6!\beta^6} + \dots \right) \\
&- \frac{\mu}{2} \sum_{m\neq 0,-2} \tilde{W}\left(\frac{m+2}{2}, \mathrm{e}\right) \left[\cos(mt) \left(1 - \frac{q^2}{2\beta^2} + \frac{q^4}{4!\beta^4} + \dots \right) \right. \\
&+ \left. \sin(mt) \left(\frac{q}{\beta} - \frac{q^3}{3!\beta^3} + \frac{q^5}{5!\beta^5} + \dots \right) \right] ,
\end{aligned}$$

where $\omega = 2\sqrt{\varepsilon a}$ is the frequency of the harmonic oscillation, $\mu \equiv \varepsilon \mathrm{e}$, while the coefficients W have been rescaled as $\tilde{W}(\frac{m+2}{2}, \mathrm{e}) = \frac{1}{\mathrm{e}} W(\frac{m+2}{2}, \mathrm{e})$. Introduce action–angle variables (I, φ) as

$$\begin{aligned}
p &= \sqrt{2I} \ \cos\varphi \\
q &= \sqrt{2I} \ \sin\varphi \ ;
\end{aligned}$$

the resulting Hamiltonian is given by

$$\begin{aligned}
&\mathcal{H}_2(I,\varphi,t) = \omega I \ - \varepsilon a \left(\frac{I^2}{16\beta^4} - \frac{5I^3}{2\cdot 6!\beta^6} + \dots \right) \\
&- \ \varepsilon a \left[- \frac{I^2}{12\beta^4} \cos 2\varphi + \frac{I^2}{48\beta^4} \cos 4\varphi + \frac{I^3}{4\cdot 6!\beta^6} \ \cdot \right. \\
&\cdot \ \left. (15\cos 2\varphi - 6\cos 4\varphi + \cos 6\varphi) + \dots \right] \\
&- \ \frac{\varepsilon \mathrm{e}}{2} \sum_{m\neq 0,-2} \tilde{W}(\frac{m+2}{2}, \mathrm{e}) \left\{ \cos(mt) \left[1 - \frac{I}{2!\beta^2}(1-\cos 2\varphi) + \frac{I^2}{8\cdot 3!\beta^4} \ \cdot \right. \right. \\
&\cdot \ \left. (3 - 4\cos 2\varphi + \cos 4\varphi) - \frac{I^3}{4\cdot 6!\beta^6} \ (10 - 15\cos 2\varphi + 6\cos 4\varphi - \cos 6\varphi) + \dots \right] \\
&+ \ \sin(mt) \left[\frac{\sqrt{2I}}{\beta} \sin\varphi - \frac{\sqrt{2}\ I^{3/2}}{12\beta^3}(3\sin\varphi - \sin 3\varphi) \right. \\
&+ \ \left. \left. \frac{\sqrt{2}I^{5/2}}{4\cdot 5!\beta^5} \ (10\sin\varphi - 5\sin 3\varphi + \sin 5\varphi) + \dots \right] \right\} ,
\end{aligned}$$

which can be written in compact form as

$$\mathcal{H}_2(I,\varphi,t) \ = \ \omega I + \varepsilon \overline{h}(I) + \varepsilon \tilde{h}(I,\varphi) + \varepsilon \mathrm{e} f(I,\varphi,t)$$

with the obvious identification of the functions $\overline{h}$, h and f. A Birkhoff normal form can be implemented to reduce the size of the perturbation $R(I,\varphi,t) \equiv \tilde{h}(I,\varphi) +$

e$f(I,\varphi,t)$. After such reduction we write the Hamiltonian in the form

$$\mathcal{H}_k(I',\varphi',t) \;=\; h_k(I';\varepsilon)+\varepsilon^{k+1}R_k(I',\varphi',t;\varepsilon)\ ,$$

where the functions h_k and R_k can be explicitly determined. The application of (computer–assisted) KAM estimates [25] allows us to establish the existence of a librational invariant torus, which confines the synchronous resonance in the phase space.

As an example, we report the results for the Rhea–Saturn system, which is observed to move in a synchronous spin–orbit resonance; for this example the stability of the synchronous resonance can be established for the realistic values of the parameters.

Theorem [24]. *Consider the system described by the Hamiltonian* (7.56) *with* $N_1=-1$, $N_2=5$ *and let* $e=0.00098$. *If* $\varepsilon_{Rhea}=3.45\cdot 10^{-4}$ *is the physical value of the perturbing parameter, then there exists an invariant torus corresponding to a libration of* 1.95^o *for any* $\varepsilon\le\varepsilon_{Rhea}$.

7.3.3 The spatial planetary three–body problem

The planetary problem concerns the study of two point–masses, say $\mathcal{P}_1$ and $\mathcal{P}_2$ with masses m_1 and m_2 of the same order of magnitude, orbiting around a central body, say $\mathcal{P}$ with mass M. It is therefore necessary to take into account the mutual interaction between $\mathcal{P}_1$ and $\mathcal{P}_2$, besides that with the central body. In order to write the Hamiltonian function, let us introduce the heliocentric positions of the planets, $\underline{r}_1,\underline{r}_2\in\mathbf{R}^3$, and the conjugated momenta referred to the center of mass, $\underline{v}_1,\underline{v}_2\in\mathbf{R}^3$. The Hamiltonian describing the motion of $\mathcal{P}_1$ and $\mathcal{P}_2$ can be decomposed as

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1\ , \tag{7.57}$$

where $\mathcal{H}_0$ is due to the decoupled Keplerian motions of the planets and $\mathcal{H}_1$ represents the interaction between $\mathcal{P}_1$ and $\mathcal{P}_2$. More precisely, one has

$$\mathcal{H}_0=\sum_{j=1}^{2}\frac{m_j+M}{2m_jM}\|\underline{v}_j\|^2-\mathcal{G}\ \frac{Mm_j}{\|\underline{r}_j\|}\ , \tag{7.58}$$

while the perturbation is given by

$$\mathcal{H}_1=\frac{\underline{v}_1\cdot\underline{v}_2}{M}-\mathcal{G}\ \frac{m_1m_2}{\|\underline{r}_1-\underline{r}_2\|}\ . \tag{7.59}$$

The preservation of the angular momentum allows us to state that the ascending nodes of the planets lie on the invariant plane perpendicular to the angular momentum and passing through the central body. The existence of invariant tori in the framework of the properly degenerate Hamiltonian (7.57), (7.58), (7.59) has been investigated in [3] under the assumption of planar motion and assuming that the ratio of the semimajor axes tends to zero. Invariant tori are shown to exist, provided that the planetary masses and the eccentricities are sufficiently small. The assumption that the ratio of the semimajor axes tends to zero has been removed in [155], where quantitative estimates have been worked out. The proper degeneracy of the Hamiltonian has been eliminated by a suitable normal form; after performing the reduction of the angular momentum, the perturbing function has been expanded using an adapted algebraic manipulator (see [110]). The result presented in [155] provides that, for sufficiently small planetary masses and eccentricities, one can apply Arnold's theorem on the existence of invariant tori, provided that the ratio α between the planetary semimajor axes satisfies $10^{-8} \leq \alpha \leq 0.8$ and that the mass ratio satisfies $0.01 \leq \frac{m_1}{m_2} \leq 100$.

The specific case of the Sun–Jupiter–Saturn planetary problem has been studied in [120]. After the Jacobi reduction of the nodes [120], the problem turns out to be described by a Hamiltonian function with four degrees of freedom, which is expanded up to the second order in the masses and averaged over the fast angles. The resulting two–degrees–of–freedom Hamiltonian describes the slow motion of the orbital parameters, and precisely of the eccentricities. The existence of invariant tori in a suitable neighborhood of an elliptic point is obtained as follows. After expressing the perturbing function in Poincaré variables, an expansion up to the order 6 in the eccentricities is performed. The computation of the Birkhoff normal form and a computer–assisted KAM theorem yield the existence of two invariant surfaces trapping the *secular* motions of Jupiter and Saturn for the astronomical values of the parameters. This approach was later extended [121] to include the description of the fast variables, like the semimajor axes and the mean longitudes of the planets. A preliminary average over the fast angles was performed without eliminating the terms with degree greater or equal than 2 with respect to the fast actions. The canonical transformations involving the secular coordinates can be adapted to produce a good initial approximation of an invariant torus for the reduced Hamiltonian of the planetary three–body problem. Afterwards the Kolmogorov normal form was constructed (so that the Hamiltonian is reduced to a harmonic oscillator plus higher–order terms) and it was numerically shown to be convergent. The numerical results on the convergence of the Kolmogorov normal form have been obtained for a planetary solar system composed by two planets with masses equal to those of Jupiter and Saturn.

7.3.4 The circular, planar, restricted three–body problem

We consider the motion of a small body ($\mathcal{P}_2$), say an asteroid, under the influence of two primaries, say the Sun ($\mathcal{P}_1$) and Jupiter ($\mathcal{P}_3$) in the framework of the circular, planar, restricted three–body problem (see Section 4.1). The Sun–Jupiter–asteroid problem was selected in [31] as a test–bench for KAM theory, which provided estimates on the mass–ratio very far from the astronomical observations; in particular, the existence of invariant tori was obtained for mass–ratios less than 10^{-333} by applying Arnold's theorem and 10^{-48} using Moser's theorem. We recall that the perturbative parameter ε coincides with the Jupiter–Sun mass ratio, which amounts to about $\varepsilon = \varepsilon_J \equiv 0.954 \cdot 10^{-3}$. The small body was chosen as the asteroid 12 Victoria, whose orbital elements are:

$$a_{\mathrm{V}} \simeq 2.335 \text{ AU} , \qquad e_{\mathrm{V}} \simeq 0.220 , \qquad i_{\mathrm{V}} \simeq 8.362^o ,$$

