4 The three–body problem and the Lagrangian solutions

The solution of the two–body problem is provided by Kepler's laws, which state that for negative energies a point–mass moves on an ellipse whose focus coincides with the other point–mass. As shown by Poincaré [149], the dynamics becomes extremely complicated when you add the gravitational influence of a third body. In Section 4.1 we shall focus on a particular three–body problem, known as the *restricted* three–body problem, where it is assumed that the mass of one of the three bodies is so small that its influence on the others can be neglected (see, e.g., [21, 44, 94, 131, 163, 169]). As a consequence the primaries move on Keplerian ellipses around their common barycenter; a simplified model consists in assuming that the primaries move on circular orbits and that the motion takes place on the same plane. Action–angle Delaunay variables are introduced for the restricted three–body problem and the expansion of the perturbing function is provided.
In the framework of the planar, circular, restricted three–body problem we derive the special solutions found by Lagrange, which are given by stationary points in the synodic reference frame (Section 4.2). The existence and stability of such solutions is also discussed in the framework of a model in which the primaries move on elliptic orbits (Section 4.3) as well as in the context of the elliptic, unrestricted three–body problem (Section 4.4).

4.1 The restricted three–body problem

Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be three bodies with masses m_1, m_2, m_3, respectively; throughout this section the three bodies are assumed to be point–masses. In the *restricted* problem one takes m_2 much smaller than m_1 and m_3, so that $\mathcal{P}_2$ does not affect the motion of $\mathcal{P}_1$ and $\mathcal{P}_3$. As a consequence we can assume that the motion of $\mathcal{P}_1$ and $\mathcal{P}_3$, to which we refer as the *primaries,* is Keplerian. Concerning the motion of $\mathcal{P}_2$ around the primaries, the region where the attraction of $\mathcal{P}_1$ or that of $\mathcal{P}_3$ is dominant is called the *sphere of influence;* an estimate of such a domain is provided in Appendix B.

4.1.1 The planar, circular, restricted three–body problem

The simplest non–trivial three–body model assumes that $\mathcal{P}_1$ and $\mathcal{P}_3$ move on a circular orbit around the common barycenter and that the motion of the three bodies takes place on the same plane. We refer to such a model as the *planar, circular,* *restricted three–body problem.* In an inertial reference frame whose origin coincides with the barycenter of the three bodies, let $\underline{\xi}_1$, $\underline{\xi}_2$, $\underline{\xi}_3 \in \mathbf{R}^2$ be the corresponding coordinates. From Newton's gravitational law one obtains that the motion of $\mathcal{P}_1$ and $\mathcal{P}_2$ is described by the equations

$$\begin{aligned} \frac{d^2\underline{\xi}_1}{dt^2} &= \mathcal{G}\frac{m_2(\underline{\xi}_2-\underline{\xi}_1)}{|\underline{\xi}_2-\underline{\xi}_1|^3}+\mathcal{G}\frac{m_3(\underline{\xi}_3-\underline{\xi}_1)}{|\underline{\xi}_3-\underline{\xi}_1|^3} \ , \\ \frac{d^2\underline{\xi}_2}{dt^2} &= -\mathcal{G}\frac{m_1(\underline{\xi}_2-\underline{\xi}_1)}{|\underline{\xi}_2-\underline{\xi}_1|^3}-\mathcal{G}\frac{m_3(\underline{\xi}_2-\underline{\xi}_3)}{|\underline{\xi}_2-\underline{\xi}_3|^3} \ . \end{aligned}$$

Next we consider a (heliocentric) reference frame with origin coinciding with $\mathcal{P}_1$; let $\underline{r}_2 \equiv \underline{\xi}_2 - \underline{\xi}_1$, $\underline{r}_3 \equiv \underline{\xi}_3 - \underline{\xi}_1$ be the relative positions with $\rho_2 \equiv |\underline{r}_2|$, $\rho_3 \equiv |\underline{r}_3|$. Then, one obtains

$$\frac{d^2\underline{r}_2}{dt^2} = -\frac{\mathcal{G}(m_1+m_2)\underline{r}_2}{\rho_2^3}-\frac{\mathcal{G}m_3\underline{r}_3}{\rho_3^3}+\frac{\mathcal{G}m_3(\underline{r}_3-\underline{r}_2)}{|\underline{r}_3-\underline{r}_2|^3} \ .$$

Setting $\mu \equiv \mathcal{G}(m_1+m_2)$ and $\varepsilon = \mathcal{G}m_3$, one has

$$\frac{d^2\underline{r}_2}{dt^2}+\frac{\mu\underline{r}_2}{\rho_2^3} = -\varepsilon\frac{\partial R}{\partial \underline{r}_2} \ ,$$

where the function R takes the form

$$R = \frac{\underline{r}_2\cdot\underline{r}_3}{\rho_3^3}-\frac{1}{|\underline{r}_3-\underline{r}_2|} \ . \tag{4.1}$$

Notice that for $\varepsilon = 0$ the dynamics reduces to the two–body problem of the motion of $\mathcal{P}_2$ around $\mathcal{P}_1$. For this reason we shall refer to ε as the perturbing parameter and to R as the perturbing function of the Keplerian motion. Recalling (3.36) we can write the three–body Hamiltonian as

$$\mathcal{H}_0(L_0,G_0,\ell_0,g_0) = -\frac{\mu^2}{2L_0^2}+\varepsilon R(L_0,G_0,\ell_0,g_0) \ , \tag{4.2}$$

where R is given by (4.1) and the functions $\underline{r}_2$, $\underline{r}_3$ must be expressed in terms of the Delaunay variables. Since the motion of $\mathcal{P}_3$ around $\mathcal{P}_1$ is circular, normalizing the time so that the angular velocity of $\mathcal{P}_3$ is equal to one, one obtains $\underline{r}_3 = (\rho_3\cos t,\ \rho_3\sin t)$. Denoting by ϑ the longitude of $\mathcal{P}_2$ and using $\underline{r}_2\cdot\underline{r}_3 = \rho_2\rho_3\cos(\vartheta-t)$, one obtains $|\underline{r}_3-\underline{r}_2| = \sqrt{\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\vartheta-t)}$. As a consequence, the perturbing function takes the form

$$R = \frac{\rho_2\cos(\vartheta-t)}{\rho_3^2}-\frac{1}{\sqrt{\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\vartheta-t)}} \ . \tag{4.3}$$

We immediately remark that R depends upon the difference $\vartheta - t$; being $\vartheta = g_0 + f$, one obtains that R depends on the difference $g_0 - t$. Therefore we perform the canonical change of variables from the Delaunay coordinates (L_0, G_0, ℓ_0, g_0) introduced in Chapter 3 to a new set of variables (L, G, ℓ, g) defined as

$$\begin{aligned} \ell &= \ell_0 \ , & L &= L_0 \ , \\ g &= g_0 - t \ , & G &= G_0 \ . \end{aligned}$$

The transformed Hamiltonian takes the form

$$\begin{aligned} \mathcal{H}(L, G, \ell, g) &= -\frac{\mu^2}{2L^2} - G \\ &+ \varepsilon \ \frac{\rho_2 \cos(g+f)}{\rho_3^2} - \frac{\varepsilon}{\sqrt{\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos(g+f)}} \ , \end{aligned}$$

where ρ_2, f are intended to be expressed in terms of the mean anomaly.

4.1.2 Expansion of the perturbing function

The perturbing function (4.3) can be written in terms of the Delaunay variables. Here we compute explicitly the first few coefficients of its Fourier–Taylor series expansion and we refer to Appendix C (see also [61, 67, 68]) for general formulae valid at any order.

