3 Kepler's problem

The *two–body problem* is the study of the motion of two material points $\mathcal{P}_1$ and $\mathcal{P}_2$, with masses respectively m_1 and m_2; when the two bodies are subject to the mutual gravitational attraction one speaks of *Kepler's problem,* whose dynamics is described by the three so–called *Kepler's laws* (see, e.g., [157]). In this chapter we concentrate on the mathematical description of the two–body problem. The starting point is the gravitational law and Newton's three laws of dynamics. The gravitational law states that two bodies attract each other through a force which is directly proportional to the product of the masses and inversely proportional to the square of their distance r:

$$\underline{F} = -\mathcal{G}\frac{m_1 m_2}{r^2}\ \underline{e}_{12}\ ,$$

where $\mathcal{G}$ is the gravitational constant, amounting to $\mathcal{G} = 6.67 \cdot 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, and $\underline{e}_{12}$ is the unit vector joining the two bodies. Newton's laws of dynamics can be stated as follows:

(i) First law (law of inertia): without external forces every body remains at rest or moves uniformly on a straight line.
(ii) Second law: the net force experienced by a body is equal to the rate of change of its momentum.
(iii) Third law (action and reaction principle): for every action, there is an equal and opposite reaction.

As a consequence of the second law, if the mass of the body is constant, one gets the fundamental principle of classical mechanics according to which the net force is equal to the product of the mass of the particle times its acceleration:

$$\underline{F}\ =\ m\underline{a}\ . \tag{3.1}$$

After the investigation of the motion of the barycenter (Section 3.1), the solution of the two–body problem (Section 3.2) will be provided in terms of the three Kepler's laws, whose solution can also be given as a time series (Section 3.3); elliptic (Section 3.4), parabolic (Section 3.5) and hyperbolic (Section 3.6) motions will be analyzed and classified according to the value of the total mechanical energy (Section 3.7). We briefly remark that the Keplerian solution is also used for mission design as for the Hohmann transfer maneuvers (Section 3.8). Action–angle variables for the two–body problem are the so–called Delaunay variables (Section 3.9), which are also used to formulate Gylden's problem concerning a two–body model with variable mass (Section 3.10).

3.1 The motion of the barycenter

We start by introducing the following notation. In an inertial reference frame $(\mathcal{O}, X, Y, Z)$ with origin in $\mathcal{O}$, let $\underline{r}_1$ and $\underline{r}_2$ be the distance vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$ from $\mathcal{O}$. Let $\underline{r} = \underline{r}_2 - \underline{r}_1$ be the relative distance between $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote by $\mathcal{B}$ the barycenter of the two bodies and let $\underline{R}$ be the distance vector of $\mathcal{B}$ from $\mathcal{O}$ (Figure 3.1). Let $\underline{F}_1$ be the force exerted by $\mathcal{P}_2$ on $\mathcal{P}_1$ and let $\underline{F}_2$ be the force exerted by $\mathcal{P}_1$ on $\mathcal{P}_2$.

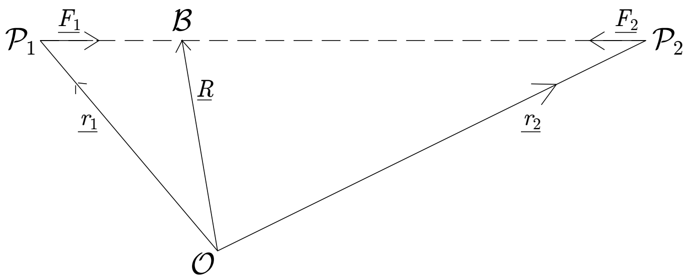

Fig. 3.1. Distance vectors in an inertial reference frame with origin in $\mathcal{O}$.

By the action and reaction principle one has

$$\underline{F}_1 = -\underline{F}_2 \ , \qquad \text{where} \qquad \underline{F}_1 = \mathcal{G}\frac{m_1 m_2}{r^2}\frac{\underline{r}}{r} \ .$$

Using (3.1), the equations of motion are given by the expressions

$$\begin{aligned} m_1 \frac{d^2\underline{r}_1}{dt^2} &= \mathcal{G}\frac{m_1 m_2}{r^2}\frac{\underline{r}}{r} \\ m_2 \frac{d^2\underline{r}_2}{dt^2} &= -\mathcal{G}\frac{m_1 m_2}{r^2}\frac{\underline{r}}{r} \ . \end{aligned} \tag{3.2}$$

Adding the above equations one obtains

$$m_1 \frac{d^2\underline{r}_1}{dt^2} + m_2 \frac{d^2\underline{r}_2}{dt^2} = \underline{0} \ ,$$

whose integration provides the relations:

$$m_1 \frac{d\underline{r}_1}{dt} + m_2 \frac{d\underline{r}_2}{dt} = \underline{C}_1 \ , \qquad m_1\underline{r}_1 + m_2\underline{r}_2 = \underline{C}_1 t + \underline{C}_2 \ ,$$

with $\underline{C}_1$, $\underline{C}_2$ being constant vectors.

Let M be the total mass, namely $M = m_1 + m_2$; the location of the barycenter is given by

$$M\underline{R} = m_1\underline{r}_1 + m_2\underline{r}_2 \ .$$

Therefore we obtain the equations

$$M\frac{d\underline{R}}{dt} = \underline{C}_1 \ , \qquad M\underline{R} = \underline{C}_1 t + \underline{C}_2 \ ,$$

which express that the barycenter moves with constant velocity.

3.2 The solution of Kepler's problem

Let us divide the first of (3.2) by m_1 and the second by m_2; subtracting the two resulting equations one obtains:

$$\frac{d^2}{dt^2}(\underline{r}_1 - \underline{r}_2) = \mathcal{G}(m_1 + m_2)\frac{\underline{r}}{r^3} \ ;$$

being $\underline{r} = \underline{r}_2 - \underline{r}_1$ one finds:

$$\frac{d^2\underline{r}}{dt^2} + \mu\frac{\underline{r}}{r^3} = \underline{0} \ , \tag{3.3}$$

where $\mu \equiv \mathcal{G}(m_1 + m_2)$. Multiplying (3.3) by the vector $\underline{r}$ one gets

$$\underline{r} \wedge \frac{d^2\underline{r}}{dt^2} = \underline{0} \ ,$$

namely

$$\underline{r} \wedge \frac{d\underline{r}}{dt} = \underline{h} \ , \tag{3.4}$$

where $\underline{h}$ is a constant vector which represents the total angular momentum; such a vector turns out to be perpendicular to the orbital plane. From (3.4) one obtains that the two bodies move at any instant on the same plane.

On such a plane of motion we introduce a reference frame $(\mathcal{P}_1, x, y, z)$ with axes parallel to the inertial frame, the z–axis being orthogonal to the plane of motion and with the origin centered in $\mathcal{P}_1$ (Figure 3.2); let us denote by $\underline{i}$, $\underline{j}$, $\underline{k}$ the unit vectors corresponding to the reference axes. Let (r, ϑ) be the polar coordinates of $\mathcal{P}_2$ with respect to $\mathcal{P}_1$. Since the coordinates of $\mathcal{P}_2$ are $(x, y, z) = (r\cos\vartheta, r\sin\vartheta, 0)$, one obtains

$$\det\left(\underline{r} \wedge \frac{d\underline{r}}{dt}\right) = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ r\cos\vartheta & r\sin\vartheta & 0 \\ \dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta & \dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta & 0 \end{pmatrix} = r^2\dot{\vartheta}\ \underline{k} \ .$$