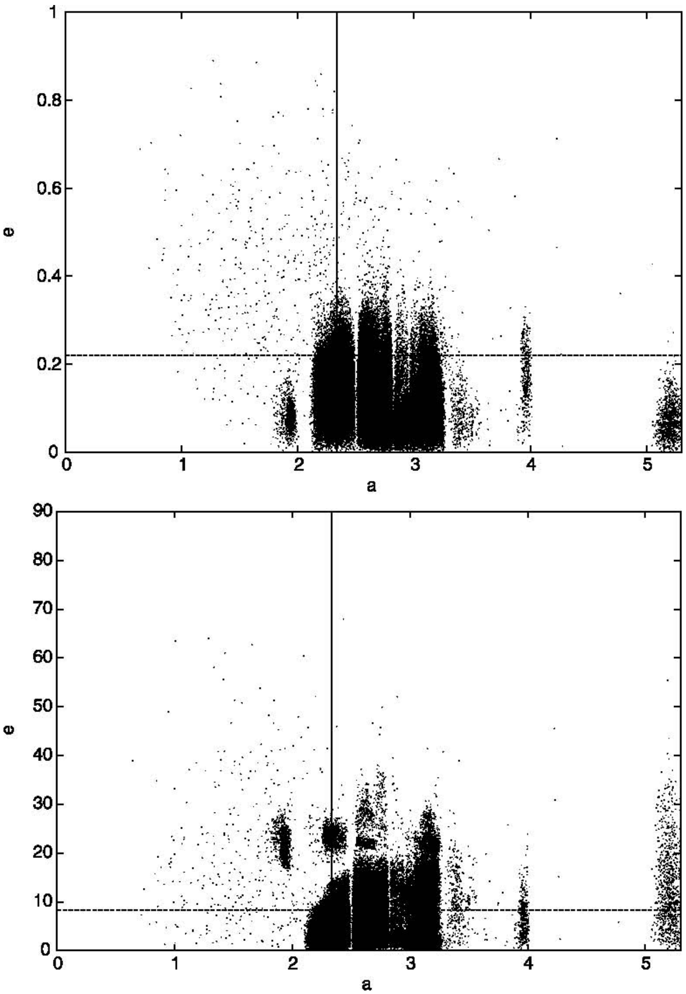

Fig. 7.2. Orbital elements of the numbered asteroids. Top: semimajor axis versus eccentricity. Bottom: semimajor axis versus inclination. The internal lines locate the position of the asteroid 12 Victoria (reprinted from [30]).

where a_V is the semimajor axis of the asteroid, e_V is the eccentricity, i_V is the inclination with respect to the ecliptic plane. Figure 7.2 shows that 12 Victoria is a typical object of the asteroidal belt[2], since the semimajor axes of most asteroids lie within the interval $1.8 \leq a \leq 3.5\ AU$, while the eccentricity is usually within $0 \leq \mathrm{e} \leq 0.35$.

The model presented above does not include many effects, most notably the eccentricity of Jupiter, the mutual inclinations, the influence of other planets, as well as dissipative effects. For consistency, the perturbing function, representing

[2] The elements of the numbered asteroids are provided by the JPL's DASTCOM database at http://ssd.jpl.nasa.gov/?sb_elem

the influence of Jupiter on the asteroid, has been expanded in the eccentricity and semimajor axes ratio, and truncated to discard all terms which are of the same order of magnitude or less than the maximum contribution due to the effects we have neglected. Indeed, in the Sun–Jupiter–Victoria model the biggest neglected contribution is due to the eccentricity of the orbit of Jupiter, which has been assumed to be zero in the present model. According to this criterion we obtain the following Hamiltonian:

$$\mathcal{H}(L,G,\ell,g) = -\frac{1}{2L^2} - G - \varepsilon\mathcal{R}(L,G,\ell,g) \ , \qquad (7.60)$$

where (L,G) are the Delaunay action variables, ℓ is the mean anomaly, g is the difference between the argument of perihelion and the true anomaly of Jupiter (see Chapter 4) and the perturbing function is given by

$$\begin{aligned}\mathcal{R}(L,G,\ell,g) &\equiv 1 + \frac{L^4}{4} + \frac{9}{64}\,L^8 + \frac{3}{8}\,L^4\mathrm{e}^2 - \left(\frac{1}{2} + \frac{9}{16}L^4\right) L^4\mathrm{e}\ \cos\ell \\ &+ \left(\frac{3}{8}\,L^6 + \frac{15}{64}L^{10}\right)\cos(\ell+g) - \left(\frac{9}{4} + \frac{5}{4}L^4\right) L^4\mathrm{e}\ \cos(\ell+2g) \\ &+ \left(\frac{3}{4}\,L^4 + \frac{5}{16}\,L^8\right)\cos(2\,\ell+2\,g) + \frac{3}{4}\,L^4\mathrm{e}\cos(3\,\ell+2\,g) \\ &+ \left(\frac{5}{8}\,L^6 + \frac{35}{128}\,L^{10}\right)\cos(3\,\ell+3\,g) + \frac{35}{64}\,L^8\cos(4\,\ell+4\,g) \\ &+ \frac{63}{128}L^{10}\cos(5\ell+5g) \ ,\end{aligned}$$

where $\mathrm{e} = \sqrt{1-\frac{G^2}{L^2}}$. Let us write (7.60) as

$$\mathcal{H}(L,G,\ell,g;\varepsilon) = \mathcal{H}_0(L,G) + \varepsilon\mathcal{R}(L,G,\ell,g) \ ,$$

where $\mathcal{H}_0(L,G) \equiv -\frac{1}{2L^2} - G$. The KAM theorem described in Section 7.2 cannot be applied, since the integrable part $\mathcal{H}_0$ is degenerate. However, it is possible to apply a different version of the theorem, which requires the *isoenergetic non–degeneracy condition* due to Arnold [6]:

$$C_E(L,G) \equiv \det\begin{pmatrix} \mathcal{H}_0'' & \mathcal{H}_0' \\ \mathcal{H}_0' & 0 \end{pmatrix} \neq 0 \qquad \text{for all} \ \ 0 < G < L \ ,$$

where $\mathcal{H}_0'$ and $\mathcal{H}_0''$ denote, respectively, the Jacobian vector and the Hessian matrix associated to $\mathcal{H}_0$. A straightforward computation shows that $C_E(L,G) = \frac{3}{L^4}$. To fix the energy level we proceed as follows (see [31]). From the physical value of the asteroid 12 Victoria, using normalized units one gets that $L_{\mathrm{V}} \simeq 0.670$, $G_{\mathrm{V}} \simeq 0.654$. Let

$$E_{\mathrm{V}}^{(0)} = -\frac{1}{2L_{\mathrm{V}}^2} - G_{\mathrm{V}} \simeq -1.768 \ , \quad E_{\mathrm{V}}^{(1)} \equiv \langle\mathcal{R}(L_{\mathrm{V}},G_{\mathrm{V}},\ell,g)\rangle \simeq -1.060 \ .$$

We define the energy level through the expression

$$E_{\rm V}^{*} = E_{\rm V}^{(0)} + \varepsilon_J E_{\rm V}^{(1)} \simeq -1.769 \ ,$$

where ε_J denotes the observed Jupiter–Sun mass–ratio. The existence of two invariant tori, bounding from above and below the observed values $L_{\rm V}$ and $G_{\rm V}$, is proven on the level set $\mathcal{H}^{-1}\big(E_{\rm V}^{*}\big)$. Setting $\tilde{L}_{\pm} = L_V \pm 0.001$, the bounding frequencies are computed as

$$\tilde{\omega}_{\pm} = \left(\frac{1}{\tilde{L}_{\pm}^{3}}, -1\right) \equiv (\tilde{\alpha}_{\pm}, -1) \ .$$

Since we need *diophantine* numbers, we proceed to compute the continued fraction expansion of $\tilde{\alpha}_{\pm}$ up to the order 5 and then we add a tail of ones to obtain the following diophantine numbers:

$$\begin{aligned} \alpha_{-} &\equiv [3; 3, 4, 2, 1^{\infty}] = 3.30976937631389\ldots \ , \\ \alpha_{+} &\equiv [3; 2, 1, 17, 5, 1^{\infty}] = 3.33955990647860\ldots \ . \end{aligned}$$

Next we introduce the frequencies

$$\omega_{\pm} \equiv (\alpha_{\pm}, -1) \ ,$$

which satisfy the diophantine condition (7.7) with $\tau = 1$ and with diophantine constants respectively equal to

$$C_{-} = 138.42 \ , \qquad C_{+} = 30.09 \ .$$

The stability of the asteroid 12 Victoria is finally obtained by proving the persistence of the unperturbed KAM tori $\mathcal{T}_0^{\pm} \equiv \{(L_{\pm}, G_{\pm})\} \times \mathbf{T}^2$ for a value of the perturbing parameter ε greater or equal than the Jupiter–Sun mass ratio.

Theorem [31]. *For $|\varepsilon| \leq 10^{-3}$ the unperturbed tori $\mathcal{T}_0^{\pm}$ can be analytically continued into invariant KAM tori $\mathcal{T}_\varepsilon^{\pm}$ for the perturbed system on the energy level $\mathcal{H}^{-1}\big(E_{\rm V}^{*}\big)$ keeping fixed the ratio of the frequencies.*

Since the orbital elements are related to the Delaunay action variables, the theorem guarantees that the semimajor axis and the eccentricity stay close to the unperturbed values within an interval of order ε (see [31] for full details on the KAM isoenergetic, computer–assisted proof).

7.4 Greene's method for the breakdown threshold

There exist different techniques which allow us to evaluate numerically the breakdown threshold of an invariant surface (see, e.g., [82, 109, 145]). One of the most accepted methods, which has been partially rigorously proved [54, 63, 127], was developed by J. Greene in [82]. His method is based on the conjecture that the breakdown of an invariant surface is closely related to the stability character of the approximating periodic orbits [92]. The key role of the periodic orbits had already been stressed by H. Poincarè in [149], who formulated the following conjecture:

"... here is a fact that I have not been able to prove rigorously, but that seems to me very reasonable. Given equations of the form (13) [Hamilton's equations] *and a particular solution of these equations, one can always find a periodic solution (whose period, it is true, can be very long) such that the difference between the two solutions may be as small as one wishes for as long as one wishes".*

Greene's algorithm for computing the breakdown threshold was originally formulated for the standard mapping, but we present it here for the spin–orbit problem, which has been assumed as a model problem throughout this chapter. Let us reduce the analysis of the differential equation (7.1) to the study of the discrete mapping obtained integrating (7.1) through an area–preserving leapfrog method:

$$\begin{aligned} y_{j+1} &= y_j - \varepsilon f_x(x_j, t_j) h \\ x_{j+1} &= x_j + y_{j+1} h \ , \end{aligned} \tag{7.61}$$

where $t_{j+1} = t_j + h$ and $h \geq 0$ denotes the integration step, $y_j \in \mathbf{R}$, $x_j \in \mathbf{T}$, $t_j \in \mathbf{T}$. We say that a periodic orbit has length q (for some positive integer q), if it closes after q iterations. We shall consider the periodic orbits which exist for all values of the parameter ε down to $\varepsilon = 0$. Analogously, we consider rotational KAM tori with the same property. In the integrable limit the rotation number is given by $\omega \equiv y_0$; if the frequency of motion is rational, say $\omega = \frac{p}{q}$ for some positive integers p and q with $q \neq 0$, then the second of (7.61) implies that

$$p = \sum_{j=1}^{q} y_j = \sum_{j=1}^{q} \frac{x_j - x_{j-1}}{h} = \frac{x_q - x_0}{h} \ .$$

If the frequency ω is irrational, the periodic orbits with frequency equal to its rational approximants $\frac{p_j}{q_j}$ are those which nearly approach the torus with rotation number ω (see Figure 7.3).