Let us introduce the Legendre polynomials $P_j(x)$ defined through the recursive relations

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_{j+1}(x) &= \frac{(2j+1)P_j(x)x - jP_{j-1}(x)}{j+1} \qquad \text{for any } j \geq 1 \ . \end{aligned}$$

Apart from a constant factor, the second term in (4.3) becomes

$$\frac{1}{\sqrt{\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3\cos(\vartheta - t)}} = \frac{1}{\rho_3} \sum_{j=0}^{\infty} P_j(\cos(\vartheta - t)) \left(\frac{\rho_2}{\rho_3}\right)^j \ ,$$

from which we obtain

$$R = -\frac{1}{\rho_3} \sum_{j=2}^{\infty} P_j(\cos(\vartheta - t)) \left(\frac{\rho_2}{\rho_3}\right)^j \ . \tag{4.4}$$

The inversion of Kepler's equation (3.24) up to the second order in the eccentricity yields

$$u = \ell + \mathrm{e}\sin\ell + \frac{\mathrm{e}^2}{2}\sin(2\ell) + O(\mathrm{e}^3) \ .$$

Using (3.23) one obtains

$$f = \ell + 2\mathrm{e}\sin\ell + \frac{5}{4}\mathrm{e}^2\sin 2l + O(\mathrm{e}^3) \ ,$$

so that

$$\vartheta - t = g + \ell + 2\mathrm{e}\sin\ell + \frac{5}{4}\mathrm{e}^2\sin 2\ell + O(\mathrm{e}^3)\ . \tag{4.5}$$

In a similar way, from $\rho_2 = a(1-\mathrm{e}\cos u)$ one obtains:

$$\rho_2 = a\ \left(1 + \frac{1}{2}\mathrm{e}^2 - \mathrm{e}\cos\ell - \frac{1}{2}\mathrm{e}^2\cos 2\ell\right) + O(\mathrm{e}^3)\ . \tag{4.6}$$

Recall that the ecccentricity is a function of the Delaunay variables through the relation $\mathrm{e} = \sqrt{1-\frac{G^2}{L^2}}$. The powers $(\frac{\rho_2}{a})^j$ for $j = 2, 3, \ldots$ admit the following expansions:

$$\begin{aligned}
\left(\frac{\rho_2}{a}\right)^2 &= 1 + \frac{3}{2}\mathrm{e}^2 - 2\mathrm{e}\cos\ell - \frac{1}{2}\mathrm{e}^2\cos 2\ell + O(\mathrm{e}^3)\\
\left(\frac{\rho_2}{a}\right)^3 &= 1 + 3\mathrm{e}^2 - 3\mathrm{e}\cos\ell + O(\mathrm{e}^3)\\
\left(\frac{\rho_2}{a}\right)^4 &= 1 + 5\mathrm{e}^2 - 4\mathrm{e}\cos\ell + \mathrm{e}^2\cos 2\ell + O(\mathrm{e}^3)\\
\left(\frac{\rho_2}{a}\right)^5 &= 1 - 5\mathrm{e}\cos\ell + \mathrm{e}^2\left(\frac{15}{2} + \frac{5}{2}\cos 2\ell\right) + O(\mathrm{e}^3)\ \ldots\ .
\end{aligned}$$

From (4.4), one gets:

$$\begin{aligned}
R\ &=\ -\tfrac{1}{\rho_3}\ [P_2(\cos(\vartheta - t))\ (\tfrac{\rho_2}{a})^2(\tfrac{a}{\rho_3})^2 + P_3(\cos(\vartheta - t))\ (\tfrac{\rho_2}{a})^3(\tfrac{a}{\rho_3})^3\\
&+P_4(\cos(\vartheta - t))\ (\tfrac{\rho_2}{a})^4(\tfrac{a}{\rho_3})^4 + P_5(\cos(\vartheta - t))\ (\tfrac{\rho_2}{a})^5(\tfrac{a}{\rho_3})^5] + \ldots\ .
\end{aligned}$$

Casting together (4.5), (4.6) and normalizing the unit of length so that $\rho_3 = 1$, one obtains the expansion

$$\begin{aligned}
R\ &=\ R_{00}(L,G) + R_{10}(L,G)\cos\ell + R_{11}(L,G)\cos(\ell + g)\\
&+\ R_{12}(L,G)\cos(\ell + 2g) + R_{22}(L,G)\cos(2\ell + 2g)\\
&+\ R_{32}(L,G)\cos(3\ell + 2g) + R_{33}(L,G)\cos(3\ell + 3g)\\
&+\ R_{44}(L,G)\cos(4\ell + 4g) + R_{55}(L,G)\cos(5\ell + 5g) + \ldots\ ,
\end{aligned} \tag{4.7}$$

where the coefficients R_{ij} are given by the following expressions:

$$\begin{aligned}
R_{00} &= -\frac{L^4}{4}\left(1 + \frac{9}{16}L^4 + \frac{3}{2}\mathrm{e}^2\right) + \ldots\ , & R_{10} &= \frac{L^4\mathrm{e}}{2}\left(1 + \frac{9}{8}L^4\right) + \ldots\\
R_{11} &= -\frac{3}{8}L^6\left(1 + \frac{5}{8}L^4\right) + \ldots\ , & R_{12} &= \frac{L^4\mathrm{e}}{4}(9 + 5L^4) + \ldots\\
R_{22} &= -\frac{L^4}{4}\left(3 + \frac{5}{4}L^4\right) + \ldots\ , & R_{32} &= -\frac{3}{4}L^4\mathrm{e} + \ldots\\
R_{33} &= -\frac{5}{8}L^6\left(1 + \frac{7}{16}L^4\right) + \ldots\ , & R_{44} &= -\frac{35}{64}L^8 + \ldots\\
R_{55} &= -\frac{63}{128}L^{10} + \ldots
\end{aligned} \tag{4.8}$$

4.1.3 The planar, elliptic, restricted three–body problem

If we assume that $\mathcal{P}_3$ orbits around $\mathcal{P}_1$ on an elliptic orbit with eccentricity e', the corresponding motion is described by a Hamiltonian function with three degrees of freedom; if ψ denotes the longitude of $\mathcal{P}_3$ and Ψ is the conjugated action variable, the Hamiltonian of the elliptic case is given by

$$\mathcal{H}(L,G,\Psi,\ell,g,\psi) = -\frac{1}{2L^2} + \Psi + \varepsilon R(L,G,\ell,g,\psi;\mathrm{e}') \ ,$$

where $R(L,G,\ell,g,\psi;\mathrm{e}')$ depends parametrically on e' and, in normalized units, ε is the primaries mass–ratio. Up to constants, the first few Fourier coefficients of the series expansion of the perturbing function are the following:

$$\begin{aligned}
R(L,G,\ell,g,\psi) \ &= \\
= \ & -\frac{L^4}{4}\left(\frac{5}{2}+\frac{9}{16}L^4-\frac{3}{2}\frac{G^2}{L^2}+\frac{3}{2}\mathrm{e}'^2\right)+L^4\frac{\mathrm{e}}{2}\left(1+\frac{9}{8}L^4\right)\cos(\ell) \\
& -\frac{3}{8}L^6\left(1+\frac{5}{8}L^4\right)\cos(\ell+g-\psi)+\frac{L^4}{4}\mathrm{e}(9+5L^4)\cos(\ell+2g-2\psi) \\
& -\frac{L^4}{4}\left(3+\frac{5}{4}L^4\right)\cos(2\ell+2g-2\psi)-\frac{3}{4}L^4\mathrm{e}\cos(3\ell+2g-2\psi) \\
& -\frac{5}{8}L^6\left(1+\frac{7}{16}L^4\right)\cos(3\ell+3g-3\psi)-\frac{35}{64}L^8\cos(4\ell+4g-4\psi) \\
& -\frac{63}{128}L^{10}\cos(5\ell+5g-5\psi)-L^4\left(\frac{3}{4}\mathrm{e}'+\frac{45}{64}L^4\mathrm{e}'\right)\cos(\psi) \\
& -L^4\left(\frac{21}{8}\mathrm{e}'+\frac{45}{32}\mathrm{e}'L^4\right)\cos(2\ell+2g-3\psi) \\
& -L^4\left(-\frac{3}{8}\mathrm{e}'+\frac{5}{32}\mathrm{e}'L^4\right)\cos(2\ell+2g-\psi)+\ldots
\end{aligned}$$

