9 Viscous Fluids

Materials are usually classified into *solids* and *fluids*, where fluids are subdivided into *liquids* and *gases*. These divisions are not always clear because there are materials which possess both solid-like and fluid-like properties. Any *fluid* is defined as a material which deforms continously as long as a shearing stress is acting. A *solid*, on the other hand, can be in equilibrium under a shear stress. Some solids have a natural configuration to which they return if an imposed stress is removed. Such a configuration can be regarded as the reference configuration. Fluids do not possess a natural configuration, i.e., they take the shape of the surrounding boundary.

In the following we differentiate between *linear viscous fluids* and *nonlinear viscous fluids* , where the latter belong to the class of *non*-NEWTON*ian fluids*.

9.1 Linear Viscous Fluids

A fluid at rest cannot sustain any shear stress, i.e., the stress state in a fluid at rest is characterized by a spherical tensor, $\boldsymbol{\sigma} \sim \boldsymbol{\delta}$, according to the constitutive equation

$$\sigma_{ij} = -p(\rho, T)\delta_{ij} \tag{9.1}$$

employed in *hydrostatics*, where the hydrostatic pressure p is related to the temperature T and the density ρ by a thermal equation of state having the form $F(p, \rho, T) = 0$. An example of an *equation of state* is the law $p = \rho RT$ of an ideal gas, where R is the special gas constant for a particular gas not to be confused with the general gas constant.

A fluid in motion ($\boldsymbol{d} \neq \boldsymbol{0}$) can sustain *viscous stress*, which can be expressed in the linear case by the linear transformation

$$\tau_{ij} = V_{ijkl}(\rho, T)d_{kl} \ . \tag{9.2}$$

This tensor is called *viscous stress tensor* or sometimes *extra stress tensor*. The cartesian components V_{ijkl} of the *viscosity tensor* $\boldsymbol{V}$ reflect the viscous properties of the fluid. In general, one can assume that fluids are *isotropic*. Then, the viscous stress tensor is a function of only one argument tensor, namely the rate-of-deformation tensor $\boldsymbol{d}$ with components (3.22), i.e.,

$$\tau_{ij} = \tau_{ij}(d_{pq}) \;, \tag{9.3}$$

and the fourth-order viscosity tensor must be an isotropic tensor, the components of which are expressed by

$$V_{ijkl} = \xi\delta_{ij}\delta_{kl} + \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \;. \tag{9.4}$$

This relation is similar to the elasticity tensor in (2.31) for isotropic linear elastic materials.

Adding (9.1) and (9.2) with (9.4), we arrive at the constitutive equation

$$\boxed{\sigma_{ij} = \left[-p(\rho, T) + \xi(\rho, T)d_{kk}\right]\delta_{ij} + 2\eta(\rho, T)d_{ij}} \tag{9.5a}$$

or

$$\tau_{ij} \equiv \sigma_{ij} + p(\rho, T)\delta_{ij} = \xi d_{kk}\delta_{ij} + 2\eta d_{ij} \;, \tag{9.5b}$$

which characterizes a NEWTON*ian fluid*.

In the special case of a *shearing flow* ($i = 1$, $j = 2$), the constitutive equation (9.5a) reduces to the simple relation $\tau = \eta\dot{\gamma}$, where $\tau = \tau_{12} \equiv \sigma_{12}$ and $\dot{\gamma} = 2d_{12}$. Thus, the parameter η in (9.5a) is the *shear viscosity*. The second parameter, ξ, in (9.5a) shall be interpreted later.

The constitutive equation (9.5a) fulfills the *principle of material frame indifference* (*objectivity*), since the rate-of-deformation tensor is an objective tensor (BETTEN, 2001a) and the right-hand side of (9.5a) is not affected by a superimposed rigid-body motion. Thus, the constitutive equation (9.5a) has the required property of being independent of any superimposed rigid-body motion. This is not true for the linear constitutive equation of an isotropic elastic material (BETTEN, 2001a).

By introducing the fourth-order spherical tensor (9.4) we have assumed that the fluid behaves *isotropic*. As a matter of fact, isotropy is a consequence of (9.2) and the requirement that the stress is not influenced by any rigid-body motion. Thus, it was not necessary to consider *isotropy* as a special assumption. However, fluids with anisotropic properties may exist, but their behavior cannot be expressed by the linear transformation (9.2).