In order to determine the linear stability of a periodic orbit, we compute the tangent space trajectory $(\partial y_j, \partial x_j)$ at (y_j, x_j), which is related to the initial conditions $(\partial y_0, \partial x_0)$ at (y_0, x_0) by

$$\begin{pmatrix} \partial y_j \\ \partial x_j \end{pmatrix} = M \begin{pmatrix} \partial y_0 \\ \partial x_0 \end{pmatrix} ,$$

where the matrix M is the product of the Jacobian of (7.61) along a full cycle of the periodic orbit:

$$M = \prod_{i=1}^{q} \begin{pmatrix} 1 & -\varepsilon f_{xx}(x_j, t_j) h \\ h & 1 - \varepsilon f_{xx}(x_j, t_j) h^2 \end{pmatrix} .$$

The eigenvalues of M are the associated Floquet multipliers (compare with Appendix D); by the area–preservation of the mapping it is $\det(M) = 1$ and denoting by $\mathrm{tr}(M)$ the trace of M, the eigenvalues are the solutions of the equation

$$\lambda^2 - \mathrm{tr}(M)\lambda + 1 = 0 \ .$$

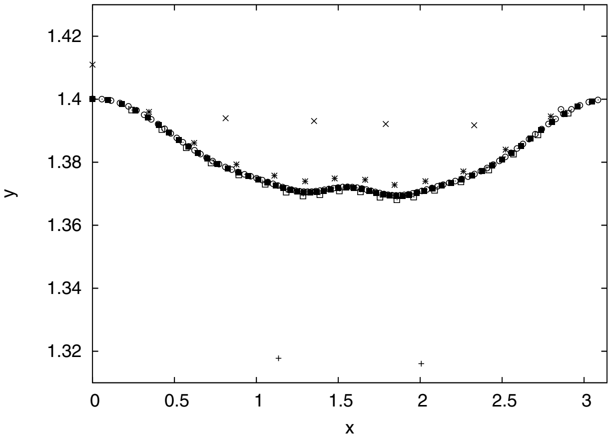

Fig. 7.3. Periodic orbits corresponding to the equations of motion associated to (7.55) approaching the torus with rotation number $\omega = 1 + \frac{1}{2+\frac{\sqrt{5}-1}{2}}$ for $\varepsilon = 0.03$ on a Poincaré section at times 2π. The graph shows the periodic orbits with frequencies 4/3 +, 7/5 ×, 18/13 ∗, 29/21 □, 76/55 ■, 123/89 ∘.

Let us introduce a quantity, called the *residue,* by means of the relation (see [82]):

$$R \equiv \frac{1}{4}(2 - \mathrm{tr}(M)) \ ,$$

where the factors 2 and 4 are introduced for convenience. The eigenvalues of M are related to the residue R by

$$\lambda \ = \ 1 - 2R \pm 2\sqrt{R^2 - R} \ .$$

When $0 < R < 1$ the eigenvalues are complex conjugates with modulus one and the orbit is stable, otherwise when $R < 0$ or $R > 1$ the periodic orbit is unstable. Due to a theorem by Poincaré, for each rational frequency the number of orbits with positive or negative residue is the same. The positive residue orbits are stable for low values of ε. The residue gets larger as the perturbing parameter increases, until it becomes greater than one, thus showing the instability of the associated periodic orbit.

According to [82], we define the *mean residue* of a periodic orbit of period p/q as the quantity

$$f\left(\frac{p}{q};\varepsilon\right) \ \equiv \ (4|R|)^{1/q} \ .$$

The definition of the mean residue for irrational frequencies ω is obtained as follows: if $\omega \equiv [a_0; a_1, \ldots, a_N, \ldots]$, then

$$f(\omega;\varepsilon) = \lim_{N\to\infty} f(\omega_N;\varepsilon) \ ,$$

where $\omega_N \equiv [a_0; a_1, \ldots, a_N]$. If ω is a noble number, say $\omega \equiv [a_0; a_1, \ldots, a_N, 1, 1, 1, \ldots]$, let $\varepsilon = \varepsilon_c(\omega)$ be such that

$$f(\omega, \varepsilon_c(\omega)) = 1 ;$$

then the corresponding residue converges to

$$R \equiv R(\omega; \varepsilon_c(\omega)) = \frac{1}{4}$$

(this assertion justifies the factor 4 introduced in the definition of the mean residue). Greene's method is based on the conjecture that a KAM rotational torus with frequency ω exists if and only if

$$f(\omega; \varepsilon) < 1$$

(see [63] for a partial proof of this statement). In Table 7.2 we consider the first few frequencies of the periodic orbits approaching the torus with frequency equal to the golden ratio. For each periodic orbit of period $\frac{p}{q}$ we report the value of the perturbing parameter $\varepsilon = \varepsilon_c(\frac{p}{q})$ at which the corresponding residue becomes bigger than $\frac{1}{4}$. As $\frac{p}{q}$ increases, the limit of the values $\varepsilon_c(\frac{p}{q})$ provides the breakdown threshold $\varepsilon_c(\omega)$ of the torus with frequency ω.

Table 7.2. Critical values $\varepsilon_c(\frac{p}{q})$ of the perturbing parameter for some periodic orbits approaching the torus with frequency equal to the golden ratio.

$\frac{p}{q}$	$\varepsilon_c(\frac{p}{q})$	$\frac{p}{q}$	$\varepsilon_c(\frac{p}{q})$
$\frac{1}{2}$	0.103	$\frac{13}{21}$	0.144
$\frac{2}{3}$	0.124	$\frac{21}{34}$	0.139
$\frac{3}{5}$	0.158	$\frac{34}{55}$	0.146
$\frac{5}{8}$	0.112	$\frac{55}{89}$	0.145
$\frac{8}{13}$	0.151	$\frac{89}{144}$	0.144

The efficiency of Greene's method strongly depends on the computational speed for the determination of the periodic orbits approaching the invariant surface. In the particular case of the spin–orbit discretized system (7.61), one can get advantage from the fact that the mapping (7.61) including the time variation $t_{j+1} = t_j + h$, herewith denoted as S, can be decomposed as the product of two involutions:

$$S = I_2 \, I_1 ,$$

where $I_1^2 = I_2^2 = 1$. In particular I_1 is given by

$$\begin{aligned} y_{j+1} &= y_j - \varepsilon f_x(x_j, t_j) h \\ x_{j+1} &= -x_j \\ t_{j+1} &= -t_j , \end{aligned}$$

while I_2 takes the form

$$\begin{aligned} y_{j+1} &= y_j \\ x_{j+1} &= -x_j + h y_j \\ t_{j+1} &= -t_j + h \ . \end{aligned}$$

The periodic orbits can be found as fixed points of one of these involutions. This decomposition of the original mapping significantly reduces the computational time for the determination of the periodic orbits, thus making easier the implementation of Greene's method.

7.5 Low–dimensional tori

For a nearly–integrable system with $m+n$ degrees of freedom, we consider the case when the unperturbed Hamiltonian is not integrable in the whole phase space, but rather on some surface foliated by invariant tori whose dimension is less than $m+n$. The proof of the existence of low–dimensional tori is based on Kolmogorov's approach under the requirement that the system satisfies two conditions, namely that it is *isotropic* and *reducible.* The theory of low–dimensional tori is very wide and heavily depends on the properties of the main frequencies of motion. Here, we just aim to give an idea of the problem, referring to [101, 119] for complete details. We start by providing the definitions of isotropic and reducible systems.

Definition. Consider an n–dimensional manifold W endowed with a symplectic non–degenerate 2–form; a submanifold U of W is called *isotropic* if the 2–form restricted to U vanishes.

Definition. Consider a nearly–integrable Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1$ with $m+n$ degrees of freedom. An invariant torus for $\mathcal{H}$ with frequency ω is called *reducible,* if in its neighborhood there exists a set of coordinates $(\underline{I}, \underline{\varphi}, \underline{z}) \in \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{R}^{2m}$, such that the unperturbed Hamiltonian takes the form

$$\mathcal{H}_0(\underline{I}, \underline{\varphi}, \underline{z}) = h(\underline{I}) + \frac{1}{2} A(\underline{I}) \underline{z} \cdot \underline{z} + \mathcal{R}_3(\underline{I}, \underline{\varphi}, \underline{z}) \ , \tag{7.62}$$

where h is a function only of $\underline{I}$, $A(\underline{I})$ is a $2m \times 2m$ symmetric matrix and $\mathcal{R}_3(\underline{I}, \underline{\varphi}, \underline{z})$ is $O(|z|^3)$.