Denoting by h the absolute value of $\underline{h}$ one has

$$r^2\dot{\vartheta} = h \ , \tag{3.5}$$

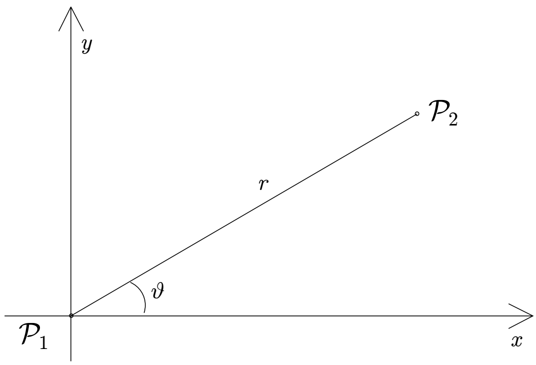

Fig. 3.2. Geometrical configuration of Kepler's problem.

which provides Kepler's second law, whose physical interpretation is the following: the areal velocity spanned by the radius vector is constant. In fact, let us evaluate the area $\delta\mathcal{A}$ spanned by the radius vector $r(t)$ at time t and by the vector $r(t+\delta t)$ at time $t+\delta t$:

$$\delta\mathcal{A} = \frac{1}{2} r(t) r(t+\delta t) \sin\delta\vartheta \ ,$$

where $\delta\vartheta$ represents the angle spanned from $r(t)$ to $r(t+\delta t)$. The variation of $\mathcal{A}$ with respect to the time is given by

$$\frac{\delta\mathcal{A}}{\delta t} = \frac{1}{2} r(t) r(t+\delta t) \frac{\sin\delta\vartheta}{\delta\vartheta} \frac{\delta\vartheta}{\delta t} \ ;$$

in the limit of δt tending to zero the areal velocity is given by

$$\dot{\mathcal{A}} = \frac{1}{2} r^2 \dot{\vartheta} \ . \tag{3.6}$$

We next consider the scalar product of (3.3) with $\underline{\dot{r}}$:

$$\underline{\dot{r}} \cdot \frac{d^2\underline{r}}{dt^2} + \mu \frac{\underline{\dot{r}} \cdot \underline{r}}{r^3} = 0 \ ,$$

which provides

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{\mu}{r} = E \ , \tag{3.7}$$

where E is a suitable real constant. Equation (3.7) provides the preservation of the total energy.

In order to solve Kepler's problem, we need to compute the radial and orthogonal components of the acceleration. In cartesian coordinates one finds

$$\begin{aligned} \ddot{x} &= \ddot{r}\cos\vartheta - 2\dot{r}\dot{\vartheta}\sin\vartheta - r\ddot{\vartheta}\sin\vartheta - r\dot{\vartheta}^2\cos\vartheta \\ \ddot{y} &= \ddot{r}\sin\vartheta + 2\dot{r}\dot{\vartheta}\cos\vartheta + r\ddot{\vartheta}\cos\vartheta - r\dot{\vartheta}^2\sin\vartheta \\ \ddot{z} &= 0 \ . \end{aligned} \tag{3.8}$$

Multiplying the first equation by $\cos\vartheta$, the second by $\sin\vartheta$ and adding the results, the radial component of the acceleration is given by

$$\ddot{r} - r\dot{\vartheta}^2 = -\frac{\mu}{r^2} \ . \tag{3.9}$$

Moreover, multiplying the second of (3.8) by $\cos\vartheta$, the first by $\sin\vartheta$ and subtracting the results, the orthogonal component is equal to

$$r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} = 0 \ .$$

Such an equation can be written in the form $\frac{d}{dt}(r^2\dot{\vartheta}) = 0$, which provides the constancy of the angular momentum h as in (3.5).

Let us introduce the quantity $\rho \equiv \frac{1}{r}$; using the constancy of the angular momentum, the radial component (3.9) can be written in terms of ρ as

$$\frac{d^2\rho}{d\vartheta^2} + \rho = \frac{\mu}{h^2} \ . \tag{3.10}$$

In fact, from

$$\frac{d\rho}{d\vartheta} = \frac{d(1/r)}{dt}\frac{r^2}{h} = -\frac{\dot{r}}{h} \ , \qquad \frac{d^2\rho}{d\vartheta^2} = -\ddot{r}\frac{r^2}{h^2} \ ,$$

one obtains the equation

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \ ,$$

while using $\dot{\vartheta} = \frac{h}{r^2}$ one gets (3.9).

The equation (3.10) can be solved to provide the variation of ρ as a function of ϑ as

$$\rho(\vartheta) = \frac{\mu}{h^2} + A\cos(\vartheta - g_0) \ ,$$

where A, g_0 are suitable constants. Recalling that $\rho = \frac{1}{r}$ and introducing the quantities $p \equiv \frac{h^2}{\mu}$, called the ellipse *parameter,* and $\mathrm{e} \equiv \frac{Ah^2}{\mu}$, called the *eccentricity,* one obtains the expression providing the radius vector as a function of the angle ϑ:

$$r = \frac{p}{1 + \mathrm{e}\cos(\vartheta - g_0)} \ . \tag{3.11}$$

The quantity g_0, usually called the *argument of perihelion,* represents the angle between the x–axis of the reference frame and the direction of the semimajor axis of the ellipse, called the *perihelion axis* (compare with Figure 3.3). Introducing the *true anomaly* f as

$$f \equiv \vartheta - g_0 \ ,$$

equation (3.11) can be equivalently written as

$$r = \frac{p}{1 + \mathrm{e}\cos f} \ . \tag{3.12}$$

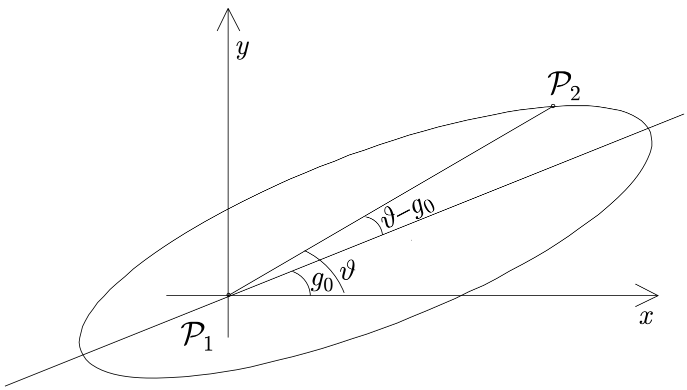

Fig. 3.3. The argument of perihelion g_0.

3.3 $\tilde{f}$ and $\tilde{g}$ series

Kepler's problem admits a solution in the form of a time series, the coefficients of such a series being functions of the mass parameter μ, of the initial values of the radius vector $\underline{r}$ and of its derivative. Inserting in (3.3) the change of time given by

$$\tau = \sqrt{\mu}\, t \ ,$$

one obtains the equation

$$\frac{d^2\underline{r}}{d\tau^2} + \frac{\underline{r}}{r^3} = \underline{0} \ . \tag{3.13}$$

For short we denote by $\underline{r}' = \frac{d\underline{r}}{d\tau}$, $\underline{r}'' = \frac{d^2\underline{r}}{d\tau^2}$ and so on. With this notation, the equation (3.13) can be written as $\underline{r}'' = -\frac{\underline{r}}{r^3}$. Differentiating (3.13) we obtain

$$\begin{aligned} \underline{r}''' &= -\left(\frac{\underline{r}'}{r^3} - \frac{3\underline{r}}{r^4}\frac{\underline{r}'\cdot\underline{r}}{r}\right) \\ \underline{r}'''' &= -\left\{\left[\frac{2\mu}{r^6} - \frac{3\underline{r}'\cdot\underline{r}'}{r^5} + \frac{15(\underline{r}'\cdot\underline{r})^2}{r^7}\right]\underline{r} - 6\frac{\underline{r}'\cdot\underline{r}}{r^5}\underline{r}'\right\} \\ &\dots \end{aligned} \tag{3.14}$$

Expanding $\underline{r}$ in Taylor series around $\tau = 0$ and setting $\underline{r}_0 = \underline{r}(0)$ (similarly for the derivatives) we obtain

$$\underline{r} = \underline{r}_0 + \underline{r}_0'\tau + \frac{1}{2}\underline{r}_0''\tau^2 + \frac{1}{3!}\underline{r}_0'''\tau^3 + \dots$$

Using (3.14) and rearranging the terms we can write

$$\underline{r} = \tilde{f}\underline{r}_0 + \tilde{g}\underline{r}_0' \ ,$$

where $\tilde{f}$ and $\tilde{g}$ are the following series in τ:

$$\begin{aligned} \tilde{f}(\tau) &= 1 - \frac{1}{2r_0^3}\tau^2 + \frac{1}{2r_0^3}\frac{\underline{r}_0'\cdot\underline{r}_0}{r_0^2}\tau^3 + \dots \\ \tilde{g}(\tau) &= \tau - \frac{1}{6r_0^3}\tau^3 + \dots \end{aligned}$$

The series $\tilde{f} = \tilde{f}(\tau)$ and $\tilde{g} = \tilde{g}(\tau)$ converge if τ is small; they can be efficiently used to determine the solution as a function of time.