4.1.4 The inclined, circular, restricted three–body problem

We assume that the motion of $\mathcal{P}_3$ around $\mathcal{P}_1$ is circular, but we let the planes of the orbits of $\mathcal{P}_2$ and $\mathcal{P}_3$ have a non–zero mutual inclination i. Using the spatial Delaunay variables (L,G,H,ℓ,g,h) introduced in Chapter 3, denoting with ψ the longitude of $\mathcal{P}_3$, the Hamiltonian function takes the form:

$$\mathcal{H}(L,G,H,\ell,g,h,\psi) = -\frac{1}{2L^2} - H + \varepsilon R(L,G,H,\ell,g,h,\psi) \ ,$$

where, setting $\gamma = \sqrt{\frac{1}{2}-\frac{H}{2G}}$, up to constants the first few terms of the Fourier expansion of the perturbing function are given by

$$
\begin{aligned}
R(L,G,H,\ell,g,h,\psi) &= -L^4\Big(\frac{1}{4}+\frac{3}{8}\mathrm{e}^2+\frac{9}{64}L^4-\frac{3}{2}\gamma^2\Big)\\
&-\Big(\frac{3}{4}-\frac{3}{2}\gamma^2+\frac{5}{16}L^4\Big)\cos(2\ell+2g+2h-2\psi)\\
&-\Big(-\frac{1}{2}\mathrm{e}+3\gamma^2\mathrm{e}-\frac{9}{16}\mathrm{e}L^4\Big)\cos(\ell)-\Big(\frac{3}{4}\mathrm{e}-\frac{3}{2}\gamma^2\mathrm{e}\Big)\cos(3\ell+2g+2h-2\psi)\\
&-\Big(-\frac{9}{4}\mathrm{e}+\frac{9}{2}\gamma^2\mathrm{e}-\frac{5}{4}\mathrm{e}L^4\Big)\cos(\ell+2g+2h-2\psi)-\frac{3}{2}\gamma^2\cos(2\ell+2g)\\
&-\frac{3}{2}\gamma^2\cos(2h-2\psi)-\frac{3}{2}\gamma^2\mathrm{e}\cos(3\ell+2g)\\
&+\frac{9}{2}\gamma^2\mathrm{e}\cos(\ell+2g)+\frac{3}{2}\gamma^2\mathrm{e}\cos(\ell+2h-2\psi)\\
&+\frac{3}{2}\gamma^2\mathrm{e}\cos(\ell-2h+2\psi)-L^6\Big(\frac{3}{8}+\frac{15}{64}L^4\Big)\cos(\ell+g+h-\psi)\\
&-\Big(\frac{5}{8}+\frac{35}{128}L^4\Big)L^6\cos(3\ell+3g+3h-3\psi)\\
&-\frac{35}{64}L^8\cos(4\ell+4g+4h-4\psi)-\frac{63}{128}L^{10}\cos(5\ell+5g+5h-5\psi)+\dots
\end{aligned}
$$

4.2 The circular, restricted Lagrangian solutions

In the framework of the restricted, planar, circular three–body problem, Euler and Lagrange proved that in a rotating reference frame the equations of motion admit the existence of equilibrium solutions, known as the collinear and triangular equilibrium points. A concrete example is provided by the Trojan and Greek groups of asteroids, which (approximately) form an equilateral triangle with Jupiter and the Sun.

The mathematical derivation of such equilibrium solutions is the following. Consider a *sidereal* reference frame $(\mathcal{O},\xi,\eta,\zeta)$, where $\mathcal{O}$ coincides with the barycenter of the three bodies, the ξ axis lies along the direction joining the bodies with masses m_1 and m_3 at time $t=0$, η is orthogonal to ξ and belongs to the orbital plane, while ζ is perpendicular to the orbital plane. Let (ξ_i,η_i,ζ_i), $i=1,3$, be the coordinates of the primaries $\mathcal{P}_1$ and $\mathcal{P}_3$. We normalize the units of measure so that the distance between the primaries is unity and that $\mathcal{G}(m_1+m_3)=1$. Without loss of generality we assume that $m_1>m_3$ and let

$$
\overline{\mu}\equiv\frac{m_3}{m_1+m_3}\ ,
$$

so that $\mu_1\equiv\mathcal{G}m_1=1-\overline{\mu}$, $\mu_3\equiv\mathcal{G}m_3=\overline{\mu}$. The equations of motion of $\mathcal{P}_2$ with coordinates (ξ,η,ζ) can be written as

$$\begin{aligned}
\ddot{\xi} &= \mu_1 \frac{\xi_1 - \xi}{r_1^3} + \mu_3 \frac{\xi_3 - \xi}{r_3^3} \\
\ddot{\eta} &= \mu_1 \frac{\eta_1 - \eta}{r_1^3} + \mu_3 \frac{\eta_3 - \eta}{r_3^3} \\
\ddot{\zeta} &= \mu_1 \frac{\zeta_1 - \zeta}{r_1^3} + \mu_3 \frac{\zeta_3 - \zeta}{r_3^3} \ ,
\end{aligned} \tag{4.9}$$

where r_1 and r_3 denote the distances from the primaries:

$$\begin{aligned}
r_1 &= \sqrt{(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2} \ , \\
r_3 &= \sqrt{(\xi_3 - \xi)^2 + (\eta_3 - \eta)^2 + (\zeta_3 - \zeta)^2} \ .
\end{aligned}$$

Let us introduce a synodic reference frame $(\mathcal{O}, x, y, z)$, rotating with the angular velocity n of the primaries, where n has been normalized to one, due to the choice of the units of measure. Let us fix the axes so that the coordinates of the primaries become $(x_1, y_1, z_1) = (-\mu_3, 0, 0)$, $(x_3, y_3, z_3) = (\mu_1, 0, 0)$. The link between the synodic and the sidereal reference frames is

$$\begin{aligned}
\xi &= \cos(t)x - \sin(t)y \\
\eta &= \sin(t)x + \cos(t)y \\
\zeta &= z \ ,
\end{aligned} \tag{4.10}$$

while the distances of $\mathcal{P}_2$ from the primaries are now given by

$$r_1 = \sqrt{(x + \mu_3)^2 + y^2 + z^2} \ , \qquad r_3 = \sqrt{(x - \mu_1)^2 + y^2 + z^2} \ . \tag{4.11}$$

Computing the second derivative of (4.10) with respect to time and inserting the result in (4.9) one obtains the equations of motion in the synodic frame:

$$\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\
\ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \\
\ddot{z} &= \frac{\partial U}{\partial z} \ ,
\end{aligned} \tag{4.12}$$

where the function U is defined as

$$U = U(x, y, z) \equiv \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_3}{r_3} \ . \tag{4.13}$$

Multiplying (4.12) by $\dot{x}$, $\dot{y}$, $\dot{z}$ and adding the results, one obtains:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} \ ; \tag{4.14}$$

notice that the left–hand side of (4.14) is equal to $\frac{1}{2}\frac{d}{dt}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, while the right–hand side is equal to $\frac{dU}{dt}$. Therefore, integrating with respect to time one gets

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - C_J \ , \tag{4.15}$$

where C_J is a constant of integration, called the *Jacobi integral.* Using (4.13) one obtains

$$C_J = x^2 + y^2 + 2\frac{\mu_1}{r_1} + 2\frac{\mu_3}{r_3} - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ . \tag{4.16}$$

Notice that (4.15) implies $2U - C_J \geq 0$. The curves of zero velocity are defined through the expression $C_J = 2U$; such a relation defines a boundary, called *Hill's surface,* which separates regions where motion is allowed or forbidden. An example of Hill's region is given in Figure 4.1.