Hamilton's equations associated to (7.62) are given by

$$\begin{aligned} \dot{\underline{z}} &= \Omega(\underline{I}) \underline{z} + O(|\underline{z}|^2) \\ \dot{\underline{I}} &= O(|\underline{z}|^3) \\ \dot{\underline{\varphi}} &= \underline{\omega}(\underline{I}) + O(|\underline{z}|^2) \ , \end{aligned}$$

where $\Omega(\underline{I}) \equiv J A(\underline{I})$, J being the standard symplectic matrix, and $\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}}$. The KAM theorem for low–dimensional tori states that, under suitable conditions on $\Omega(\underline{I})$ and on $\omega(\underline{I})$, one can prove the existence of isotropic, reducible,

n–dimensional invariant tori on which a quasi–periodic motion takes place. The invariant tori are *elliptic* if the eigenvalues of $\Omega(\underline{I})$ are purely imaginary, while they are *hyperbolic* if $\Omega(\underline{I})$ has no purely imaginary eigenvalues. As already mentioned, the proofs of the existence of low–dimensional tori may vary according to the assumptions on the frequencies Ω, $\underline{\omega}$ and we refer to the specialized literature for further details (see, e.g., [2]). Here we just mention how a parametrization in the style of (7.10) can be found for lower–dimensional tori. To see how it works, let us consider a concrete example, and precisely the four–dimensional standard map described by the equations

$$\begin{aligned} y_{n+1} &= y_n + \varepsilon f_1(x_n, z_n, \lambda) \\ x_{n+1} &= x_n + y_{n+1} \\ w_{n+1} &= w_n + \varepsilon f_2(x_n, z_n, \lambda) \\ z_{n+1} &= z_n + w_{n+1} \ , \end{aligned} \tag{7.63}$$

where $(y_n, w_n) \in \mathbf{R}^2$, $(x_n, z_n) \in \mathbf{T}^2$, $\varepsilon > 0$ is the perturbing parameter and $\lambda > 0$ is the coupling parameter. From (7.63) it follows that

$$\begin{aligned} x_{n+1} - 2x_n + x_{n-1} = \varepsilon f_1(x_n, z_n, \lambda) \\ z_{n+1} - 2z_n + z_{n-1} = \varepsilon f_2(x_n, z_n, \lambda) \ . \end{aligned}$$

Let us parametrize a one–dimensional invariant torus with frequency ω by means of the equations

$$\begin{aligned} x_n &= \vartheta + u_1(\vartheta; \varepsilon, \lambda) \\ z_n &= \vartheta + u_2(\vartheta; \varepsilon, \lambda) \ , \end{aligned}$$

where $\vartheta_{n+1} = \vartheta_n + \omega$. One finds that the unknown functions u_1 and u_2 must satisfy the equations

$$\begin{aligned} u_1(\vartheta + \omega) - 2u_1(\vartheta) + u_1(\vartheta - \omega) &= \varepsilon f_1(\vartheta + u_1(\vartheta; \varepsilon, \lambda), \vartheta + u_2(\vartheta; \varepsilon, \lambda), \lambda) \\ u_2(\vartheta + \omega) - 2u_2(\vartheta) + u_2(\vartheta - \omega) &= \varepsilon f_2(\vartheta + u_1(\vartheta; \varepsilon, \lambda), \vartheta + u_2(\vartheta; \varepsilon, \lambda), \lambda) \ , \end{aligned}$$

whose solution describes the low–dimensional torus with frequency ω (see [100]).

Within the spatial three–body problem the existence of low–dimensional tori has been investigated in [99]. In particular, the three–body model studied in [99] admits four degrees of freedom after having performed the reduction of the nodes. Solutions with two or three rationally independent frequencies have been proved, provided the mutual inclinations i_i, i_2 satisfy the condition (see [99])

$$\cos^2(i_1 + i_2) < \frac{3}{5} \ .$$

The existence of quasi–periodic motions with a number of frequencies less than the number of degrees of freedom has been studied also in [113]; in particular, the solutions of the planar three–body problem such that the mean value of the difference of the perihelia is zero have been investigated. The planetary planar $(N+1)$–body problem has been analyzed in [16] and [17], where the existence of N–dimensional elliptic (i.e. linearly stable) tori is shown. Around the elliptic tori there exists a set of positive measure of maximal tori. The proof is based on an elliptic KAM theorem under suitable non–degeneracy conditions (i.e., the so–called Melnikov conditions).

7.6 A dissipative KAM theorem

Let us consider the dissipative spin–orbit equation that we write in compact form as (compare with (5.21))

$$\ddot{x} + \eta(\dot{x} - \nu) + \varepsilon f_x(x,t) = 0 \ , \tag{7.64}$$

where $f_x(x,t) \equiv \varepsilon(\frac{a}{r})^3 \sin(2x - 2f)$, $\eta \equiv K_d \bar{L}(\mathrm{e})$, $\nu \equiv \frac{\bar{N}(\mathrm{e})}{\bar{L}(\mathrm{e})}$. We immediately remark that for $\eta \neq 0$ and $\varepsilon = 0$ the torus $\mathcal{T}_0 \equiv \{y = \nu\} \times \{(\vartheta, \tau) \in \mathbf{T}^2\}$ is a global attractor and the flow on $\mathcal{T}_0$ is given by $(\vartheta, \tau) \to (\vartheta + \nu t, \tau + t)$. This is easily seen from the fact that the solution of (7.64) for $\varepsilon = 0$ is given by

$$x(t) = x_0 + \nu(t - t_0) + \frac{1 - e^{-\eta(t-t_0)}}{\eta} (v_0 - \nu) \ ,$$

where $x_0 \equiv x(t_0)$ and $v_0 \equiv \dot{x}(t_0)$. An invariant attractor with frequency ω is parametrized by

$$x(t) = \vartheta + u(\vartheta, t) \ , \tag{7.65}$$

where $u = u(\vartheta, t)$ is a real analytic function for $(\vartheta, t) \in \mathbf{T}^2$ and $\dot{\vartheta} = \omega$. The existence of the invariant attractor with frequency ω for (7.64) is provided by the following

Theorem [32]. *Assume that ω is diophantine; then, there exists $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ and for any $0 \leq \eta < 1$, there exists a function $u = u(\vartheta, t)$ with $\langle u \rangle = 0$ and $1 + u_\vartheta \neq 0$, such that (7.65) is a solution of (7.64) provided*

$$\nu = \omega \ (1 + \langle (u_\vartheta)^2 \rangle) \ . \tag{7.66}$$

The proof of the theorem is based on the following ideas (we refer to [32] for full details). Let us start by introducing the operator $\partial_\omega \equiv \omega \frac{\partial}{\partial \vartheta} + \frac{\partial}{\partial t}$, so that $\dot{x} = \omega + \partial_\omega u$ and $\ddot{x} = \partial_\omega^2 u$. The solution (7.65) is quasi–periodic if the function u satisfies

$$\partial_\omega^2 u + \eta \partial_\omega u + \varepsilon f_x(\vartheta + u, t) + \gamma = 0 \ , \qquad \gamma \equiv \eta(\omega - \nu) \ . \tag{7.67}$$

The unknowns u, γ must satisfy the compatibility condition

$$\eta\omega \ \langle (u_\vartheta)^2 \rangle + \gamma = 0 \ , \tag{7.68}$$

which is equivalent to (7.66). The proof of the existence of the quasi–periodic attractor is perturbative in ε, but uniform in η; the conservative KAM torus bifurcates in the attractor as far as $\eta \neq 0$. For the spin–orbit problem, one has to keep in mind that in place of η and ν one should consider the dissipative constant K_d and the eccentricity e. As a consequence, the theorem is stated for any $0 \le K_d < 1$ and besides the existence of a function $u = u(\vartheta, t)$, one needs to find a function $\mathrm{e} = \mathrm{e}(K_d, \omega, \varepsilon) = \nu_{\mathrm{e}}^{-1}(\omega) + O(\varepsilon^2)$ to satisfy the compatibility condition (7.66).

Coming back to equation (7.67), let us introduce the operators

$$D_\eta u \equiv \partial_\omega u + \eta u \ , \qquad \Delta_\eta u \equiv D_\eta \partial_\omega u = \partial_\omega D_\eta u \ .$$

Then, (7.67) becomes

$$\mathcal{F}_\eta(u;\gamma) \equiv \Delta_\eta u + \varepsilon f_x(\vartheta + u, t) + \gamma = 0 \ .$$

In particular, if $u = \sum_{(n,m)\in \mathbf{Z}^2} \hat{u}_{n,m} e^{i(n\vartheta + mt)}$, then

$$\begin{aligned} \partial_\omega u &= \sum_{(n,m)\in\mathbf{Z}^2} i(\omega n + m)\hat{u}_{n,m} e^{i(n\vartheta+mt)} \\ D_\eta u &= \sum_{(n,m)\in\mathbf{Z}^2} [i(\omega n + m) + \eta]\hat{u}_{n,m} e^{i(n\vartheta+mt)} \ ; \end{aligned}$$

being $|i(\omega n + m) + \eta| \ge |\eta| > 0$, then D_η is invertible with

$$D_\eta^{-1} u = \sum_{(n,m)\in\mathbf{Z}^2} \frac{\hat{u}_{n,m} e^{i(n\vartheta+mt)}}{i(\omega n + m) + \eta} \ .$$

Having introduced the norm $\|u\|_\xi \equiv \sum_{(n,m)\in\mathbf{Z}^2} |\hat{u}_{n,m}| e^{(|n|+|m|)\xi}$, one can state the following

Theorem. *Let $0 < \xi < \bar{\xi} \le 1$, $0 \le \eta < 1$; let ω be diophantine and define M such that*

$$\|\varepsilon f_{xxx}\|_{\bar{\xi}} \le M \ .$$

Assume that there exists an approximate solution $v = v(\vartheta, t; \eta)$, $\beta = \beta(\eta)$ such that v_ϑ is bounded and invertible; let the error function $\chi = \chi(\vartheta, t; \eta) \equiv \mathcal{F}_\eta(v; \beta)$ satisfy a smallness requirement of the form

$$\mathcal{D}\ \|\chi\|_\xi \le 1 \ ,$$

where $\mathcal{D}$ depends upon ξ, M, as well as upon the norms of v and of its derivatives. Then, there exist $u = u(\vartheta, t; \eta) \in C^\infty$ and $\gamma = \gamma(\eta) \in C^\infty$, which solve $\mathcal{F}_\eta(u;\gamma) = 0$.

The proof is constructive and the solution is obtained as the limit of a sequence of approximate solutions (v_j, β_j), quadratically converging to the solution (u, γ). We sketch here the proof as a sequence of five main steps, referring to [32] for complete details.