3.4 Elliptic motion

We prove that for eccentricities between 0 and 1 ($0 \leq e < 1$) the motion takes place on an ellipse. We consider a reference frame centered in $\mathcal{P}_1$ and with the abscissa coinciding with the perihelion line. Having denoted by r the size of the radius vector joining $\mathcal{P}_1$ and $\mathcal{P}_2$, and by f the angle spanned by the radius vector with respect to the perihelion axis, the coordinates of $\mathcal{P}_2$ are given by

$$(x, y) = (r\cos f, r\sin f) \ .$$

Therefore from (3.12) we obtain $p = r + \mathrm{e}x$; taking the square of such equation and recalling that $r = \sqrt{x^2 + y^2}$, one obtains

$$x^2(1 - \mathrm{e}^2) + 2p\mathrm{e}\ x + y^2 = p^2\ .$$

This equation can be written as

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1\ , \tag{3.15}$$

where

$$x_0 = -\frac{p\mathrm{e}}{1 - \mathrm{e}^2}\ , \qquad a = \frac{p}{1 - \mathrm{e}^2}\ , \qquad b = \frac{p}{\sqrt{1 - \mathrm{e}^2}}\ . \tag{3.16}$$

Notice that (3.15) describes an ellipse with semimajor axes a and b oriented according to the x and y axes. Moreover we find that the quantity $\mathrm{e} = \sqrt{1 - \frac{b^2}{a^2}}$ coincides with the *eccentricity* of the ellipse.

We have thus proved Kepler's first law, which states the following: assuming that $\mathcal{P}_1$ coincides with the Sun and $\mathcal{P}_2$ with a planet, then the motion of the planet takes place on an ellipse with the Sun located at one of the two foci.

From the second of (3.16) and from $p = \frac{h^2}{\mu}$, one obtains $h = \sqrt{\mu a(1 - \mathrm{e}^2)}$. From (3.5) and (3.6) the angular momentum h is equal to twice the areal velocity; denoting by T the period of revolution, being πab the area of the ellipse, one obtains that $h = \frac{2}{T}\pi ab$. Using the relation $b = a\sqrt{1 - \mathrm{e}^2}$ one gets that the period of revolution and the semimajor axis are linked by the expression:

$$T^2 = \frac{4\pi^2}{\mu}\ a^3\ . \tag{3.17}$$

Equation (3.17) provides the content of Kepler's third law.

We are finally in the position to summarize Kepler's laws, which were proved in the present and previous sections.

First law. The orbit of each planet around the Sun is an ellipse with the Sun at one focus.

Second law. The radius vector sweeps equal areas in equal intervals of time.

Third law. The square of the period of revolution is proportional to the third power of the semimajor axis.

We remark that, among other consequences, Kepler's third law allows us to estimate the mass of a planet, once the orbital elements of one of its satellites are known. More precisely, let us denote by m_{Sun}, $m_{\mathcal{P}}$, $m_{\mathcal{S}}$ the masses of the Sun, of the planet $\mathcal{P}$ and of its satellite $\mathcal{S}$. Let $a_{\mathcal{P}}$, $a_{\mathcal{S}}$ and $T_{\mathcal{P}}$, $T_{\mathcal{S}}$ be, respectively, the semimajor axes and the periods of the planet around the Sun, and of the satellite around the planet; we assume that these quantities can be obtained by direct measurements. Applying Kepler's third law to the pairs Sun–planet and planet–satellite, one obtains

$$\mathcal{G}(m_{Sun} + m_{\mathcal{P}}) = 4\pi^2 \frac{a_{\mathcal{P}}^3}{T_{\mathcal{P}}^2}\ , \qquad \mathcal{G}(m_{\mathcal{P}} + m_{\mathcal{S}}) = 4\pi^2 \frac{a_{\mathcal{S}}^3}{T_{\mathcal{S}}^2}\ .$$

The ratio of the two equations provides

$$\frac{m_{\mathcal{P}} + m_{\mathcal{S}}}{m_{Sun} + m_{\mathcal{P}}} = \left(\frac{a_{\mathcal{S}}}{a_{\mathcal{P}}}\right)^3 \left(\frac{T_{\mathcal{P}}}{T_{\mathcal{S}}}\right)^2 .$$

Assuming that the mass of the satellite is negligible with respect to that of the planet and that the mass of the Sun is known, the previous equation provides an estimate for the mass of the planet as

$$m_{\mathcal{P}} = m_{Sun} \frac{\left(\frac{a_{\mathcal{S}}}{a_{\mathcal{P}}}\right)^3 \left(\frac{T_{\mathcal{P}}}{T_{\mathcal{S}}}\right)^2}{1 - \left(\frac{a_{\mathcal{S}}}{a_{\mathcal{P}}}\right)^3 \left(\frac{T_{\mathcal{P}}}{T_{\mathcal{S}}}\right)^2} . \tag{3.18}$$

For example, let us take $\mathcal{P}$ as Jupiter and $\mathcal{S}$ as its satellite Io; their elements are $a_{\mathcal{P}} = 7.78 \cdot 10^8$ km, $T_{\mathcal{P}} = 4331.87$ days, $a_{\mathcal{S}} = 421\,800$ km, $T_{\mathcal{S}} = 1.769$ days, while $m_{Sun} = 2 \cdot 10^{30}$ kg. The expression (3.18) provides an estimate for the mass of Jupiter equal to $1.9 \cdot 10^{27}$ kg in full agreement with the experimental data.

To conclude the description of the elliptic motion, we provide the formula for the squared velocity which, expressed in terms of the polar coordinates, takes the form,

$$v^2 = \dot{r}^2 + r^2 \dot{f}^2 .$$

From (3.12) and (3.5) one finds

$$\dot{r} = \frac{h}{p} \mathrm{e} \sin f , \qquad r\dot{f} = \frac{h}{p}(1 + \mathrm{e} \cos f) .$$

Computing the square, adding the two equations and using $\frac{h^2}{\mu} = p = a(1 - \mathrm{e}^2)$ one obtains

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a}\right) .$$

We remark that at perihelion $r = a(1 - \mathrm{e})$ so that the velocity $v^2 = \frac{\mu}{a} \frac{1+\mathrm{e}}{1-\mathrm{e}}$ is maximum, while at aphelion $r = a(1 + \mathrm{e})$ so that $v^2 = \frac{\mu}{a} \frac{1-\mathrm{e}}{1+\mathrm{e}}$ and the velocity is minimum. We also remark that for $\mathrm{e} = 0$ the orbit reduces to a circle.