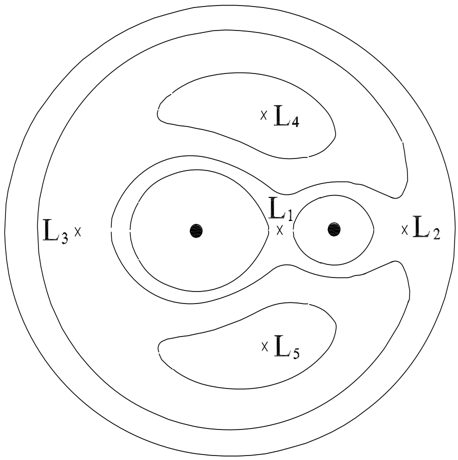

Fig. 4.1. The triangular and collinear equilibrium points with an example of Hill's surfaces.

Let us now turn to the determination of the position of the equilibrium points in the planar case with $z = 0$ [142], since we assumed that the motion of the three bodies takes place on the same plane. Recalling (4.11) and using $\mu_1 + \mu_3 = 1$ one has:

$$\mu_1 r_1^2 + \mu_3 r_3^2 = x^2 + y^2 + \mu_1 \mu_3 \ .$$

Inserting such an expression in U one has

$$U = \mu_1 \left(\frac{r_1^2}{2} + \frac{1}{r_1} \right) + \mu_3 \left(\frac{r_3^2}{2} + \frac{1}{r_3} \right) - \frac{1}{2} \mu_1 \mu_3 \ .$$

The equilibrium points are the solutions of the system obtained imposing that the partial derivatives of (4.13) with respect to x and y are zero:

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial U}{\partial r_1} \frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_3} \frac{\partial r_3}{\partial x} \\
&= \mu_1 \left(r_1 - \frac{1}{r_1^2} \right) \frac{x + \mu_3}{r_1} + \mu_3 \left(r_3 - \frac{1}{r_3^2} \right) \frac{x - \mu_1}{r_3} = 0 \\
\frac{\partial U}{\partial y} &= \frac{\partial U}{\partial r_1} \frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_3} \frac{\partial r_3}{\partial y} \\
&= \mu_1 \left(r_1 - \frac{1}{r_1^2} \right) \frac{y}{r_1} + \mu_3 \left(r_3 - \frac{1}{r_3^2} \right) \frac{y}{r_3} = 0 \ .
\end{aligned} \tag{4.17}$$

A solution of (4.17) is obtained by solving the equations

$$r_1 - \frac{1}{r_1^2} = 0 \ , \qquad\qquad r_3 - \frac{1}{r_3^2} = 0 \ ,$$

from which one obtains $r_1 = r_3 = 1$, namely

$$(x + \mu_3)^2 + y^2 = 1 \ , \qquad\qquad (x - \mu_1)^2 + y^2 = 1 \ .$$

Solving these equations, one finds the equilibrium solutions

$$\left(\frac{1}{2} - \mu_3, \frac{\sqrt{3}}{2}\right) \ , \qquad\qquad \left(\frac{1}{2} - \mu_3, -\frac{\sqrt{3}}{2}\right) \ ,$$

which correspond to the *triangular* Lagrangian solutions, usually denoted as L_4 and L_5 (see Figure 4.1).

Other solutions are obtained observing that $y = 0$ solves the second of (4.17); in particular, there exist three *collinear* equilibrium solutions usually denoted as L_1, L_2, L_3, where L_1 is located between the primaries, while L_2 and L_3 are outside the primaries. We derive in detail the location of L_1; the same procedure can be straightforwardly extended to L_2 and L_3.

At L_1 we have $y = 0$ and $r_1 = x + \mu_3$, $r_3 = -x + \mu_1$, so that $r_1 + r_3 = 1$; moreover, $\frac{\partial r_1}{\partial x} = -\frac{\partial r_3}{\partial x} = 1$. Replacing in $\frac{\partial U}{\partial x} = 0$, one obtains

$$\mu_1\left(1 - r_3 - \frac{1}{(1 - r_3)^2}\right) - \mu_3\left(r_3 - \frac{1}{r_3^2}\right) = 0 \ ,$$

from which one gets

$$\frac{\mu_3}{3\mu_1} = r_3^3\ \frac{1 - r_3 + \frac{r_3^2}{3}}{(1 + r_3 + r_3^2)(1 - r_3)^3} \ .$$

Define $\alpha \equiv (\frac{\mu_3}{3\mu_1})^{1/3}$; developing α in Taylor series, one finds

$$\alpha = r_3 + \frac{1}{3}r_3^2 + \frac{1}{3}r_3^3 + \frac{53}{81}r_3^4 + \ldots$$

Inverting such relation, for example using the Lagrange inversion method [142], one has

$$r_3 = \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + \ldots \qquad (4.18)$$

Since r_3 represents the distance along the x–axis from the body with mass m_3, the solution (4.18) provides the location of the equilibrium point L_1 as a function of the mass ratio α. Similar computations can be performed for L_2 such that $r_1 = x + \mu_3$ and $r_3 = x - \mu_1$ with $r_1 - r_3 = 1$, and for L_3 such that $r_1 = -x - \mu_3$ and $r_3 = -x + \mu_1$ with $r_3 - r_1 = 1$.

To give a concrete example, in the Moon–Earth system the location of the equilibrium points is the following: L_1 lies at $3.26 \cdot 10^5$ km from the Earth, L_2 is at $4.49 \cdot 10^5$ km, L_3 is about $3.82 \cdot 10^5$ km from the Earth, while L_4 and L_5 are the triangular positions at $3.84 \cdot 10^5$ km, being located on the Moon's orbit.

We conclude with a discussion on the linear stability of the equilibrium positions (see [142]). Let us denote by (x_ℓ, y_ℓ) one of the five stationary solutions $(L_1, \ldots, L_5)$; let (δ_x, δ_y) be a small displacement from the equilibrium and let $(x, y) \equiv (x_\ell + \delta_x, y_\ell + \delta_y)$. Let us insert such coordinates in (4.12) and expand the derivatives of U in a neighborhood of the equilibrium solution. Using the notation

$$U_{xx} = \frac{\partial^2 U(x_\ell, y_\ell)}{\partial x^2}\ , \qquad U_{xy} = \frac{\partial^2 U(x_\ell, y_\ell)}{\partial x \partial y}\ , \qquad U_{yy} = \frac{\partial^2 U(x_\ell, y_\ell)}{\partial y^2}\ ,$$

the equations for the variations (δ_x, δ_y) can be written as

$$\begin{pmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \ddot{\delta}_x \\ \ddot{\delta}_y \end{pmatrix} = A \begin{pmatrix} \delta_x \\ \delta_y \\ \dot{\delta}_x \\ \dot{\delta}_y \end{pmatrix}\ ,$$

where

$$A \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}\ .$$

The eigenvalues of A are the solutions of the secular equation $\det(A - \lambda I_4) = 0$ (where I_4 is the 4×4 identity matrix), namely

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + (U_{xx}U_{yy} - U_{xy}^2) = 0\ .$$

This equation admits four roots:

$$\begin{aligned}
\lambda_{1,2} &= \pm\left[\frac{1}{2}(U_{xx} + U_{yy} - 4) - \frac{1}{2}[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)]^{\frac{1}{2}}\right]^{\frac{1}{2}} \\
\lambda_{3,4} &= \pm\left[\frac{1}{2}(U_{xx} + U_{yy} - 4) + \frac{1}{2}[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)]^{\frac{1}{2}}\right]^{\frac{1}{2}}\ .
\end{aligned}$$

The equilibrium solution is stable, if the eigenvalues are purely imaginary.