Step 1. Establish some properties of the operators D_η, Δ_η as well as of their derivatives and inverse functions, providing formulae of the form

$$\|D_\eta^{-s}\,\partial_\vartheta^p u\|_{\xi-\delta} \le \sigma_{p,s}(\delta)\,\|u\|_\xi \ ,$$

for some $0<\delta<\xi$ and for $p,s\in\mathbf{Z}_+$, where

$$\sigma_{p,s}(\delta) \equiv \sup_{(n,m)\in\mathbf{Z}^2\backslash\{\underline{0}\}} \Big(|i(\omega n+m)+\eta|^{-s}|n|^p e^{-\delta(|n|+|m|)}\Big) \ ,$$

which can be bounded as

$$\sigma_{p,s}(\delta) \le \left(\frac{s\tau+p}{e}\right)^{s\tau+p} C^s\delta^{-(s\tau+p)} \ .$$

It turns out that $\langle(1+u_\vartheta)\mathcal{F}_\eta(u;\gamma)\rangle = \eta\omega\langle(u_\vartheta)^2\rangle+\gamma$; if $\mathcal{F}_\eta(u;\gamma)=0$ one finds the compatibility condition (7.68).

Step 2. Given an approximate solution (v,β) of $\mathcal{F}_\eta(u;\gamma)=0$, a quadratically smaller approximation (v',β') is found by a Newton iteration scheme. More precisely, starting from

$$\chi \equiv \mathcal{F}_\eta(v;\beta) = \Delta_\eta v + \varepsilon f_x(\vartheta+v,t)+\beta \ ,$$

one looks for a solution

$$v' = v+\tilde v \ , \qquad \beta' = \beta+\tilde\beta \ ,$$

such that $\tilde v,\tilde\beta = O(\|\chi\|)$, $\mathcal{F}_\eta(v';\beta') = O(\|\chi\|^2)$. In order to find $\tilde v$ and $\tilde\beta$, setting $V\equiv 1+v_\vartheta$ let us introduce the quantities

$$Q_1 \equiv \varepsilon[f_x(\vartheta+v+\tilde v,t) - f_x(\vartheta+v,t) - f_{xx}(\vartheta+v,t)\tilde v] \ , \qquad Q_2 \equiv V^{-1}\chi_\vartheta\ \tilde v \ ;$$

it follows that

$$\mathcal{F}_\eta(v';\beta') \equiv \mathcal{F}_\eta(v+\tilde v;\beta+\tilde\beta) = \chi+\tilde\beta+A_{\eta,v}\tilde v+Q_1+Q_2$$

with $A_{\eta,v}\tilde v \equiv V^{-1}\ D_\eta\Big(V^2 D_0(V^{-1}\tilde v)\Big)$. One can find explicit expressions for $\tilde v$, $\tilde\beta$, such that they satisfy the relation

$$\chi+\tilde\beta+A_{\eta,v}\tilde v = 0 \ ;$$

the latter equation provides $\chi' \equiv \mathcal{F}_\eta(v+\tilde v,\beta+\tilde\beta) = Q_1+Q_2$, so that the new error term is quadratically smaller.

Step 3. Given the estimates on the norms of v_θ, $\tilde v$, $\tilde v_\theta$, $\tilde\beta$, a KAM algorithm is implemented to compute an estimate on the norm of the error function χ' of the form

$$\|\chi'\|_{\xi-\delta} \le C_1\delta^{-s}\|\chi\|_\xi^2 \ ,$$

for some C_1, $s>0$.

Step 4. Implement a KAM algorithm which provides that under smallness conditions on the parameters there exists a sequence (v_j, β_j) of approximate solutions, which converges to the true solution:

$$(u, \gamma) \equiv \lim_{j\to\infty} (v_j, \beta_j) \ ,$$

where (u, γ) satisfy $\mathcal{F}_\eta(u; \gamma) = 0$.

Step 5. A local uniqueness is shown by proving that if there exists a solution $\xi(t) = \vartheta + w(\vartheta, t)$ with $\dot{\vartheta} = \omega$ and $\langle w \rangle = 0$, then $w \equiv u$, while ν coincides with (7.66).

7.7 Converse KAM

Converse KAM theory provides upper bounds on the perturbing parameter ensuring the non–existence of invariant tori. Following [126, 128, 129] (see also [6]) we adopt the Lagrangian formulation as follows. As in the previous sections, we are concerned with applications to the spin–orbit model; therefore we introduce a one–dimensional, time–dependent Lagrangian function of the form $\mathcal{L} = \mathcal{L}(x, y, t)$, where $x \in \mathbf{T}$, $y \in \mathbf{R}$. We assume that the Lagrangian function satisfies the so–called *Legendre condition,* which requires that $\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2}$ is everywhere positive. A function $x = x(t)$ is an orbit for $\mathcal{L}$ if for any $t_0 < t_1$ and for any variation $\delta x = \delta x(t)$ such that $\delta x(t_0) = \delta x(t_1) = 0$, the variation $\delta \mathcal{A}$ of the action is zero, where

$$\mathcal{A}(x) \equiv \int_{t_0}^{t_1} \mathcal{L}(x(t), \dot{x}(t), t) \ dt \ . \tag{7.69}$$

A trajectory $x = x(t)$ has *minimal action* if for any $t_0 < t_1$ and $\tilde{x}(t)$ such that $\tilde{x}(t_0) = x(t_0)$, $\tilde{x}(t_1) = x(t_1)$, then $\mathcal{A}(x) \leq \mathcal{A}(\tilde{x})$. The minimal action is *non–degenerate* if for any $t_0 < t_1$, then $\delta^2 \mathcal{A}$ is positive definite for any variation δx such that $\delta x(t_0) = \delta x(t_1) = 0$.
The Legendre transformation allows us to introduce the Hamiltonian function $\mathcal{H} = \mathcal{H}(y, x, t)$ associated to $\mathcal{L}$, where $y \in \mathbf{R}$ is the momentum associated to x. A *Lagrangian graph* is described by a C^1–generating function $\mathcal{S} = \mathcal{S}(x, t)$ such that $y = \mathcal{S}_x(x, t)$, $T = \mathcal{S}_t(x, t)$, where T is the the variable conjugated to the time in the extended phase space. We now give a characterization of Lagrangian graphs and rotational tori.

Proposition [129]. *An invariant rotational two–dimensional torus for* $\mathcal{H}_1(y, x, T, t) \equiv \mathcal{H}(y, x, t) + T$ *with* $\mathcal{H}_{yy}$ *positive definite is a Lagrangian graph.*

Moreover, we have the following

Lemma [129]. *If* Σ *is an invariant surface for the Hamiltonian* $\mathcal{H}_1(y, x, T, t) \equiv \mathcal{H}(y, x, t) + T$ *such that locally* $y = \mathcal{S}_x(x, t)$, *then* Σ *is a Lagrangian graph.*

In order to introduce a converse KAM criterion, we need the following theorem due to K. Weierstrass (see [129]).

Theorem. *If Σ is an invariant Lagrangian graph for a Lagrangian system satisfying the Legendre condition, then any orbit on Σ has a non–degenerate minimal action.*

From the Weierstrass theorem it follows that if the orbit segment $x = x(t)$ for $t \in [t_0, t_1]$ is not a non–degenerate minimum for $\mathcal{A}$, then it is not contained in any invariant Lagrangian graph. In practice, one should compute the quantity $\delta^2\mathcal{A}$ for some variation δx with $\delta x(t_0) = \delta x(t_1) = 0$ and check whether it fails to be positive definite. This method allows us to give an elementary analytical estimate, which can be explicitly computed. Following [40], let us consider the spin–orbit equation (5.18) that we write as

$$\ddot{x} + \varepsilon \sum_{m=1}^{N} \alpha_m(\mathrm{e}) \sin(2x - mt) = 0 \tag{7.70}$$

for some $N > 0$; the coefficients $\alpha_m(\mathrm{e})$ are trivially related to the coefficients $W(\frac{m}{2}, \mathrm{e})$ in (5.18). We apply the criterion based on the Weierstrass theorem to the model described by (7.70). The Lagrangian function associated to (7.70) has the form

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}\dot{x}^2 + \frac{\varepsilon}{2} \sum_{m=1}^{N} \alpha_m(\mathrm{e}) \cos(2x - mt) \ .$$

The second variation of the action is given by

$$\delta^2\mathcal{A} = \int_{t_0}^{t_1} \left[\delta\dot{x}^2 - 2\varepsilon \sum_{m=1}^{N} \alpha_m(\mathrm{e}) \cos(2x - mt)\delta x^2 \right] dt \ .$$

Consider the deviation $\delta x(t) = \cos\frac{t}{4\tau}$ such that $\delta x(\pm 2\pi\tau) = 0$; notice that $\int_0^{2\pi\tau} \delta x^2 = \pi\tau$, $\int_0^{2\pi\tau} \delta\dot{x}^2 = \frac{\pi}{16\tau}$. Writing (7.70) as

$$\ddot{x} = g(x, t) \equiv -\varepsilon \sum_{m=1}^{N} \alpha_m(\mathrm{e}) \sin(2x - mt)$$

and assuming the initial conditions $x(0) = 0$, $\dot{x}(0) = v_0$, the solution of (7.70) can be written in integral form as

$$x(t) = v_0 t + \int_0^t (t - s) g(x(s), s) \ ds \ .$$

Let G be a bound on $g(x, t)$, i.e. $|g(x, t)| \leq \varepsilon \sum_{m=1}^{N} |\alpha_m(\mathrm{e})| \equiv G$; as a first approximation we can use the inequality

$$|x(t) - v_0 t| \leq \frac{G}{2} t^2 \ .$$

Since $\cos\vartheta \geq 1 - \frac{1}{2}\vartheta^2$, we obtain

$$\cos(2x - mt) \geq 1 - \frac{1}{2}(|m - 2v_0|t + Gt^2)^2 \ .$$

Therefore the second variation of the action for the variation $\delta x(t) = \cos\frac{t}{4\tau}$, $-2\pi\tau \le t \le 2\pi\tau$, is bounded by

$$\delta^2\mathcal{A} \le \frac{\pi}{8\tau} - 4\varepsilon \sum_{m=1}^{N} |\alpha_m(\mathrm{e})| \int_0^{2\pi\tau} \left[1 - \frac{1}{2}(|m - 2v_0|t + Gt^2)^2\right] \delta x^2 \; dt$$

$$\le \frac{\pi}{8\tau} - 4G\pi\tau + 2\varepsilon \sum_{m=1}^{N} |\alpha_m(\mathrm{e})| \int_0^{\frac{\pi}{2}} \Big[|m - 2v_0|^2 (4\tau)^3 \vartheta^2 \cos^2\vartheta$$

$$+G^2(4\tau)^5 \vartheta^4 \cos^2\vartheta + 2|m - 2v_0| G (4\tau)^4 \vartheta^3 \cos^2\vartheta\Big] \; d\vartheta \; .$$