For the sake of practical applications it may be useful to split the constitutive equation (9.5a) into a scalar and a deviatoric part. To do this, we firstly equalize the trace

$$\sigma_{kk} \equiv 3\bar{\sigma} = -3\bar{p} \tag{9.6}$$

and the trace of (9.5a) arriving at the scalar relation

$$\boxed{\bar{p} = p(\rho, T) - \left(\xi + \frac{2}{3}\eta\right) d_{kk}} \ . \tag{9.7a}$$

The deviatoric equation can be derived from the stress deviator (2.23) by taking the relations (9.5a), (9.6), and (9.7a) into account:

$$\boxed{\sigma'_{ij} = 2\eta d'_{ij}} \ , \tag{9.7b}$$

in which $d'_{ij} := d_{ij} - d_{kk}\delta_{ij}/3$ are the components of rate-of-deformation deviator $\boldsymbol{d}'$. In a similar way we arrive from (9.5b) at the following two equations

$$\bar{\tau} \equiv \frac{1}{3}\tau_{kk} = \left(\xi + \frac{2}{3}\eta\right) d_{kk} \quad \text{and} \quad \tau'_{ij} = 2\eta d'_{ij} \ . \tag{9.8a,b}$$

By analogy with the *bulk modulus*, also called *volume elasticity modulus* (BETTEN, 2001a),

$$K \equiv E_{\text{Vol}} := \sigma_{\text{Vol}}/\varepsilon_{\text{Vol}} \equiv \bar{\sigma}/\varepsilon_{kk} \ , \tag{9.9}$$

we define the *bulk viscosity* (*volume viscosity*) as the quotient from viscous volume stress τ_{Vol} and volume strain rate d_{Vol}:

$$\eta_{\text{Vol}} := \tau_{\text{Vol}}/d_{\text{Vol}} \equiv \bar{\tau}/d_{kk} \ , \tag{9.10}$$

so that we find by considering (9.8a) the result

$$\eta_{\text{Vol}} = \xi + \frac{2}{3}\eta \ . \tag{9.11}$$

Hence, the parameter ξ is immaterial since the constitutive equations (9.7a,b) and (9.8a,b) can be expressed according to

$$\boxed{\bar{p} = p(\rho, T) - \eta_{\text{Vol}} d_{kk}} \ , \qquad \boxed{\sigma'_{ij} = 2\eta d'_{ij}} \ , \tag{9.12a,b}$$

and

$$\bar{\tau} = \eta_{\text{Vol}} d_{kk} \equiv p(\rho, T) - \bar{p} \ , \qquad \tau'_{ij} = 2\eta d'_{ij} \ , \tag{9.13a,b}$$

respectively, by taking (9.11) into account. The relation (9.12a) associates the mean normal stress $\sigma_{kk}/3 = -\bar{p}$ with the *thermodynamic pressure* $p(\rho, T)$ and the *bulk viscosity* η_{Vol}, while the equation (9.12b) relates the *shear effect* of the motion with the *stress deviator*.

The *volume viscosity* η_{Vol} takes into account the molecular degrees of freedom and vanishes for one-atomic gases. Experimental investigations have shown that the volume viscosity (9.11) is very small or even negligible. Thus, in such cases it is justified in assuming the STOKES *condition*

$$\boxed{\xi + \frac{2}{3}\eta = 0} \; . \tag{9.14}$$

On this condition or in an *incompressible* NEWTON*ian fluid* ($d_{kk} = 0$) the mean pressure $\bar{p}$ in (9.7a) equals the *thermodynamic pressure* $p(\rho, T)$ at all times. For *nonlinear viscous fluids* (section 9.2), the assumption of incompressibility does not imply $\bar{p} = p$.

Assuming the STOKES condition (9.14), we immediately arrive from (9.5a) at the constitutive equation

$$\boxed{\sigma_{ij} = -p\delta_{ij} - \frac{2}{3}\eta d_{kk}\delta_{ij} + 2\eta d_{ij} \equiv -p\delta_{ij} + 2\eta d'_{ij}} \; , \tag{9.15}$$

which describes the so called STOKES *fluid*.

The importance of the *volume viscosity* (9.10), (9.11) becomes also visible when discussing the *dissipation power*

$$\dot{D} := \tau_{ij} d_{ji} = \xi d_{kk}^2 + 2\eta d_{ij} d_{ji} \; , \tag{9.16a}$$

which can be deduced from (9.5b) and represented in the form

$$\dot{D} = \eta_{\mathrm{Vol}} d_{kk}^2 + 2\eta d'_{ij} d'_{ji} \; , \tag{9.16b}$$

if we split the rate-of-deformation tensor $\boldsymbol{d}$ into the deviator (d'_{ij}) and the spherical tensor ($d_{kk}\delta_{ij}/3$). We can also express (9.16b) as

$$\dot{D} = \eta_{\mathrm{Vol}} I_1^2 + 4\eta I'_2 \; , \tag{9.16c}$$

where the invariants $I_1 \equiv d_{kk}$ and $I'_2 \equiv d'_{ij} d'_{ji}/2$ have been introduced. According to (9.16b,c), the dissipation power can be decomposed into two parts characterizing the *volume change* (without change of shape) and the *distortion*, respectively.