3.4.1 Mean and eccentric anomaly

If T denotes the period of revolution of $\mathcal{P}_2$ around $\mathcal{P}_1$, we introduce the *mean motion* as

$$n \equiv \frac{2\pi}{T} . \tag{3.19}$$

The angular momentum can be expressed in terms of the mean motion as $h = \frac{2}{T}\pi a^2 \sqrt{1 - \mathrm{e}^2} = na^2 \sqrt{1 - \mathrm{e}^2}$. Let t_0 be the time of passage at perihelion; we define the *mean anomaly* ℓ_0 as the angle described by the radius vector rotating around the focus with mean angular velocity n during the interval $t - t_0$:

$$\ell_0 \equiv n(t - t_0) .$$

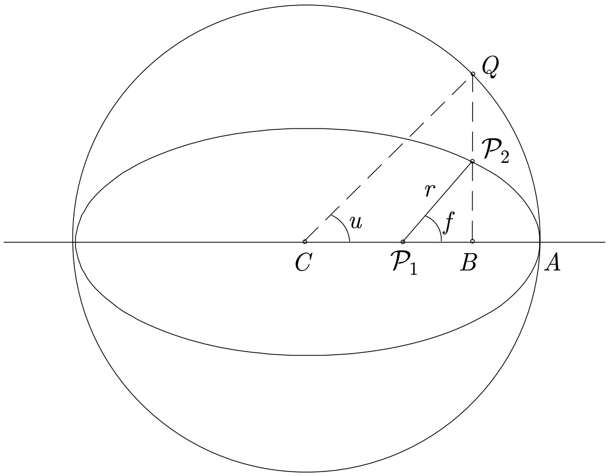

Fig. 3.4. The eccentric anomaly.

We next introduce a quantity u called the *eccentric anomaly:* draw the circle with radius equal to the semimajor axis of the ellipse (see Figure 3.4); from the instantaneous position of $\mathcal{P}_2$ on the ellipse, draw the perpendicular to the semimajor axis until it meets the circle and let u be the angle QCA formed by the direction to the center and the direction corresponding to the semimajor axis.

The mathematical relations within the true, mean and eccentric anomalies can be easily derived from the geometry of the problem. With reference to Figure 3.4 one has: $\mathcal{P}_1 B = CB - C\mathcal{P}_1 = a\cos u - a\mathrm{e}$ and, since $\mathcal{P}_1 B = r\cos f$, it follows that

$$r\cos f = a(\cos u - \mathrm{e}) \ . \tag{3.20}$$

By elementary properties of the ellipse one has $\frac{\mathcal{P}_2 B}{QB} = \frac{b}{a}$, namely $\frac{r\sin f}{a\sin u} = \frac{b}{a}$; by this relation one has:

$$r\sin f = a\sqrt{1-\mathrm{e}^2}\sin u \ . \tag{3.21}$$

Computing the square of (3.20), (3.21) and adding the two equations one obtains

$$r^2 = a^2 + a^2\mathrm{e}^2\cos^2 u - 2a^2\mathrm{e}\cos u \ ,$$

from which it follows that

$$r = a(1 - \mathrm{e}\cos u) \ ; \tag{3.22}$$

this relation provides the radius vector as a function of the eccentric anomaly.

Taking into account that $2r\sin^2\frac{f}{2} = r(1-\cos f)$ and using (3.20), (3.21), one obtains

$$\begin{aligned} 2r\sin^2\frac{f}{2} &= a(1+\mathrm{e})(1-\cos u) \\ 2r\cos^2\frac{f}{2} &= a(1-\mathrm{e})(1+\cos u) \ ; \end{aligned}$$

computing the ratio of the two equations one gets

$$\tan\frac{f}{2} = \sqrt{\frac{1+\mathrm{e}}{1-\mathrm{e}}}\ \tan\frac{u}{2}\ , \tag{3.23}$$

which provides the true anomaly as a function of the eccentric anomaly.

Let us now derive the relation between the eccentric and mean anomalies; this formula is known as *Kepler's equation.*

From Kepler's second law we can state that the ratio bewteen the area of the region defined by $\mathcal{P}_1\mathcal{P}_2A$ and the area of the ellipse amounts to $\frac{t-t_0}{T}$; recalling the definition of the mean anomaly one has that

$$area(\mathcal{P}_1\mathcal{P}_2A) = \frac{1}{2}ab\ell_0\ .$$

On the other hand this area can be obtained as the sum of the area $\mathcal{P}_1\mathcal{P}_2B$ and of the area $B\mathcal{P}_2A$, where the area $B\mathcal{P}_2A$ is equal to $\frac{b}{a}$ times the area of QBA; therefore one has the sequence of relations:

$$\begin{aligned}
area(\mathcal{P}_1\mathcal{P}_2A) &= area(\mathcal{P}_1\mathcal{P}_2B) + \frac{b}{a}area(QBA) \\
&= area(\mathcal{P}_1\mathcal{P}_2B) + \frac{b}{a}(area(QCA) - area(QCB)) \\
&= \frac{1}{2}r^2\sin f\cos f + \frac{b}{a}\left(\frac{1}{2}a^2u - \frac{1}{2}a^2\sin u\cos u\right) \\
&= \frac{1}{2}ab(u - \mathrm{e}\sin u)\ .
\end{aligned}$$

One thus obtains that the relation between ℓ_0 and u is given by

$$\ell_0 = u - \mathrm{e}\sin u\ , \tag{3.24}$$

which is known as *Kepler's equation.* It is now necessary to solve this equation to get u as a function of the time, being $\ell_0 = n(t-t_0)$. Once such equation is solved, and therefore $u = u(t)$ is obtained, one inserts the resulting expression in (3.22) and (3.23) to obtain the variation with time of the radius vector and the true anomaly, thus providing the solution of the equation of motion.

3.4.2 Solution of Kepler's equation

In order to find the eccentric anomaly as a function of the time, it is necessary to solve the implicit Kepler's equation (3.24). An approximate solution can be computed as long as the eccentricity e is small. Indeed, the inversion of (3.24) provides u as a function of ℓ_0 as a series in the eccentricity:

$$\begin{aligned}
u &= \ell_0 + \mathrm{e}\sin u \\
&= \ell_0 + \mathrm{e}\sin(\ell_0 + \mathrm{e}\sin u) \\
&= \ell_0 + \mathrm{e}\sin(\ell_0 + \mathrm{e}\sin(\ell_0 + \mathrm{e}\sin u)) \\
&= \ell_0 + \left(\mathrm{e} - \frac{\mathrm{e}^3}{8}\right)\sin\ell_0 + \frac{1}{2}\mathrm{e}^2\sin(2\ell_0) + \frac{3}{8}\mathrm{e}^3\sin(3\ell_0) + O(\mathrm{e}^4)\ ,
\end{aligned}$$

where $O(\mathrm{e}^4)$ denotes a quantity of order e^4. The complete solution can be expressed as

$$u = \ell_0 + \mathrm{e}\sum_{k=1}^{\infty} \frac{1}{k}\Big[J_{k-1}(k\mathrm{e}) + J_{k+1}(k\mathrm{e})\Big]\sin(k\ell_0)\ , \tag{3.25}$$

where $J_k(x)$ are the *Bessel's functions* of order k and argument x; they are defined by the relation

$$J_k(x) \equiv \frac{1}{2\pi}\int_0^{2\pi} \cos(kt - x\sin t)dt\ .$$

The functions $J_k(x)$ can be developed as follows:

$$\begin{aligned} J_0(x) &\equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\Big(\frac{x}{2}\Big)^{2m} \\ J_k(x) &\equiv \Big(\frac{x}{2}\Big)^k \frac{1}{k!}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\prod_{j=1}^{m}(k+j)}\Big(\frac{x}{2}\Big)^{2m}\ . \end{aligned} \tag{3.26}$$

Notice that equations (3.25), (3.26) provide the solution of Kepler's equation with arbitrary precision.

3.5 Parabolic motion

When $\mathrm{e} = 1$ one gets the open trajectory described by the equation

$$r = \frac{p}{1+\cos f}\ . \tag{3.27}$$

From (3.27) it follows that $r + r\cos f = p$, namely $r + x = p$; using $r = \sqrt{x^2+y^2}$ one obtains $y^2 = -2px + p^2$, namely

$$x = -\frac{y^2}{2p} + \frac{p}{2}\ ,$$

which describes a parabola in the plane (y, x) with vertex coinciding with $(\frac{p}{2}, 0)$ (see Figure 3.5).