Let us define the quantity

$$I_n \equiv 2 \int_0^{\frac{\pi}{2}} \vartheta^n \cos^2\vartheta \; d\vartheta \; ;$$

then, one obtains

$$\frac{\delta^2\mathcal{A}}{\tau} \le \frac{\pi}{8\tau^2} - 4G\pi + \frac{\varepsilon}{\tau} \sum_{m=1}^{N} |\alpha_m(\mathrm{e})| \cdot \Big[|m - 2v_0|^2 (4\tau)^3 I_2$$

$$+2|m - 2v_0| G (4\tau)^4 I_3 + G^2 (4\tau)^5 I_4\Big] \equiv \Phi(\varepsilon, v_0, \tau) \; . \quad (7.71)$$

The non–existence criterion is fulfilled whenever one can find $\tau > 0$ such that $\Phi(\varepsilon, v_0, \tau) < 0$, so that $\delta^2\mathcal{A} < 0$. Denote by ε_{NE} the value of the perturbing parameter at which this condition first occurs. As concrete examples we consider the orbital eccentricity of the Moon (e = 0.0549) and of Mercury (e = 0.2056); moreover we consider $v_0 = 1$ and $v_0 = 1.5$, corresponding, respectively, to the 1:1 and 3:2 resonance. The results of the implementation of the Weierstrass criterion based on the estimate (7.71) are provided in Table 7.3, where $N = 7$ has been taken in (7.70) (see [40]). Though the estimates are rather crude and could be further refined, they show how to find by simple explicit computations the regions of non–existence of rotational invariant tori.

Table 7.3. The non–existence criterion based on the Weierstrass theorem provides the following values associated to the Moon with eccentricity e = 0.0549 and to Mercury with eccentricity e = 0.2056 (reprinted, with permission, from [40], Copyright 2007, American Institute of Physics).

	Moon	Mercury
$v_0 = 1$	$\varepsilon_{NE} \simeq 0.15$	$\varepsilon_{NE} \simeq 0.82$
$v_0 = 1.5$	$\varepsilon_{NE} \simeq 0.77$	$\varepsilon_{NE} \simeq 0.58$

7.7.1 Conjugate points criterion

A method of investigating the non–existence of invariant tori has been formulated in [129] for conservative systems, based on the following

Definition. Let $(x, y) : [t_0, t_1] \to \mathbf{T} \times \mathbf{R}$ be an orbit; the times t_0 and t_1 are said to be *conjugate,* if there exists a non–zero tangent orbit $(\delta x, \delta y)$, such that $\delta x(t_0) = \delta x(t_1) = 0$.

We also introduce the twist property as follows. Let us write (7.70) as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\varepsilon \sum_{m=1}^{N} \alpha_m(e) \sin(2x - mt) \ . \end{aligned} \tag{7.72}$$

We say that (7.72) satisfies the *twist property* if there exists a constant $A > 0$ such that

$$\frac{\partial \dot{x}}{\partial y} \geq A$$

(in our case $A = 1$). A result due to K. Jacobi shows that minimizing orbits (with respect to the action (7.69)) have no conjugate points. This leads to the following non–existence criterion, which can be formulated to encompass also the dissipative context [40].

Conjugate points criterion: The existence of conjugate points implies that the orbit does not belong to any rotational invariant torus, otherwise the forward orbit starting from the initial *vertical* vector $(0, 1)$ at $t = t_0$ is prevented from crossing the tangent to the torus and the twist property implies that $\delta x(t) > 0$ for all $t > t_0$.

For the conservative case with time–reversal symmetry and initial conditions on the symmetry line $x = 0$, the backward trajectory and the backward tangent orbit are determined by reflecting the forward ones. We can conclude that the times $\pm t$ are conjugate whenever the tangent orbit of the *horizontal* vector $(\delta x(0), \delta y(0)) = (1, 0)$ satisfies $\delta x(t) = 0$. This remark considerably decreases the computational time, also due to the fact that close to a suitable symmetry line the rotation of the tangent orbits is strongest and it is convenient to select orbit segments which straddle it symmetrically.

The dissipative case does not admit time–reversal symmetry and it is necessary to integrate backward and forward orbits. However, one can choose $t_0 = 0$ and avoid backward integration, thus integrating just forwards from the vertical vector $(\delta x(0), \delta y(0)) = (0, 1)$ and then looking for a change of sign of $\delta x(t)$. We report in Figure 7.4 an application of the conjugate points criterion for the dissipative spin–orbit problem and for different values of the eccentricity. A grid of 500×500 points over $y(0) \in [0.2, 2]$ and $\varepsilon \in [0, 0.1]$ has been computed.

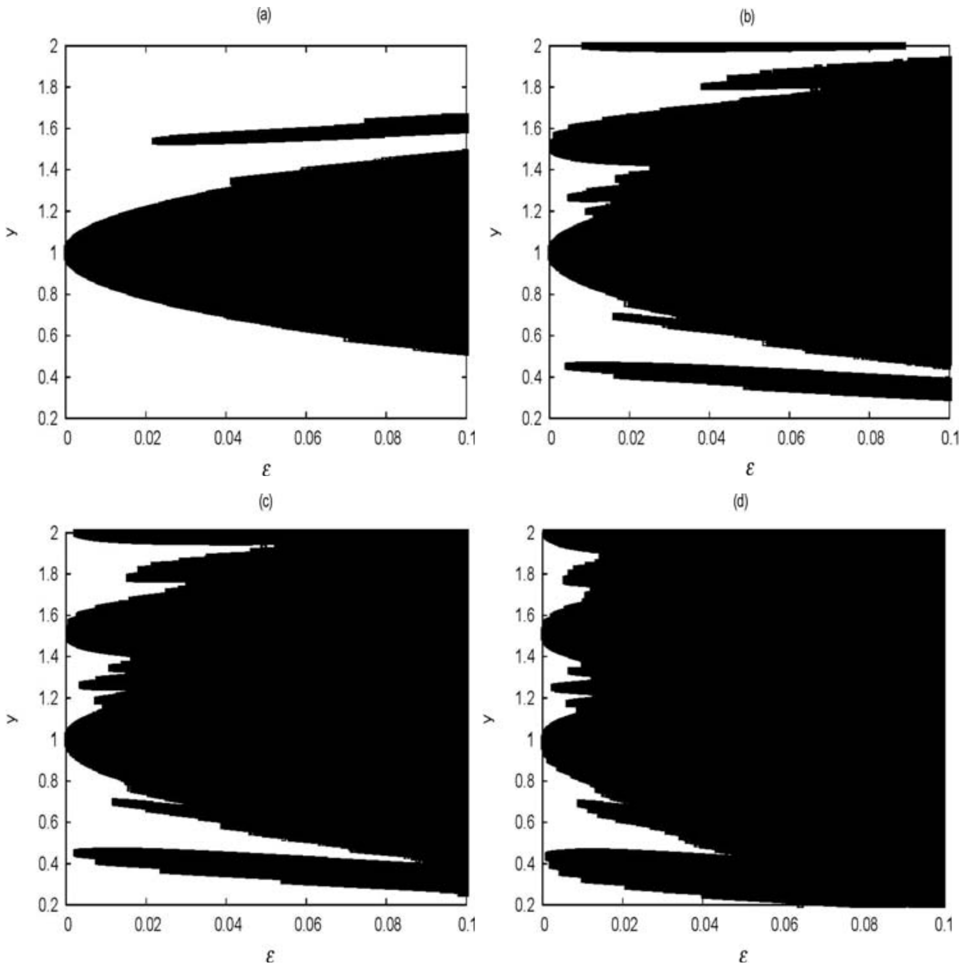

Fig. 7.4. The black region denotes the non–existence of rotational invariant tori for $K_d = 10^{-3}$. (a) e = 0.001, (b) e = 0.0549, (c) e = 0.1, (d) e = 0.2056.

7.7.2 Cone-crossing criterion

Without using time–reversal symmetry and without taking initial conditions on a symmetry line, the conjugate points criterion with $t_0 = -t_1$ can be applied, provided one computes the slope of an initial tangent vector, say $(\delta x(0), \delta y(0))$, such that $\delta x(\pm t_1) = 0$ simultaneously. To this end one can compute the monodromy matrix M at times $\pm t$ by integrating the equations

$$\dot{M} = F(x, y, t)M \ ,$$

where $M(0)$ equals the identity matrix and $F(x, y, t)$ denotes the Jacobian of the vector field. Then, the initial condition $(\delta x(0), \delta y(0)) \equiv (\xi, \eta)$ satisfies the relations

$$M_{11}(t)\xi + M_{12}(t)\eta = 0 ,$$
$$M_{11}(-t)\xi + M_{12}(-t)\eta = 0 .$$

There exists a non–zero solution if and only if

$$C(t) \equiv M_{11}(t)M_{12}(-t) - M_{12}(t)M_{11}(-t) = 0 .$$

Therefore we conclude that the times $\pm t$, $t > 0$, are conjugate if and only if $C(t) = 0$. A result by Birkhoff states that a rotational invariant torus is a graph of a Lipschitz function.

If the initial condition is on a rotational invariant torus, one can determine upper and lower bounds on the slope of the initial tangent vector, providing the so–called local *Lipschitz cone* [165]. The condition $C(t) = 0$ corresponds to the equality of the upper and lower bounds; for larger t the upper bound becomes less than the lower bound. However, this is in contrast with the existence of a rotational invariant torus through that initial point, thus yielding the so–called *cone–crossing criterion* [128] as a method to establish the non–existence of rotational invariant tori.

The practical implementation of the criterion is the following. First we remark that it is more convenient to integrate the equation for the inverse monodromy matrix $N(t) = M(t)^{-1}$. Starting from $(x(0), y(0))$, let $(x(\pm t), y(\pm t))$ be the corresponding forward and backward trajectories; then integrate the equations backwards and forwards in time

$$\dot{N}(t) = -N(t)\ F(x(t), y(t), t)$$

with $N(0)$ being the identity matrix. For any $t > 0$, let

$$w^{\pm}(t) = N(\mp t)\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \pm N_{12}(\mp t) \\ \pm N_{22}(\mp t) \end{pmatrix}$$

be tangent vectors at $(x(0), y(0))$, which give a local Lipschitz cone through the initial condition. Let $C(t) = w^{-}(t) \wedge w^{+}(t)$; then $C(0) = 0$ and $\dot{C}(0) > 0$. Finally, if there exists a time $t' > 0$ such that $C(t') \leq 0$, then the orbit starting from $(x(0), y(0))$ does not belong to an invariant rotational torus.