For the collinear equilibrium position L_1, one has $y_\ell = 0$, $r_1 = x_\ell + \mu_3$, $r_3 = -x_\ell + \mu_1$; defining $M \equiv \frac{\mu_1}{r_1^3} + \frac{\mu_3}{r_3^3}$, the characteristic equation becomes

$$\lambda^4 + (2 - M)\lambda^2 + (1 + M - 2M^2) = 0\ .$$

Therefore the product of the four eigenvalues amounts to $1 + M - 2M^2$, with the constraints $\lambda_1 = -\lambda_2$, $\lambda_3 = -\lambda_4$. The eigenvalues are purely imaginary provided that $\lambda_1^2 = \lambda_2^2 < 0$ and $\lambda_3^2 = \lambda_4^2 < 0$, which imply that $1 + M - 2M^2 > 0$, namely $-\frac{1}{2} < M < 1$. These inequalities would guarantee the stability of the equilibrium point; however, computing M at the collinear point L_1 one finds that $M > 1$. In fact, in the case of L_1 we know that $r_1 < 1$ and $r_3 < 1$, so that $M > \mu_1 + \mu_3 = 1$.

We conclude that the collinear point L_1 is unstable for any value of the masses. The same conclusion holds for L_2 and L_3.

Concerning the triangular equilibrium positions one has $x_\ell = \frac{1}{2} - \mu_3$, $y_\ell = \pm\frac{\sqrt{3}}{2}$, $r_1 = r_3 = 1$. Computing the derivatives of U at the equilibria, one obtains

$$U_{xx} = \frac{3}{4}\ , \qquad U_{yy} = \frac{9}{4}\ , \qquad U_{xy} = \pm\frac{3\sqrt{3}}{4}(1-2\mu_3)\ .$$

The eigenvalues become

$$\begin{aligned}\lambda_{1,2} &= \pm\frac{\sqrt{-1-\sqrt{1-27(1-\mu_3)\mu_3}}}{\sqrt{2}}\ ,\\ \lambda_{3,4} &= \pm\frac{\sqrt{-1+\sqrt{1-27(1-\mu_3)\mu_3}}}{\sqrt{2}}\ .\end{aligned}$$

The eigenvalues are purely imaginary provided

$$1-27(1-\mu_3)\mu_3 \geq 0\ ; \tag{4.19}$$

recalling that we assumed $m_1 > m_3$, so that $\mu_1 > \mu_3$ with $\mu_1 + \mu_3 = 1$, taking into account the inequality (4.19) one obtains

$$\mu_3 \leq \frac{27-\sqrt{621}}{54} \simeq 0.0385\ . \tag{4.20}$$

In conclusion, if the masses verify (4.20), then the triangular equilibrium solutions are linearly stable.

4.3 The elliptic, restricted Lagrangian solutions

Consider the planar motion of a body $\mathcal{P}_2(x,y)$ of mass μ_2 in the gravitational field of two primaries, $\mathcal{P}_1(x_1,y_1)$ and $\mathcal{P}_3(x_3,y_3)$ with masses μ_1 and μ_3, which are assumed to move on elliptic orbits around their common center of mass $\mathcal{O}$; let f denote the true anomaly of the common ellipse and let $r = \frac{a(1-\mathrm{e}^2)}{1+\mathrm{e}\cos f}$ be the distance between $\mathcal{P}_1$ and $\mathcal{P}_3$. In an inertial barycentric reference frame, the cartesian equations of the motion of $\mathcal{P}_2$ are given by

$$\begin{aligned}\ddot{x} &= -\frac{\mu_1(x-x_1)}{r_1^3} - \frac{\mu_3(x-x_3)}{r_3^3}\\ \ddot{y} &= -\frac{\mu_1(y-y_1)}{r_1^3} - \frac{\mu_3(y-y_3)}{r_3^3}\ ,\end{aligned}$$

where $r_1 = \sqrt{(x-x_1)^2+(y-y_1)^2}$, $r_3 = \sqrt{(x-x_3)^2+(y-y_3)^2}$; the above equations are associated to the Lagrangian function

$$\mathcal{L}(\dot{x},\dot{y},x,y,r,f) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \frac{\mu_1}{r_1} + \frac{\mu_3}{r_3}\ ,$$

where the coordinates of the primaries, (x_1, y_1) and (x_3, y_3), depend upon r and f in the following way:

$$\begin{aligned} x_1 &= -\mu_3 r \cos f\ , \qquad & x_3 = \mu_1 r \cos f\ , \\ y_1 &= -\mu_3 r \sin f\ , \qquad & y_3 = \mu_1 r \sin f\ . \end{aligned}$$

Next we move to a barycentric reference frame $(\mathcal{O}, \xi, \eta)$ rotating with variable angular velocity, such that at each instant of time the rotation angle is equal to f with $\dot f = \frac{h}{r^2}$, h being the angular momentum and having assumed $\mathcal{G}(m_1+m_3) = 1$. The transformation equations are given by

$$\begin{aligned} x &= \xi \cos f - \eta \sin f \\ y &= \xi \sin f + \eta \cos f\ . \end{aligned}$$

Thus, the primaries oscillate on the ξ–axis and have coordinates

$$\begin{aligned} \xi_1 &= -\mu_3 r\ , \qquad & \xi_3 = \mu_1 r\ , \\ \eta_1 &= 0\ , \qquad & \eta_3 = 0\ . \end{aligned}$$

The new Lagrangian function takes the form:

$$\mathcal{L}(\dot\xi, \dot\eta, \xi, \eta, r, f) = \frac{1}{2}(\dot\xi^2 + \dot\eta^2) + \frac{1}{2}(\xi^2 + \eta^2)\dot f^2 + (\xi\dot\eta - \dot\xi\eta)\dot f + \frac{\mu_1}{r_1} + \frac{\mu_3}{r_3}\ .$$

The transformation to the so–called rotating–pulsating coordinates (X, Y) is achieved through the further change of variables:

$$\begin{aligned} \xi &= rX \\ \eta &= rY\ ; \end{aligned}$$

the primaries are now in a fixed position with coordinates $(X_1, Y_1) = (-\mu_3, 0)$, $(X_3, Y_3) = (\mu_1, 0)$ and the Lagrangian function takes the form

$$\begin{aligned} \mathcal{L}(\dot X, \dot Y, X, Y, r, f) &= \frac{r^2}{2}(\dot X^2 + \dot Y^2) + r\dot r(X\dot X + Y\dot Y) \\ &+(X^2+Y^2)\left(1 + \frac{\dot r^2}{2} + \frac{h^2}{2r^2}\right) + h(X\dot Y - Y\dot X) + \frac{1}{r}\left(\frac{\mu_1}{r_1} + \frac{\mu_3}{r_3}\right)\ . \end{aligned}$$