Notice that equation (3.27) can be written in the form

$$r = \frac{p}{2}\left(1 + \tan^2\frac{f}{2}\right)\ .$$

Using (3.5), (3.27), one has:

$$\Big(\frac{p}{2}\Big)^2 \frac{1}{\cos^4\frac{f}{2}}\dot f = \sqrt{p\mu}\ ,$$

whose integration yields

$$2\left(\frac{\mu}{p^3}\right)^{1/2}(t - t_0) = \tan\frac{f}{2} + \frac{1}{3}\tan^3\frac{f}{2}\ , \tag{3.28}$$

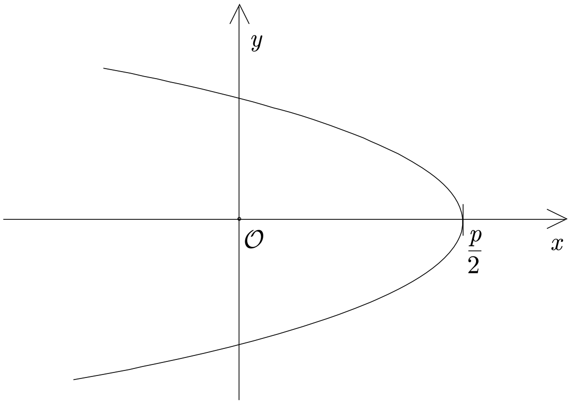

Fig. 3.5. The parabolic solution of Kepler's problem.

where t_0 is the time of passage at perihelion. The solution of equation (3.28), known as *Barker's equation,* provides the variation of the true anomaly as a function of time in the case of a parabolic orbit.

3.6 Hyperbolic motion

For $\mathrm{e} > 1$, we write the polar equation as

$$r = \frac{a(\mathrm{e}^2 - 1)}{1 + \mathrm{e}\cos f} \ . \tag{3.29}$$

Using the relations

$$x = r\cos f \ , \qquad r = \sqrt{x^2 + y^2} \ , \qquad b = a\sqrt{\mathrm{e}^2 - 1} \ ,$$

we obtain

$$x^2(\mathrm{e}^2 - 1) - y^2 + a^2(\mathrm{e}^2 - 1)^2 - 2a\mathrm{e}(\mathrm{e}^2 - 1)x = 0 \ ; \tag{3.30}$$

since $b = a\sqrt{\mathrm{e}^2 - 1}$, the equation (3.30) becomes

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1 \ ,$$

where we have introduced $x_0 = a\mathrm{e}$. From the angular momentum integral the velocity can be written as

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right) \ .$$

Notice that r tends to infinity with a non–zero velocity given by $v^2 = \frac{\mu}{a}$.

From (3.29) one has

$$\cos f = \frac{a(\mathrm{e}^2 - 1)}{\mathrm{e}r} - \frac{1}{\mathrm{e}} \ . \tag{3.31}$$

From (3.17) and (3.19) one has $n^2 a^3 = \mu$; computing the derivative of (3.31) with respect to r and using the angular momentum integral in the form $h = \sqrt{\mu p} = r^2 \dot{f}$ as well as $p = a(\mathrm{e}^2 - 1)$, one finds

$$n\frac{dt}{dr} = \frac{r}{a\sqrt{(a+r)^2 - a^2\mathrm{e}^2}} \ .$$

Introducing the hyperbolic eccentric anomaly u_h such that

$$r \equiv a(\mathrm{e}\cosh u_h - 1) \ , \tag{3.32}$$

one obtains

$$n\frac{dt}{du} = \mathrm{e}\cosh u_h - 1$$

whose integration provides the hyperbolic Kepler's equation

$$\ell_0 = \mathrm{e}\sinh u_h - u_h \ ,$$

where ℓ_0 is the mean anomaly. Notice that such equation is not periodic and that the solution tends quickly to infinity. From (3.29) and (3.32) one gets

$$\frac{\mathrm{e}^2 - 1}{1 + \mathrm{e}\cos f} = \mathrm{e}\cosh u_h - 1 \ ;$$

using the formulae

$$\cos f = \frac{1 - \tan^2 \frac{f}{2}}{1 + \tan^2 \frac{f}{2}} \ , \qquad \cosh u_h = \frac{1 + \tanh^2 \frac{u_h}{2}}{1 - \tanh^2 \frac{u_h}{2}} \ ,$$

one obtains the relation between the true and eccentric anomaly in the case of hyperbolic motion:

$$\tan\frac{f}{2} = \sqrt{\frac{\mathrm{e}+1}{\mathrm{e}-1}} \ \tanh\frac{u_h}{2} \ .$$

Numerical methods for solving Kepler's equation in the hyperbolic case were developed for example in [72, 135].

3.7 Classification of the orbits

According to the value of the parameter e (the eccentricity) the trajectory coincides with the following conic sections:

(i) $\mathrm{e} = 0$: the trajectory is a circle;
(ii) $0 < \mathrm{e} < 1$: the trajectory is an ellipse;
(iii) $\mathrm{e} = 1$: the trajectory is a parabola;
(iv) $\mathrm{e} > 1$: the trajectory is a hyperbola.

The same classification of the orbits can be inferred as a function of the energy. In polar coordinates the energy is given by (see (3.7))

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2) - \frac{\mu}{r} \ ,$$

where $\mu \equiv \mathcal{G}(m_1 + m_2)$; using the angular momentum integral we can write

$$E = \frac{1}{2}\dot{r}^2 + V_e(r) \ ,$$

where $V_e(r)$ is the effective potential given by

$$V_e(r) = \frac{h^2}{2r^2} - \frac{\mu}{r} \ .$$

Then, we obtain

$$\frac{dr}{dt} = \sqrt{2(E - V_e(r))} \ .$$

Through the angular momentum integral one gets

$$\vartheta - \vartheta_0 = h \int \frac{dr}{r^2\sqrt{2(E - V_e(r))}} = \arccos \frac{\frac{r_0}{r} - 1}{\sqrt{1 - \frac{E}{E_0}}} \ ,$$

where r_0 is such that $V_e(r_0)$ is minimum and $E_0 = E(r_0)$, namely

$$r_0 = \frac{h^2}{\mu} \ , \qquad E_0 = -\frac{\mu^2}{2h^2} \ .$$

Recalling (3.11) we find

$$p = r_0 \ , \qquad \mathrm{e} = \sqrt{1 - \frac{E}{E_0}} \ , \qquad \vartheta_0 = g_0 \ .$$

In summary we obtain that the parameter e is related to the energy E by

$$\mathrm{e} = \sqrt{1 + \frac{2h^2E}{\mu^2}} \ .$$

According to the classification of the orbits in terms of the eccentricity we obtain the following classification of the trajectories in terms of the energy:

(i) $E = -\frac{\mu^2}{2h^2}$ (i.e. $\mathrm{e} = 0$): the trajectory is a circle;
(ii) $E < 0$ (i.e. $0 < \mathrm{e} < 1$): the trajectory is an ellipse;
(iii) $E = 0$ (i.e. $\mathrm{e} = 1$): the trajectory is a parabola;
(iv) $E > 0$ (i.e. $\mathrm{e} > 1$): the trajectory is a hyperbola.

3.8 Spacecraft transfers

As a practical implementation of Keplerian orbits, we consider the problem of spacecraft transfers. The transfer of a spacecraft from one orbit to another is obtained by implementing proper orbital maneuvers (see, e.g., [51]). The classical ones are the so–called *Hohmann transfer* and *bi–elliptic Hohmann transfer* maneuvers, which are based on a careful combination of suitable Keplerian elliptic orbits. Impulse maneuvers require a short firing of the on–board engines, so to allow for a change of sign and direction of the velocity vector. A Hohmann transfer requires two impulse maneuvers for transferring the spacecraft from one circular orbit of radius r_A to another coplanar circular orbit of radius r_B, through an elliptic orbit which is tangent to both circles at their periapses (see Figure 3.6(a)). The changes of velocities required at the periapses can be easily computed using the angular momentum integral. Bi–elliptic Hohmann transfers between the circles of radii r_A and r_B are constructed using two semi–ellipses as in Figure 3.6(b). The first semi–ellipse allows us to reach a point C outside the external circle (see Figure 3.6(b)), while the second semi–ellipse joins with the target point B on the external circle.