7.7.3 Tangent orbit indicator

Based on the conjugate points criterion, we introduce an indicator of chaos, which can be used as a complementary tool to Lyapunov exponents, frequency analysis, FLIs, etc. (see Chapter 2). We start by remarking that through the change of sign of $\delta x(t)$ we can distinguish between rotational tori, librational tori and chaos. Starting from a horizontal tangent vector, for a librational torus the δx–component oscillates around zero (a linear increase is observed when starting from the vertical tangent vector). The results are shown in Figure 7.5(*a*,*b*), obtained by integrating (7.70) through a fourth–order symplectic Yoshida's method [175] shortly recalled in Appendix F. Notice that the first crossing occurs at $t = 3.39$. A similar behavior is observed for the chain of islands of Figure 7.5(*c*,*d*). Oscillations with large

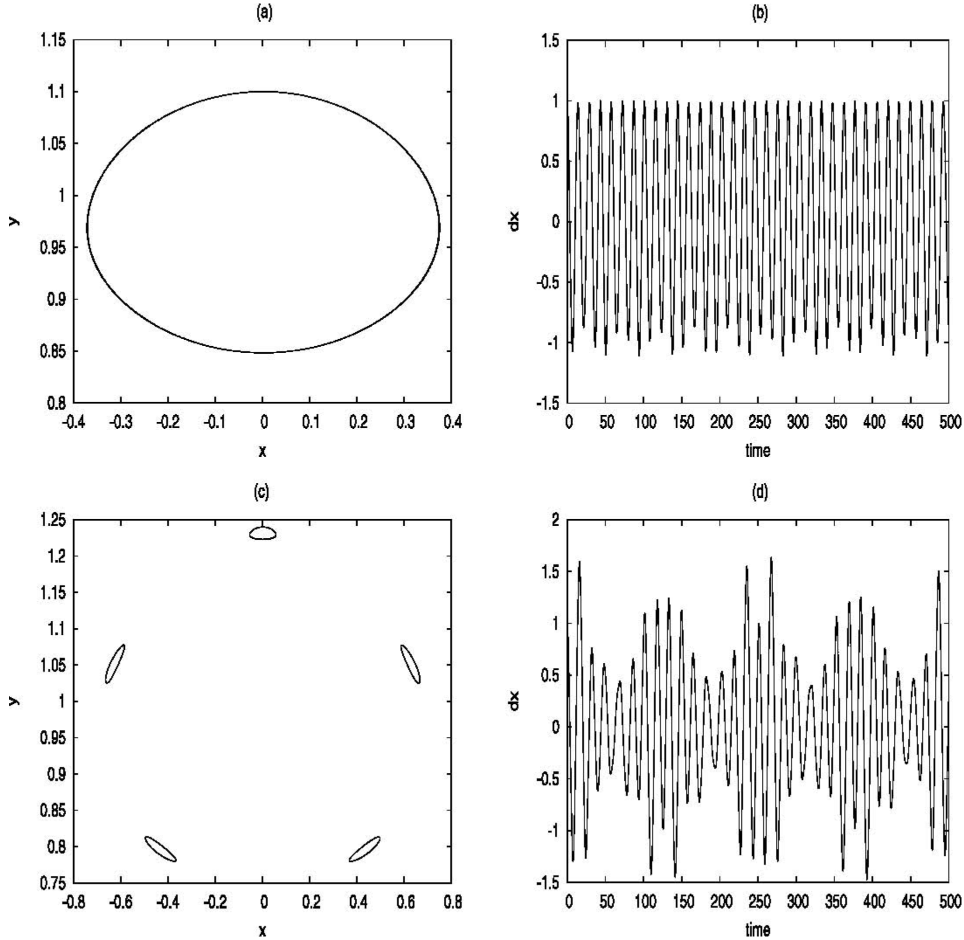

Fig. 7.5. Analysis of (7.70) with $\varepsilon = 0.1$, e $= 0.0549$ (after [40]). The left column shows the Poincaré section on the plane $t = 0$; the right column shows the implementation of the conjugate points method from the horizontal tangent vector. (a, b) refer to an example of a librational invariant torus for the initial conditions $x = 0$, $y = 1.1$. (c, d) refer to an example with a chain of islands for the initial conditions $x = 0$, $y = 1.24$. (e, f) refer to an example of chaotic motion for the initial conditions $x = 0$, $y = 1.3$. (g, h) refer to an example of a rotational invariant torus for the initial conditions $x = 0$, $y = 1.8$ (reprinted, with permission, from [40], Copyright 2007, American Institute of Physics).

amplitudes are observed for chaotic motions as shown in Figure 7.5(e,f). Finally, rotational invariant tori are characterized by positive oscillations of δx far from zero (see Figure 7.5(g,h)).

This scenario leads to the introduction of the so–called *tangent orbit indicator* by computing the average of $\delta x(t)$ over a finite interval of time. The resulting value characterizes the dynamics as follows: a zero value denotes a librational regime, a

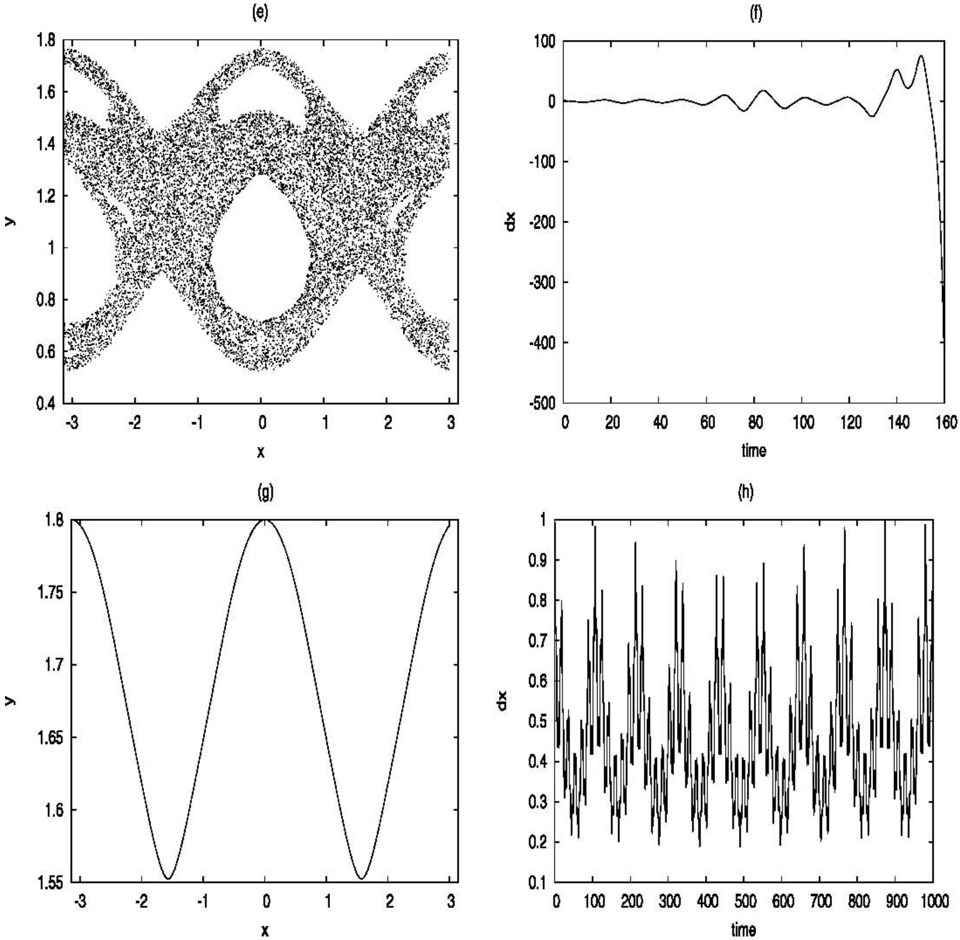

Fig. 7.5. (continued).

moderate value is associated to rotational tori, high values correspond to chaotic motions.

As an example we report in Figure 7.6 (top panels) the computation of the tangent orbit indicators with horizontal initial tangent vector over a grid of 500×500 initial conditions in x and y for the equation (7.70). Figure 7.6 (bottom panels) provides the tangent orbit indicator in the plane y–ε for a fixed x_0. A black color denotes tangent orbit indicators close to zero, grey stands for moderate values, while white corresponds to large values. The results are in full agreement with those obtained implementing other techniques, like frequency analysis or the computation of the FLIs introduced in Chapter 2 (see [37]).