Finally, we change the time taking the true anomaly as independent variable through the transformation

$$dt = \frac{1}{h} r^2\ df\ .$$

Denoting by $X' \equiv \frac{dX}{df}$ and $Y' \equiv \frac{dY}{df}$, the new Lagrangian function is given by

$$\mathcal{L}(X', Y', X, Y, r, f) = \frac{1}{2}(X'^2+Y'^2)+XY'-YX'+\frac{r}{2h^2}(X^2+Y^2)+\frac{r}{h^2}\left(\frac{\mu_1}{r_1^2}+\frac{\mu_3}{r_3^2}\right)\ .$$

The corresponding equations of motion take a form similar to that of the circular case (see (4.12)), being

$$\begin{aligned} X'' - 2Y' &= \Omega_X \\ Y'' + 2X' &= \Omega_Y \ , \end{aligned} \tag{4.21}$$

where we define $\Omega = \Omega(X, Y, f)$ as

$$\Omega = \frac{1}{1 + \mathrm{e}\cos f}\left[\frac{1}{2}(X^2 + Y^2) + \frac{\mu_1}{r_1} + \frac{\mu_3}{r_3} + \frac{1}{2}\mu_1\mu_3\right]$$

and Ω_X, Ω_Y denote the derivatives with respect to X, Y, respectively. Let Ω_0 be defined through the relation

$$\Omega_0 \equiv (1 + \mathrm{e}\cos f)\Omega \ .$$

The equivalent of the Jacobi integral is obtained from (4.21) multiplying the first equation by X' and the second by Y'; adding the results one obtains:

$$\left(\frac{dX}{df}\right)^2 + \left(\frac{dY}{df}\right)^2 = 2\int(\Omega_X dX + \Omega_Y dY) \ . \tag{4.22}$$

Let us write the derivative of Ω with respect to the true anomaly as

$$\Omega_f = \frac{\mathrm{e}\sin f}{(1 + \mathrm{e}\cos f)^2}\ \Omega_0 \ .$$

Then, (4.22) becomes:

$$\begin{aligned} \left(\frac{dX}{df}\right)^2 + \left(\frac{dY}{df}\right)^2 &= 2\int(d\Omega - \Omega_f df) \\ &= 2\Omega - 2\mathrm{e}\int \frac{\Omega_0 \sin f}{(1 + \mathrm{e}\cos f)^2} df - C_{\mathrm{e}} \ , \end{aligned}$$

where C_{e} is a constant which reduces to the Jacobi integral in the circular case $\mathrm{e} = 0$.

The stationary solutions of (4.21) are given by

$$\frac{\partial\Omega}{\partial X} = 0 \ , \qquad \frac{\partial\Omega}{\partial Y} = 0$$

or equivalently by

$$\frac{\partial\Omega_0}{\partial X} = 0 \ , \qquad \frac{\partial\Omega_0}{\partial Y} = 0 \ ,$$

which imply that the solutions of the elliptic problem coincide with those of the circular case. In particular, the triangular solutions are located at $(\frac{1}{2} - \mu_3, \pm\frac{\sqrt{3}}{2})$, which pulsate as the coordinates. In order to analyze the stability, one starts by introducing a displacement (δ_X, δ_Y) from the libration points, say $X \equiv X_\ell + \delta_X$, $Y \equiv Y_\ell + \delta_Y$, where (X_ℓ, Y_ℓ) coincides with one of the five stationary solutions; the linearized equations can be written as

$$\begin{aligned} \delta_X'' - 2\delta_Y' &= \frac{1}{1+\mathrm{e}\cos f}\ [\Omega^{(\ell)}_{0,XX}\delta_X + \Omega^{(\ell)}_{0,XY}\delta_Y] \\ \delta_Y'' + 2\delta_X' &= \frac{1}{1+\mathrm{e}\cos f}\ [\Omega^{(\ell)}_{0,YX}\delta_X + \Omega^{(\ell)}_{0,YY}\delta_Y]\ , \end{aligned}$$

where $\Omega^{(\ell)}_{0,XX}$ denotes the second derivative of Ω_0 with respect to X computed at the stationary solution (X_ℓ, Y_ℓ) (similarly for the other derivatives). A numerical procedure based on Floquet theory (see Appendix D) and on the computation of the characteristic exponents (see [52]) provides the domain of the linear stability in the parameter plane (μ, e).

Figure 4.2 shows the regions of linear stability of the triangular solutions (compare with [52]): at $\mu \simeq 0.028$ one has linear instability for any value of the eccentricity; at $\mu \simeq 0.038$ one has instability also for $\mathrm{e} = 0$, while the eccentricity can have a stabilizing effect up to $\mu \simeq 0.047$ (notice that the point D in Figure 4.2 has coordinates $D(0.047, 0.314)$). The collinear points are always unstable, as in the circular case, for any value of the eccentricity and of the mass parameter.

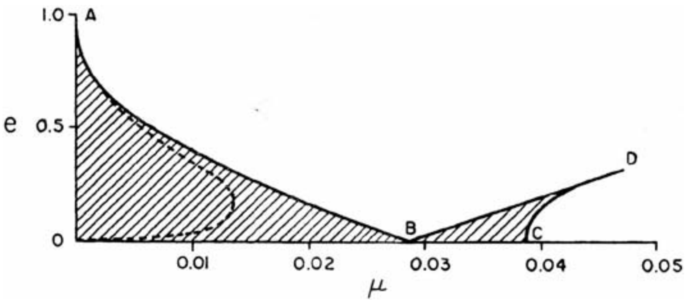

Fig. 4.2. The shaded area denotes a region of equilibrium of the elliptic, restricted triangular solutions as the parameters μ and e are varied (after [52]; reproduced by permission of the AAS).

4.4 The elliptic, unrestricted triangular solutions

Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be three bodies of masses m_1, m_2, m_3 which are subject to the mutual gravitational attraction; we assume that the three bodies move in the same plane and we denote the position vectors in an inertial reference frame by means of the two–dimensional vectors $\underline{q}_1$, $\underline{q}_2$, $\underline{q}_3$. The equations of motion can be written as

$$m_i \ddot{\underline{q}}_i = \frac{\partial U}{\partial \underline{q}_i}\ , \qquad i = 1, 2, 3\ , \tag{4.23}$$

where

$$U(\underline{q}) = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{\|\underline{q}_j - \underline{q}_i\|}\ .$$

Following [154] (see also [134]) the generalization of the Lagrangian solutions of the restricted case is obtained by looking for a periodic homographic[1] solution of the form

$$\underline{q}_i(t) = \psi(t)\underline{z}_i \ , \qquad\qquad i = 1, 2, 3, \tag{4.24}$$

where $\underline{z}_i$ are constant vectors and $\psi(t)$ is an unknown function, which can be found as follows. Inserting (4.24) in (4.23) one obtains

$$m_i \underline{z}_i \ddot{\psi}(t) = \sum_{1 \le j \le 3, j \ne i} \frac{m_i m_j \psi(t)(\underline{z}_j - \underline{z}_i)}{|\psi(t)|^3 \|\underline{z}_j - \underline{z}_i\|^3} \ , \qquad\qquad i = 1, 2, 3,$$

which can be split as

$$\begin{aligned} \ddot{\psi}(t) &= -\nu \frac{\psi(t)}{|\psi(t)^3|} \\ \sum_{1 \le j \le 3, j \ne i} \frac{m_i m_j (\underline{z}_j - \underline{z}_i)}{\|\underline{z}_j - \underline{z}_i\|^3} + \nu m_i \underline{z}_i &= \underline{0} \ , \end{aligned} \tag{4.25}$$

where ν is a real constant. From the first equation we recognize that $\psi(t)$ is a solution of a Keplerian motion; summing the second equation in (4.25) over $i = 1, 2, 3$, one obtains

$$\sum_{i=1}^{3} m_i \underline{z}_i = \underline{0} \ , \tag{4.26}$$

showing that the center of mass is located at the origin of the reference frame. Let d be the length of the sides of the triangular solution; the scaling factor ν can be set to one by a proper choice of d. In fact, the first component of the second equation in (4.25) is given by

$$\nu \underline{z}_1 = \frac{1}{d^3} \left[m_2(\underline{z}_1 - \underline{z}_2) + m_3(\underline{z}_1 - \underline{z}_3) \right] = \frac{M}{d^3} \underline{z}_1 \ ,$$

where $M = m_1 + m_2 + m_3$ denotes the total mass. Setting

$$d = M^{\frac{1}{3}} \ , \tag{4.27}$$

we obtain $\nu = 1$.