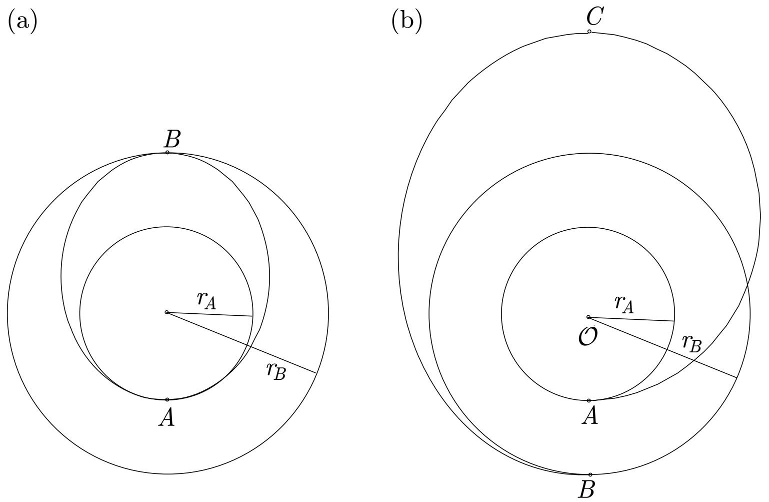

Fig. 3.6. (a) A Hohmann transfer from the circular orbit of radius r_A to the circular orbit of radius r_B. (b) A bi–elliptic Hohmann transfer from the point A on the circle of radius r_A to the point B on the circle of radius r_B.

3.9 Delaunay variables

Classical action–angle variables (see [73] and Appendix A) for the two–body problem are known as Delaunay variables [18, 169]. We present their detailed derivation for the planar motion and we provide the results for the spatial case. Let (r, ϑ) be the polar coordinates as in Figure 3.2 and let (p_r, p_ϑ) be the conjugated momenta;

it is readily seen that $p_\vartheta = h$. The Hamiltonian function[1] governing the two–body motion is given by

$$\mathcal{H}(p_r, p_\vartheta, r, \vartheta) = \frac{1}{2}\left(p_r^2 + \frac{p_\vartheta^2}{r^2}\right) - \frac{\mu}{r} \ .$$

Being ϑ a cyclic variable, we introduce the effective potential (see Figure 3.7) as

$$V_e(r) = \frac{p_\vartheta^2}{2r^2} - \frac{\mu}{r} \ , \tag{3.33}$$

so that the Hamiltonian can be written as a one–dimensional Hamiltonian of the form

$$\mathcal{H}(p_r, r) = \frac{p_r^2}{2} + V_e(r) \ . \tag{3.34}$$

For a fixed value E of the energy, let $r_\pm = r_\pm(E)$ be the roots of $V_e(r) = E$, so that

$$E - V_e(r) = -\frac{E}{r^2}(r_+ - r)(r - r_-) \qquad \text{with} \qquad r_\pm(E) = \frac{\mu \pm \sqrt{\mu^2 + 2Ep_\vartheta^2}}{-2E} \ .$$

The period of the motion can be expressed as

$$T(E) = 2\int_{r_-(E)}^{r_+(E)} \frac{dr}{\sqrt{2(E - V_e(r))}} = 2\pi\mu\left(\frac{1}{-2E}\right)^{3/2} \ .$$

By Kepler's third law (3.17) one obtains the following relation between the semi-major axis and the energy:

$$a = -\frac{\mu}{2E} \ . \tag{3.35}$$

Let us define the action variable L_0 as

$$L_0 \equiv \sqrt{-\frac{\mu^2}{2E}}$$

which in view of (3.35) provides

$$L_0 = \sqrt{\mu a} \ .$$

On the other hand, since (3.34) does not depend on ϑ, we can define another action variable as

$$G_0 \equiv p_\vartheta \ ;$$

using the expression for the angular momentum $h = \sqrt{\mu a(1 - \mathrm{e}^2)}$ and being $p_\vartheta = h$, one gets

$$G_0 = L_0\sqrt{1 - \mathrm{e}^2} \ .$$

[1] See Appendix A for a basic introduction to Hamiltonian dynamics.

Notice that one can express the elliptic elements, namely semimajor axis and eccentricity, in terms of the Delaunay action variables as

$$a = \frac{L_0^2}{\mu} \ , \qquad \mathrm{e} = \sqrt{1 - \frac{G_0^2}{L_0^2}} \ .$$

The Hamiltonian function expressed in terms of the action variables becomes

$$\mathcal{H} = \mathcal{H}(L_0) = -\frac{\mu^2}{2L_0^2} \ . \tag{3.36}$$

As for the angle variables, we proceed as follows. Using the relations

$$p_r = p_r(L_0, G_0, r) = \sqrt{-\frac{\mu^2}{L_0^2} + \frac{2\mu}{r} - \frac{G_0^2}{r^2}} \ , \qquad p_\vartheta = G_0 \ ,$$

we introduce the generating function defining the Delaunay variables as

$$\Phi(L_0, G_0, r, \vartheta) = \int p_r dr + \int p_\vartheta d\vartheta = \int \sqrt{-\frac{\mu^2}{L_0^2} + \frac{2\mu}{r} - \frac{G_0^2}{r^2}} dr + G_0 \vartheta \ .$$

The angle variable conjugated to L_0 is defined as

$$\ell_0 = \frac{\partial \Phi}{\partial L_0} = \int \frac{\mu^2}{L_0^3 \sqrt{-\frac{\mu^2}{L_0^2} + \frac{2\mu}{r} - \frac{G_0^2}{r^2}}} dr \ .$$

Using (3.22) it follows that ℓ_0 coincides with the mean anomaly, namely

$$\ell_0 = u - \mathrm{e} \sin u \ .$$

The angle variable conjugated to G_0 is computed as

$$g_0 = \frac{\partial \Phi}{\partial G_0} = \vartheta - \int \frac{G_0}{r^2 \sqrt{-\frac{\mu^2}{L_0^2} + \frac{2\mu}{r} - \frac{G_0^2}{r^2}}} dr \ .$$

Using (3.22) one finds that $g_0 = \vartheta - f$, which coincides with the argument of perihelion.

In the spatial case, namely when the three bodies are not constrained to move on the same plane, one needs to add a third pair of action–angle variables. Indeed, in polar coordinates (r, ϑ, φ) the spatial two–body Hamiltonian is given by

$$\mathcal{H}(p_r, p_\vartheta, p_\varphi, r, \vartheta, \varphi) = \frac{1}{2} \left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \vartheta} \right) - \frac{\mu}{r} \ ,$$

where $(p_r, p_\vartheta, p_\varphi)$ are conjugated to (r, ϑ, φ). Define

$$G_0 = \sqrt{p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2 \vartheta}} \ , \qquad H_0 = p_\varphi$$

and let the energy be

$$E = \frac{1}{2}\left(p_r^2 + \frac{G_0^2}{r^2}\right) - \frac{\mu}{r}\ .$$

One easily finds that

$$p_\vartheta = \pm\sqrt{G_0^2 - \frac{H_0^2}{\sin^2\vartheta}}\ , \qquad p_r = \pm\sqrt{2\left(E + \frac{\mu}{r}\right) - \frac{G_0^2}{r^2}}\ .$$