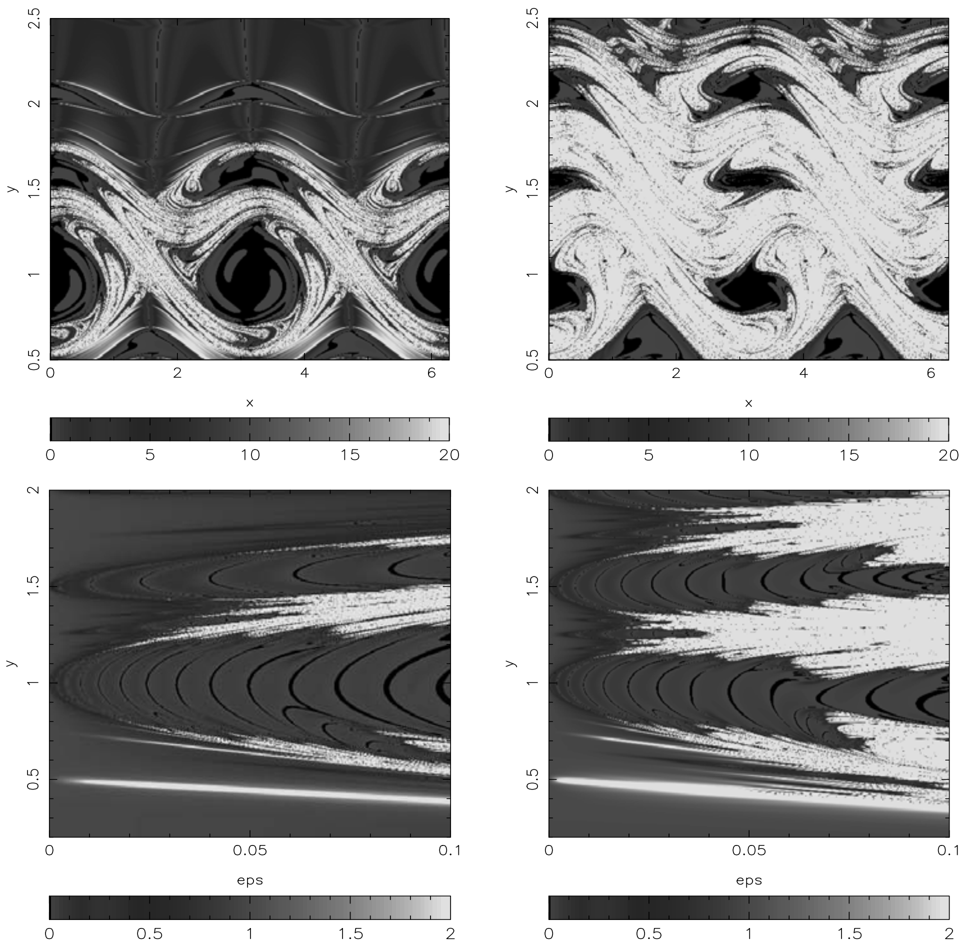

Fig. 7.6. Tangent orbit indicator associated to (7.70) for $\varepsilon = 0.1$ from the initial horizontal tangent vector. Top left: graph in the plane x–y with e = 0.0549; top right: graph in the plane x–y with e = 0.2056; bottom left: graph in the plane ε–y with e = 0.0549; bottom right: graph in the plane ε–y with e = 0.2056. (Reprinted, with permission, from [40], Copyright 2007, American Institute of Physics.)

7.8 Cantori

Let $\mathcal{L} = \mathcal{L}(\underline{x}, \underline{X})$ be a Lagrangian function with $\underline{x} \in \mathbf{T}^n$ and $\underline{X} \equiv \dot{\underline{x}} \in \mathbf{R}^n$. For a function $v = v(\underline{\vartheta})$, let $D_{\underline{\omega}}$ be the operator defined as $D_{\underline{\omega}} v = \underline{\omega} \cdot \frac{\partial v}{\partial \underline{\vartheta}}$. An n–dimensional torus is described by the equations $\underline{x} = \underline{x}(\underline{\vartheta})$, $\underline{X} = D_{\underline{\omega}} \underline{x}(\underline{\vartheta})$; let a variation be described as $\underline{x}(\underline{\vartheta}) + \delta \underline{x}(\underline{\vartheta})$, $D_{\underline{\omega}} \underline{x}(\underline{\vartheta}) + D_{\underline{\omega}} \delta \underline{x}(\underline{\vartheta})$. Let us introduce the functional

$$\mathcal{A}_{\underline{\omega}} \equiv \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} \mathcal{L}(\underline{x}(\underline{\vartheta}), D_{\underline{\omega}} \underline{x}(\underline{\vartheta}))\ d\underline{\vartheta}\ .$$

A variational principle can be stated as follows

Theorem [6]. *A smooth surface is an invariant torus with frequency* $\underline{\omega}$ *if and only if it is a stationary point of the functional* $\mathcal{A}_{\underline{\omega}}$.

A solution of the variational problem is a so–called *cantorus,* which is defined as follows (see [43]). Let us consider the case $n = 2$. We introduce the following definition (see [7, 132]).

Definition. An *Aubry–Mather set* is an invariant set, which is obtained embedding a Cantor subset in the phase space of the standard two–dimensional torus.

Let us consider a one–dimensional, time–dependent Hamiltonian of the form $\mathcal{H} = \mathcal{H}(y, x, t)$. Assume it admits two invariant tori described by $y = y_0$, $y = y_1$ with $y_0 < y_1$. Denote by $\underline{\Phi} = \underline{\Phi}(y, x) \equiv (\Phi_1(y, x), \Phi_2(y, x))$ the Poincaré map associated to $\mathcal{H}$ at times 2π, which we assume to satisfy the so–called twist condition namely $\frac{\partial \Phi_2(y,x)}{\partial y} > 0$; the mapping $\underline{\Phi}$ is area preserving and it leaves invariant the circles $y = y_0$, $y = y_1$ as well as the annulus between them. Let $\omega_0 \equiv \omega(y_0)$ and $\omega_1 \equiv \omega(y_1)$ be the frequencies corresponding to y_0 and y_1. By the twist condition one has that $\omega_0 < \omega_1$. The Aubry–Mather theorem can be stated as follows.

Theorem [6]. *For any irrational* $\omega \in (\omega_0, \omega_1)$*, there exists an Aubry–Mather set* with rotation number ω, which is a subset of a closed curve parametrized by $x = \vartheta + u(\vartheta)$, $y = v(\vartheta)$, where $\vartheta \in \mathbf{T}$ is such that $\vartheta' = \vartheta + \omega$, u is monotone and u, v are 2π–periodic.

If the functions u and v are continuous, then the original Hamiltonian system admits a two–dimensional invariant torus with frequency ω. On the other hand, if u and v are discontinuous, then the original Hamiltonian system admits a cantorus, whose gaps coincide with the discontinuities of u and v. We remark that a Cantor set does not divide the phase space into invariant regions, since the orbits can diffuse through the gaps of the Cantor set. However, the leakage cannot be easy and the cantorus can still act as a barrier over long time scales [153].

The numerical detection of cantori is rather difficult and they are often approximated by high–order periodic orbits [49, 85]. In very peculiar examples, an analytic expression of the cantori can be given. This is the case of the conservative sawtooth map, which is described by the equations

$$\begin{aligned} y_{n+1} &= y_n + \lambda f(x_n) \\ x_{n+1} &= x_n + y_{n+1} \ , \end{aligned} \tag{7.73}$$

where $x_n \in \mathbf{T}$, $y_n \in \mathbf{R}$, $\lambda \in \mathbf{R}_+$ denotes the perturbing parameter and the perturbation f on the covering $\mathbf{R}$ of $\mathbf{T}$ is defined as

$$\begin{aligned} f(x) &\equiv \mathrm{mod}(x, 1) - \frac{1}{2} && \text{if } 0 < \mathrm{mod}(x, 1) < 1 \\ f(x) &\equiv 0 && \text{if } x \in \mathbf{Z} \ . \end{aligned} \tag{7.74}$$

The mapping (7.73) is area–preserving; for $\lambda > 0$ there do not exist invariant circles and the phase space is filled by cantori and periodic orbits. Since $x_{n+1} - x_n = y_{n+1}$,

$x_n - x_{n-1} = y_n$, one obtains

$$x_{n+1} - 2x_n + x_{n-1} = \lambda f(x_n) \ .$$

Let us parametrize a solution with frequency $\omega \in \mathbf{R}$ as

$$x(\vartheta) = \vartheta + u(\vartheta) \ , \qquad \vartheta \in \mathbf{T} \ , \tag{7.75}$$

where $\vartheta' = \vartheta + \omega$. Then, the function u must satisfy the equation

$$u(\vartheta + \omega) - 2u(\vartheta) + u(\vartheta - \omega) = \lambda \ f(\vartheta + u(\vartheta)) \ .$$

We can determine $u(\vartheta)$ by expanding it as

$$u(\vartheta) = \sum_{n=-\infty}^{\infty} a_n f(\vartheta + n\omega) \tag{7.76}$$

for some coefficients a_n which are given by

$$a_n = -\alpha \ \rho^{-|n|} \ ,$$

with

$$\alpha \equiv \left(1 + \frac{4}{\lambda}\right)^{-1/2} , \qquad \rho = 1 + \frac{\lambda}{2} + \left(\lambda + \frac{\lambda^2}{4}\right)^{1/2} .$$

In fact, inserting the series expansion (7.76) in (7.73) we obtain

$$\sum_j (a_{j-1} - 2a_j + a_{j+1}) f(\vartheta + j\omega) = \lambda f(\vartheta + u(\vartheta)) \ .$$

Being $f(\vartheta + u(\vartheta)) = \vartheta + u(\vartheta) - \frac{1}{2} = f(\vartheta) + u(\vartheta)$, one finds the following recursive relations

$$\begin{aligned} a_{-1} - 2a_0 + a_1 &= \lambda(1 + a_0) && j = 0 \\ a_{j-1} - 2a_j + a_{j+1} &= \lambda a_j && j \neq 0 \ . \end{aligned} \tag{7.77}$$

Let us write $a_j = -\alpha\rho^{-|j|}$; from the first of (7.77) for $j = 0$ one has $-\alpha\rho^{-1} + 2\alpha - \alpha\rho^{-1} = \lambda(1 - \alpha)$, namely

$$\alpha = \frac{\lambda\rho}{2\rho - 2 + \lambda\rho} \ . \tag{7.78}$$

Equation (7.77) for $j \neq 0$ implies that $-\alpha\rho^{-|j-1|} + 2\alpha\rho^{-|j|} - \alpha\rho^{-|j+1|} = -\lambda\alpha\rho^{-|j|}$, namely $\rho^2 - (2 + \lambda)\rho + 1 = 0$ with solution

$$\rho = 1 + \frac{\lambda}{2} + \left(\lambda + \frac{\lambda^2}{4}\right)^{1/2} .$$

Replacing this expression for ρ in (7.78) one obtains $\alpha = (1 + \frac{4}{\lambda})^{-1/2}$.

Taking advantage of the solution parametrized as in (7.75) with u given by (7.76), one can prove the existence of cantori for the sawtooth map through the following proposition as stated in [42].

Proposition. *Let ω be irrational, let*

$$\tilde{M}_\omega \equiv \{(x(\vartheta), x(\vartheta+\omega)) \ : \ \vartheta \in \mathbf{R}\}$$

and let $M_\omega \equiv \tilde{M}_\omega / \mathbf{Z}$. Then, M_ω is a Cantor set.

A proof of the existence of cantori in the dissipative sawtooth map, defined by the equations

$$\begin{aligned} y_{n+1} &= b y_n + c + \lambda f(x_n) \\ x_{n+1} &= x_n + y_{n+1} \ , \end{aligned}$$

for $b \in \mathbf{R}$, $c \in \mathbf{R}$ and $f(x)$ as in (7.74) is provided in [39].