If $\underline{p}_i$ $(i = 1, 2, 3)$ denote the momenta conjugated to $\underline{q}_i$, the Hamiltonian governing the three–body problem can be written as

$$\begin{aligned} \mathcal{H}_1(\underline{p}_1, \underline{p}_2, \underline{p}_3, \underline{q}_1, \underline{q}_2, \underline{q}_3) &= \frac{\|\underline{p}_1\|^2}{2m_1} + \frac{\|\underline{p}_2\|^2}{2m_2} + \frac{\|\underline{p}_3\|^2}{2m_3} \\ &- \frac{m_1 m_2}{\|\underline{q}_2 - \underline{q}_1\|} - \frac{m_1 m_3}{\|\underline{q}_3 - \underline{q}_1\|} - \frac{m_2 m_3}{\|\underline{q}_3 - \underline{q}_2\|} \ . \end{aligned} \tag{4.28}$$

[1] A homographic solution is a configuration which remains similar to itself all times.

The center of mass and the total linear momentum can be eliminated through a transformation to *Jacobi coordinates:*

$$\begin{aligned} \underline{u}_1 &= \underline{q}_2 - \underline{q}_1 & \underline{v}_1 &= -\frac{m_2}{m_1+m_2}\underline{p}_1 \\ \underline{u}_2 &= \underline{q}_3 - \frac{1}{m_1+m_2}(m_1\underline{q}_1 + m_2\underline{q}_2) & \underline{v}_2 &= -\frac{m_3}{M}(\underline{p}_1+\underline{p}_2+\underline{p}_3)+\underline{p}_3 \\ \underline{u}_3 &= \frac{1}{M}(m_1\underline{q}_1 + m_2\underline{q}_2 + m_3\underline{q}_3) & \underline{v}_3 &= \underline{p}_1+\underline{p}_2+\underline{p}_3 \ . \end{aligned} \tag{4.29}$$

An alternative reduction is obtained through the transformation to heliocentric coordinates as in Appendix E. Recalling (4.24) and (4.26), we obtain

$$\sum_{i=1}^{3} m_i\underline{q}_i = \psi(t)\sum_{i=1}^{3} m_i\underline{z}_i = \underline{0} \ , \qquad \sum_{i=1}^{3} \underline{p}_i = \dot{\psi}(t)\ \sum_{i=1}^{3} m_i\underline{z}_i = \underline{0} \ ,$$

which imply the elimination of the center of mass and of the total linear momentum, since the above equations yield that $\underline{u}_3 = \underline{0}$ and $\underline{v}_3 = \underline{0}$. Under the transformation (4.29) the Hamiltonian (4.28) becomes:

$$\mathcal{H}_2(\underline{v}_1,\underline{v}_2,\underline{u}_1,\underline{u}_2) = \frac{\|\underline{v}_1\|^2}{2M_1} + \frac{\|\underline{v}_2\|^2}{2M_2} - \frac{m_1m_2}{\|\underline{u}_1\|} - \frac{m_1m_3}{\|\underline{u}_2+M_3\underline{u}_1\|} - \frac{m_2m_3}{\|\underline{u}_2+M_4\underline{u}_1\|} \ , \tag{4.30}$$

where $M_1 \equiv \frac{m_1m_2}{m_1+m_2}$, $M_2 \equiv \frac{m_3(m_1+m_2)}{M}$, $M_3 \equiv \frac{m_2}{m_1+m_2}$, $M_4 \equiv -\frac{m_1}{m_1+m_2}$. Transforming to polar coordinates and making use of the constancy of the angular momentum allows us to reduce to three degrees of freedom. More precisely, we start by performing a coordinate change from $(\underline{u}_i,\underline{v}_i) \in \mathbf{R}^4$ to $(r_i,\vartheta_i,R_i,\Theta_i) \in \mathbf{R}^4$, defined as

$$\underline{u}_i = \begin{pmatrix} r_i\cos\vartheta_i \\ r_i\sin\vartheta_i \end{pmatrix} \ , \qquad \underline{v}_i = \begin{pmatrix} R_i\cos\vartheta_i - \frac{\Theta_i}{r_i}\sin\vartheta_i \\ R_i\sin\vartheta_i + \frac{\Theta_i}{r_i}\cos\vartheta_i \end{pmatrix} \ , \qquad i=1,2 \ .$$

The Hamiltonian (4.30) becomes

$$\begin{aligned} \mathcal{H}_3(R_1,R_2,\Theta_1,\Theta_2,r_1,r_2,\vartheta_1,\vartheta_2) &= \frac{1}{2M_1}\left(R_1^2+\frac{\Theta_1^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2+\frac{\Theta_2^2}{r_2^2}\right) \\ &- \frac{m_1m_2}{r_1} - \frac{m_1m_3}{\rho_1} - \frac{m_2m_3}{\rho_2} \ , \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} \rho_1 &\equiv \sqrt{r_2^2 + M_3^2r_1^2 + 2M_3r_1r_2\cos(\vartheta_2-\vartheta_1)} \ , \\ \rho_2 &\equiv \sqrt{r_2^2 + M_4^2r_1^2 + 2M_4r_1r_2\cos(\vartheta_2-\vartheta_1)} \ . \end{aligned}$$

Since (4.31) depends on ϑ_1, ϑ_2 through the difference $\vartheta_2-\vartheta_1$, we can perform the canonical change of variables

$$\begin{aligned} \xi &= \vartheta_1 & \Xi &= \Theta_1+\Theta_2 \\ \lambda &= \vartheta_2-\vartheta_1 & \Lambda &= \Theta_2 \ , \end{aligned}$$

which makes the Hamiltonian (4.31) independent of ξ. Therefore, setting $\Xi = h$, where h denotes the constant angular momentum, the transformed Hamiltonian becomes:

$$\begin{aligned}\mathcal{H}_4(R_1, R_2, \Lambda, r_1, r_2, \lambda) &= \frac{1}{2M_1}\left(R_1^2 + \frac{(h-\Lambda)^2}{r_1^2}\right) + \frac{1}{2M_2}\left(R_2^2 + \frac{\Lambda^2}{r_2^2}\right) \\ &- \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2}\ , \end{aligned} \tag{4.32}$$

where

$$\delta_1 \equiv \sqrt{r_2^2 + M_3^2 r_1^2 + 2M_3 r_1 r_2 \cos\lambda}\ , \qquad \delta_2 \equiv \sqrt{r_2^2 + M_4^2 r_1^2 + 2M_4 r_1 r_2 \cos\lambda}\ .$$