Having set

$$\vartheta_- = \arcsin\frac{H_0}{G_0}\ , \qquad \vartheta_+ = 2\pi - \vartheta_-$$

and

$$r_\pm = -\frac{1}{2E}\left(\mu \pm \sqrt{\mu^2 + 2EG_0^2}\right)\ ,$$

the action variables can be defined as

$$\begin{aligned} A_1 &\equiv \frac{1}{2\pi}\int_0^{2\pi} p_\varphi d\varphi = H_0 \\ A_2 &\equiv \frac{1}{2\pi}\int_{\vartheta_-}^{\vartheta_+} p_\vartheta d\vartheta = G_0 - |H_0| \\ A_3 &\equiv \frac{1}{2\pi}\int_{r_-}^{r_+} p_r dr = -G_0 + L_0\ . \end{aligned}$$

Being $L_0^2 = -\frac{\mu^2}{2E}$, the new Hamiltonian is given by

$$\mathcal{H}(A_1, A_2, A_3) = -\frac{\mu^2}{2(A_1 + A_2 + A_3)^2}\ .$$

Let α_1, α_2, α_3 be the conjugated angle variables. The relation with the Delaunay variables is obtained through the symplectic change of coordinates

$$\begin{aligned} L_0 &= |A_1| + A_2 + A_3 & \ell_0 &= \alpha_3 \\ G_0 &= |A_1| + A_2 & g_0 &= \alpha_2 - \alpha_3 \\ H_0 &= |A_1| & h_0 &= \alpha_1 - \alpha_2\ , \end{aligned}$$

where it can be shown (see, e.g., [53]) that H_0 is related to G_0 by

$$H_0 = G_0 \cos i\ ,$$

being i the inclination of the orbital plane with respect to a fixed inertial reference frame. The angle variable h_0 conjugated to H_0 coincides with the *longitude of the ascending node,* namely the angle formed by the x–axis of the reference frame with the line of nodes given by the intersection of the orbital plane with the xy–reference plane. The Hamiltonian function of the spatial case becomes

$$\mathcal{H} = \mathcal{H}(L_0) = -\frac{\mu^2}{2L_0^2}\ .$$

We remark that if the eccentricity is zero, then $G_0 = L_0$ and the argument of perihelion is not defined; similarly, when the inclination is zero, then $H_0 = G_0$ and the longitude of the ascending node is not defined. In these cases it is convenient to introduce the *modified* Delaunay variables defined as

$$\begin{aligned} \Lambda &= L_0 & \lambda &= \ell_0 + g_0 + h_0 \\ \Gamma &= L_0 - G_0 = L_0\Big(1 - \sqrt{1 - \mathrm{e}^2}\,\Big) & \gamma &= -g_0 - h_0 \\ \Phi &= G_0 - H_0 = 2G_0 \sin^2 \frac{i}{2} & \varphi &= -h_0 \ . \end{aligned}$$

The singularities are now represented by $\Gamma = 0$ and $\Phi = 0$, for which γ and φ are not defined. We remark that for small values of the eccentricity and of the inclination, it is often convenient to introduce the so–called *Poincaré variables* defined as

$$\begin{aligned} p_1 &= \sqrt{2\Gamma}\cos\gamma & p_2 &= \sqrt{2\Phi}\cos\varphi \\ q_1 &= \sqrt{2\Gamma}\sin\gamma & q_2 &= \sqrt{2\Phi}\sin\varphi \ . \end{aligned}$$

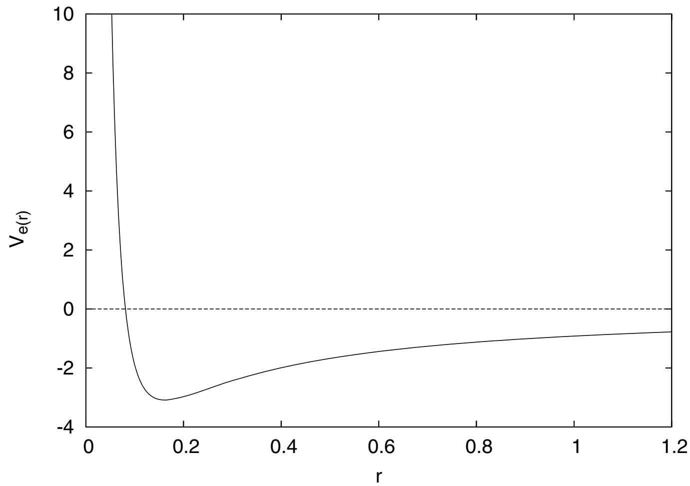

Fig. 3.7. Graph of the effective potential $V_e(r)$ given in (3.33) for $p_\vartheta = 0.4025$ and $\mu = 1$.

3.10 The two–body problem with variable mass

3.10.1 The rocket equation

In this section we study the two–body problem formed by a planet and a satellite and we assume that the mass of the satellite is not constant, but varies with time. For example, we can imagine dealing with an artificial satellite, whose mass variation is due to the loss of fuel. We assume that the decrease of the mass of the rocket is constant, namely

$$\frac{dm}{dt} = -b \tag{3.37}$$

for some positive constant b. Let us denote by $\underline{v}_p$ the exhaust velocity of the expelled particles with respect to the spacecraft; let $\Delta t = t - t_0$ with t_0 being the initial time and let $\Delta\underline{v} = \underline{v}(t) - \underline{v}(t_0)$ be the variation of the velocity. Let $m\underline{v}$ be the momentum at time t, and let $(m - b\Delta t)(\underline{v} + \Delta\underline{v})$ be the momentum at time $t + \Delta t$. Without external forces acting on the rocket (as in the case of high–thrust engines), the total change of linear momentum is given by (see [152])

$$(m - b\Delta t)(\underline{v} + \Delta\underline{v}) + b\Delta t(\underline{v} + \underline{v}_p) - m\underline{v} = \underline{0} \ .$$

In the limit for Δt tending to zero, one gets the *rocket equation* (see, e.g., [152]):

$$m\frac{d\underline{v}}{dt} = -b\underline{v}_p \ . \tag{3.38}$$

Recalling (3.37) and assuming $\underline{v}_p$ constant, the solution of (3.38) is given by

$$\Delta\underline{v} = -\underline{v}_p \log \frac{m(t_0)}{m} \ ,$$

where $m(t_0)$ is the initial mass. Notice that the quantity $\Delta\underline{v}$ is the velocity needed for the maneuver, which depends on the rate of mass loss.

3.10.2 Gylden's problem

A physical sample described by a two–body problem with variable mass is composed by a planet orbiting around a central star which varies its mass [58, 88, 90, 116]. Following classical results by Jeans [98] one can assume a mass variation according to the law

$$\frac{dm}{dt} = -\alpha m^j \ , \tag{3.39}$$

where α is usually small and j varies in the interval $[1.4, 4.4]$. For example, in the case of the Sun, the decrease of mass by radiation implies that α is of the order of 10^{-12} or 10^{-13}, where the units of measure have been assumed as the solar mass, the astronomical unit and the year; a bigger α must be adopted in the case of corpuscolar emission.

Denoting by $\underline{v}_C$ the velocity of the center of mass and by $\underline{F}$ the sum of all external forces, the equation of motion is given by

$$\frac{d}{dt}(m\underline{v}_C) = \underline{F} \ .$$

For a point within the body let $\underline{v}$ be its velocity and let $\underline{v}_m$ be the relative velocity of the escaping or incident mass with respect to the center of mass [91]; then, we can write the equation of motion as

$$m(t)\frac{d\underline{v}}{dt} = \underline{F} + \underline{v}_m \frac{dm(t)}{dt} \ . \tag{3.40}$$