Remark. In the planetary case one assumes that one mass is much larger than the others, say $m_1 = \mu_1$, $m_2 = \varepsilon\mu_2$, $m_3 = \varepsilon\mu_3$, where ε is a small quantity and μ_i $(i = 1, 2, 3)$ is of the order of unity. Then, applying the change of variables

$$\tilde{R}_i = \frac{R_i}{\varepsilon}\ , \qquad \tilde{r}_i = r_i\ , \qquad \tilde{\Lambda} = \frac{\Lambda}{\varepsilon}\ , \qquad \tilde{\lambda} = \lambda\ , \qquad \tilde{h} = \frac{h}{\varepsilon}\ ,$$

one obtains the Hamiltonian

$$\begin{aligned}\mathcal{H}_5\big(\tilde{R}_1, \tilde{R}_2, \tilde{\Lambda}, \tilde{r}_1, \tilde{r}_2, \tilde{\lambda}\big) &= \frac{\varepsilon}{2M_1}\left(\tilde{R}_1^2 + \frac{(\tilde{h}-\tilde{\Lambda})^2}{\tilde{r}_1^2}\right) + \frac{\varepsilon}{2M_2}\left(\tilde{R}_2^2 + \frac{\tilde{\Lambda}^2}{\tilde{r}_2^2}\right) \\ &- \frac{\mu_1\mu_2}{\tilde{r}_1} - \frac{\mu_1\mu_3}{\tilde{\delta}_1} - \varepsilon\frac{\mu_2\mu_3}{\tilde{\delta}_2}\ , \end{aligned} \tag{4.33}$$

where $\tilde{\delta}_i$ are the quantities δ_i with r_i, λ replaced by $\tilde{r}_i$, $\tilde{\lambda}$. Observing that

$$\frac{\varepsilon}{M_1} = \frac{1}{\mu_2} + O(\varepsilon)\ , \qquad \frac{\varepsilon}{M_2} = \frac{1}{\mu_3} + O(\varepsilon)\ ,$$

one finds that the Hamiltonian (4.33) can be written as

$$\begin{aligned}\mathcal{H}_6(\tilde{R}_1, \tilde{R}_2, \tilde{\Lambda}, \tilde{r}_1, \tilde{r}_2, \tilde{\lambda}) &= \frac{1}{2\mu_2}\left(\tilde{R}_1^2 + \frac{(\tilde{h}-\tilde{\Lambda})^2}{\tilde{r}_1^2}\right) + \frac{1}{2\mu_3}\left(\tilde{R}_2^2 + \frac{\tilde{\Lambda}^2}{\tilde{r}_2^2}\right) \\ - \frac{\mu_1\mu_2}{\tilde{r}_1} - \frac{\mu_1\mu_3}{\tilde{\delta}_1} + \varepsilon F\big(\tilde{R}_1, \tilde{R}_2, \tilde{\Lambda}, \tilde{r}_1, \tilde{r}_2, \tilde{\lambda}\big)\ , \end{aligned} \tag{4.34}$$

for a suitable function $F = F(\tilde{R}_1, \tilde{R}_2, \tilde{\Lambda}, \tilde{r}_1, \tilde{r}_2, \tilde{\lambda})$. The Hamiltonian (4.34) is equal to the sum of two decoupled Kepler's motions, perturbed by a function of order ε, which can be considered as a small parameter. This model fits the planetary case where one mass (corresponding to the Sun) is much larger than those of the other bodies (the planets), which can be assumed to be of the same order of magnitude.

Coming back to the Lagrangian positions, let us denote by $\gamma = \gamma(t)$ the periodic orbit corresponding to the triangular configuration with sides of length d as in (4.27). Following [154] the stability of such configurations is investigated by linearizing the equations of motion associated to the Hamiltonian function (4.32) around the periodic solution. One obtains a time–dependent, periodic, linear system in the overall set of variables $\underline{z} \in \mathbf{R}^6$ of the form:

$$\dot{\underline{z}} = J D^2 \mathcal{H}_4(\gamma(t)) \underline{z} \ , \tag{4.35}$$

where J is the 6×6–dimensional matrix $J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$ (being I_n with $n \in \mathbf{Z}_+$ the $n \times n$ identity matrix) and $D^2\mathcal{H}_4(\gamma(t))$ denotes the Hessian of $\mathcal{H}_4$ computed along the periodic orbit. Let T be its period; the linear stability analysis involves the determination of the monodromy matrix $C = Z(T)$, where Z is the 6×6–dimensional matrix, solution of (4.35) with initial data $Z(0) = I_6$. The eigenvalues of C are the so–called *characteristic multipliers,* which are symmetric about the unit circle, due to the Hamiltonian character of the dynamics. The system is linearly stable if all multipliers have modulus one. In particular, two multipliers are unity: one of them is associated to the periodic orbit and the other to the Hamiltonian. Therefore, the linear stability is determined by the remaining four eigenvalues. Indeed, a suitable change of variables allows us to decouple the system: one part is associated to the unitary eigenvalues and a second part is a 4×4–dimensional system associated to the other eigenvalues (see [154]). In the latter case, the secular equation of order 4 depends on two parameters: the eccentricity and the mass parameter β defined as

$$\beta \equiv 27 \ \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} \ .$$

In the circular case the characteristic multipliers can be analytically computed and it is shown that they are purely imaginary if $0 \leq \beta < 1$; this is a classical result, already obtained by E.J. Gascheau in the 19*th* century ([74], see also [156]).

In the elliptic case the characteristic multipliers are obtained through a numerical integration; the results show that the triangular configuration becomes unstable as the eccentricity increases (see [154]). In particular, the stability is lost through a *period–doubling bifurcation* (namely two multipliers collide at -1 and move off along the real axis). For $\beta = \frac{3}{4}$ the system becomes unstable for any value of the eccentricity; afterwards there is an interval where the stability is maintained locally for non–zero values of the eccentricity, even though the circular solution is unstable. Finally the stability is lost through a *Krein bifurcation,* according to which two multipliers collide on the unitary circle and move off in the complex plane (see Figure 4.3).

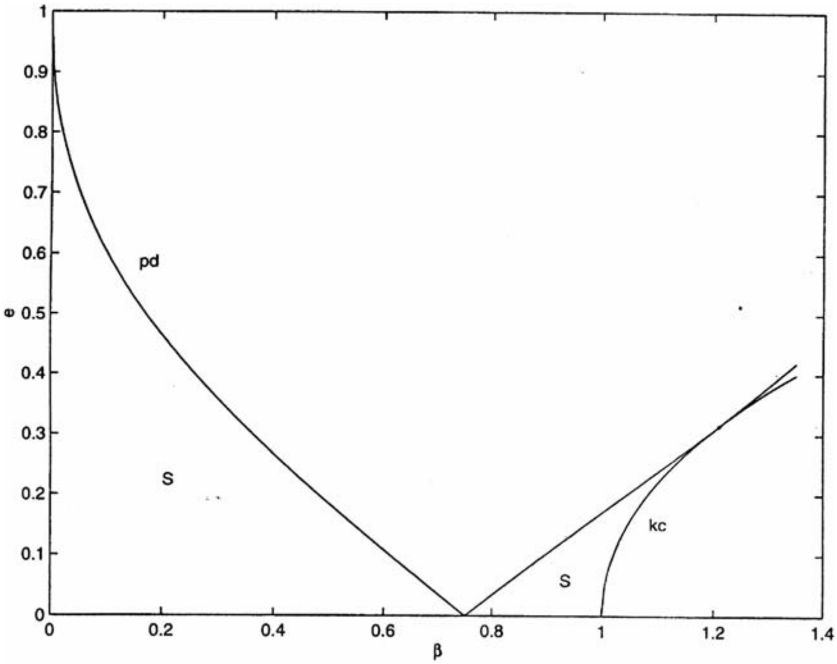

Fig. 4.3. The linear stability of the elliptic, unrestricted, triangular solutions within the plane (β,e). The meaning of the labels is the following: S denotes a linear stability region, pd is the period doubling curve ending at $\beta = \frac{3}{4}$, kc is the Krein collision curve starting at $\beta = 1$ (reprinted from [154] with permission from Elsevier).