When the mass is ejected isotropically, for example as for the solar wind, the sum of the contributions of the second term of the right–hand side of (3.40) cancels out and the equation of motion reduces to the so–called *Gylden's equation*

$$m(t)\frac{d\underline{v}}{dt} = \underline{F} \ . \tag{3.41}$$

If the body travels within a stationary cloud and accumulates mass, then it is $\underline{v}_m = -\underline{v}$ and the equation of motion (3.40) reduces to the so–called *Levi–Civita's equation*

$$\frac{d}{dt}(m(t)\underline{v}) = \underline{F} \ .$$

In the rest of this section we will concentrate on the analysis of Gylden's equation (3.41). Let us denote by $\underline{x} \in \mathbf{R}^3$ the two–body relative position vector and by $\underline{X} \in \mathbf{R}^3$ the conjugated momentum vector. In suitable units of measure let us write the Hamiltonian function associated to (3.41) as

$$\mathcal{H}_0(\underline{X}, \underline{x}, t) = \frac{1}{2}\underline{X} \cdot \underline{X} - \frac{k(t)}{\|\underline{x}\|} \ , \tag{3.42}$$

where in (3.41) we assumed $\underline{F} = -\frac{k(t)}{\|\underline{x}\|^3}\underline{x}$ ($\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^3$) with $k(t)$ taking into account the mass variation (eventually one can assume that the dependence upon time is due to a time variation of the gravitational constant). According to [56] we assume that

$$k(t) \equiv \frac{k_0}{\varepsilon(t)} \ , \tag{3.43}$$

where $\varepsilon(t_0) = 1$ for some initial time t_0; we also assume that at any time $\varepsilon(t)$ is positive and that $\dot{\varepsilon}(t) \neq 0$. From (3.42) the equations of motion read as

$$\begin{aligned} \dot{\underline{x}} &= \underline{X} \\ \dot{\underline{X}} &= -k(t)\frac{\underline{x}}{\|\underline{x}\|^3} \ , \end{aligned}$$

from which we obtain that the angular momentum vector $\underline{h} = \underline{x} \wedge \underline{X}$ is constant (as it follows from $\dot{\underline{h}} = \dot{\underline{x}} \wedge \underline{X} + \underline{x} \wedge \dot{\underline{X}}$). Let us show that a suitable coordinate and time transformation gives (3.42) in the form of a perturbed Kepler's problem. Let $(\underline{y}, \underline{Y})$ denote a new set of variables obtained through the generating function

$$\Phi_1(\underline{X}, \underline{y}, t) \equiv \varepsilon \underline{y} \left(\underline{X} - \frac{1}{2}\dot{\varepsilon}\underline{y} \right) \ ,$$

which provides the change of coordinates:

$$\underline{x} = \varepsilon \underline{y} \ , \qquad \underline{X} = \frac{1}{\varepsilon}\underline{Y} + \dot{\varepsilon}\underline{y} \ .$$

Denoting by $\delta(t) \equiv \varepsilon^3 \ddot{\varepsilon}$, the Hamiltonian in the new variables takes the form

$$\mathcal{H}_1(\underline{Y}, \underline{y}, t) = \frac{1}{\varepsilon^2}\Big(\frac{1}{2}\underline{Y} \cdot \underline{Y} - \frac{k_0}{\|\underline{y}\|} + \frac{1}{2}\delta(t)\|\underline{y}\|^2\Big) \ .$$

Next we perform a change of time according to

$$dt = \varepsilon^2 d\tau \ ; \tag{3.44}$$

setting $\delta(\tau) = \delta(t(\tau))$ through the transformation (3.44), the new Hamiltonian becomes

$$\mathcal{H}_2(\underline{Y}, \underline{y}, \tau) = \frac{1}{2}\underline{Y} \cdot \underline{Y} - \frac{k_0}{\|\underline{y}\|} + \frac{1}{2}\delta(\tau)\|\underline{y}\|^2 \ , \tag{3.45}$$

with associated Hamilton's equations

$$\frac{d\underline{y}}{d\tau} = \frac{\partial \mathcal{H}_2}{\partial \underline{Y}} \ , \qquad \frac{d\underline{Y}}{d\tau} = -\frac{\partial \mathcal{H}_2}{\partial \underline{y}} \ .$$

From (3.43) and (3.44) we obtain

$$\frac{\dot{k}}{k} = -\frac{\dot{\varepsilon}}{\varepsilon} \ , \qquad \frac{\ddot{\tau}}{\dot{\tau}} = -\frac{2\dot{\varepsilon}}{\varepsilon} \ ,$$

which yield

$$\frac{\ddot{\tau}}{\dot{\tau}} = \frac{2\dot{k}}{k} \ ,$$

usually referred to as the *law of marginal acceleration in ephemeris time* [56].

Let us express the Hamiltonian (3.45) in terms of the Delaunay variables introduced in Section 3.9. To this end, we set $\underline{y} = (r\cos\vartheta, r\sin\vartheta)$, $\underline{Y} = (p_r\cos\vartheta - \frac{p_\vartheta}{r}\sin\vartheta, p_r\sin\vartheta + \frac{p_\vartheta}{r}\cos\vartheta)$ and we perform a change of variables from $(p_r, p_\vartheta, r, \vartheta)$ to Delaunay variables (L_0, G_0, ℓ_0, g_0) through the generating function

$$\Phi_2(L_0, G_0, r, \vartheta, t) = \int_{r_0}^{r} \sqrt{-\frac{k^2}{L_0^2} + \frac{2k}{r} - \frac{G_0^2}{r^2}}\ dr + G_0\vartheta \ ,$$

where r_0 is a root of the function $A(L_0, G_0, r) \equiv -\frac{k^2}{L_0^2} + \frac{2k}{r} - \frac{G_0^2}{r^2}$. With the present notation the semimajor axis and the eccentricity of the osculating Keplerian orbit are related to the action Delaunay variables by

$$a = \frac{L_0^2}{k} \ , \qquad \mathrm{e} = \sqrt{1 - \frac{G_0^2}{L_0^2}} \ .$$

The time derivative of the generating function is given by

$$\begin{aligned} \frac{\partial \Phi_2}{\partial t} &= \frac{\dot{k}}{k}\left[k\int_{r_0}^{r}\frac{dr}{r\sqrt{A(L_0, G_0, r)}} - \frac{k}{a}\int_{r_0}^{r}\frac{dr}{\sqrt{A(L_0, G_0, r)}}\right] \\ &= \frac{\dot{k}}{k}\left[\frac{L_0^3}{k^2}\frac{k}{a}u - \frac{L_0^3}{k^2}\frac{k}{a}(u - \mathrm{e}\sin u)\right] \\ &= \frac{\dot{k}}{k}L_0\mathrm{e}\sin u \ . \end{aligned}$$

Finally, the one–dimensional, time–dependent Hamiltonian function describing Gylden's problem is given by

$$\mathcal{H}_3(L_0, \ell_0, t; G_0) = \mathcal{H}_2 + \frac{\partial \Phi_2}{\partial t} = -\frac{k^2}{2L_0^2} + \frac{\dot{k}}{k} L_0 \mathrm{e} \sin u \ ,$$

where u is related to ℓ_0 by Kepler's equation $\ell_0 = u - \mathrm{e} \sin u$, which can be inverted to provide $\sin u = \sin \ell_0 + \frac{\mathrm{e}}{2} \sin 2\ell_0 - \frac{\mathrm{e}^2}{8}(\sin \ell_0 - 3 \sin 3\ell_0) + O(\mathrm{e}^3)$. Here L_0, G_0, ℓ_0 should be interpreted as the osculating elements of the Keplerian motion having $k = k(t)$ constant.

In the following example we choose $j = 3$ in (3.39), so that the variation of the mass is given by the equation $\dot{m} = -\alpha m^3$, whose integration provides $m(t) = \frac{1}{\sqrt{2\alpha t}}$. We assume that the gravitational constant does not vary with time and we normalize it to one, so that $k(t)$ coincides with $m(t)$. The Hamiltonian function of Gylden's problem, depending parametrically on the eccentricity e and on the perturbing parameter α, turns out to be:

$$\begin{aligned} \mathcal{H}_G(L_0, \ell_0, t; \mathrm{e}, \alpha) &= -\frac{m(t)^2}{2L_0^2} - \alpha m(t)^2 L_0 \mathrm{e} \Big(\sin \ell_0 + \frac{\mathrm{e}}{2} \sin 2\ell_0 \\ &- \frac{\mathrm{e}^2}{8} (\sin \ell_0 - 3 \sin 3\ell_0) \Big) \ . \end{aligned}$$