Based upon the *second law of thermodynamics*, dissipation is required to be nonnegative. Thus, we deduce from (9.16b,c):

$$\eta_{\text{Vol}} \geq 0 \quad \text{and} \quad \eta \geq 0 \;, \tag{9.17a,b}$$

or, considering (9.11), we find:

$$\xi \geq -\frac{2}{3}\eta \;. \tag{9.17c}$$

Now, let us discuss an *extension flow* (DIN 13 342) characterized by the uniaxial stress component

$$\tau_{11} = (1-2\nu)\xi d_{11} + 2\eta d_{11} \;, \tag{9.18}$$

which results from (9.5b) by inserting $i = j = 1$, where

$$\nu := -d_{22}/d_{11} = -d_{33}/d_{11} \tag{9.19}$$

is the isotropic *transverse contraction ratio*. In contrast to the *shear viscosity* η in (9.5a,b) and the *volume viscosity* (9.10), we define an *extension viscosity* according to

$$\eta_D := \tau_{11}/d_{11} \;. \tag{9.20}$$

Hence, we deduce from (9.18) for a NEWTONian fluid

$$\eta_D = (1-2\nu)\xi + 2\eta \;. \tag{9.21}$$

On the other hand, we follow from (9.8b) the relation

$$\tau'_{11} = 2\eta d'_{11} \;, \tag{9.22}$$

so that for the extension flow

$$\tau_{ij} = \text{diag}\{\tau_{11}, 0, 0\} \;, \qquad d_{ij} = \text{diag}\{d_{11}, -\nu d_{11}, -\nu d_{11}\} \tag{9.23a,b}$$

with $\tau'_{11} = 2\tau_{11}/3$ and $d'_{11} = 2(1+\nu)d_{11}/3$ the extension viscosity (9.20) yields

$$\eta_D = 2(1+\nu)\eta \;. \tag{9.24}$$

This result corresponds with the similar relation of the linear theory of elasticity,

$$E = 2(1+\nu)G \;, \tag{9.24*}$$

where E and G are the elasticity and the shear modulus, respectively.

For an incompressible ($\nu = 1/2$) NEWTONian fluid, we read from (9.24) the TROUTON *number* (1906)

$$N_{\mathrm{Tr}} := \eta_D/\eta = 3 \ . \tag{9.25}$$

Combining (9.21) and (9.24), the parameter ξ may be expressed as

$$\xi = \frac{2\nu}{1-2\nu}\eta \ , \tag{9.26}$$

so that the *volume viscosity* (9.11) becomes

$$\eta_{\mathrm{Vol}} = \frac{2}{3}\frac{1+\nu}{1-2\nu}\eta \ , \tag{9.27}$$

and by eliminating the shear viscosity η from (9.24) and (9.27) we arrive at the relation

$$\eta_{\mathrm{Vol}} = \frac{1}{3(1-2\nu)}\eta_D \ , \tag{9.28}$$

which corresponds with the similar formula of elasticity,

$$K \equiv E_{\mathrm{Vol}} = \frac{1}{3(1-2\nu)}E \ , \tag{9.28*}$$

where E_{Vol} is the *volume elasticity modulus*, most called *bulk modulus*.

In a similar way, by eliminating the transvection ratio ν, we arrive from (9.27) and (9.28) at the formula

$$\eta_D = 9\eta_{\mathrm{Vol}}\eta/\left(3\eta_{\mathrm{Vol}} + \eta\right) \ , \tag{9.29}$$

which contains the TROUTON *number* (9.25) for $3\eta_{\mathrm{Vol}} \gg \eta$, while for $3\eta_{\mathrm{Vol}} \ll \eta$ th relation $\eta_D = 9\eta_{\mathrm{Vol}}$ follows.

Experimental investigations on *non*-NEWTON*ian fluids* have shown that the extension viscosity or the shear viscosity is a function of the strain rate, $\eta_D = \eta_D(d)$, or of the shear rate, $\eta = \eta(\dot\gamma)$, respectively. For example, LAUN and MÜNSTEDT (1978) have carried out experiments on the LDPE melt IUPACA at $T = 150°\mathrm{C}$. The results are illustrated in Fig. 9.1.

We see that for small deformation rates the viscosities are approaching the TROUTON *number* (9.25):

$$\lim_{d\to 0}\eta_D(d) = 3\lim_{\dot\gamma\to 0}\eta(\dot\gamma) \tag{9.30}$$

Further experiments on *non*-NEWTON*ian fluids* are carried out by BALLMANN (1965), MEISSNER (1971; 1972), STEVENSON (1972), ASTARITA

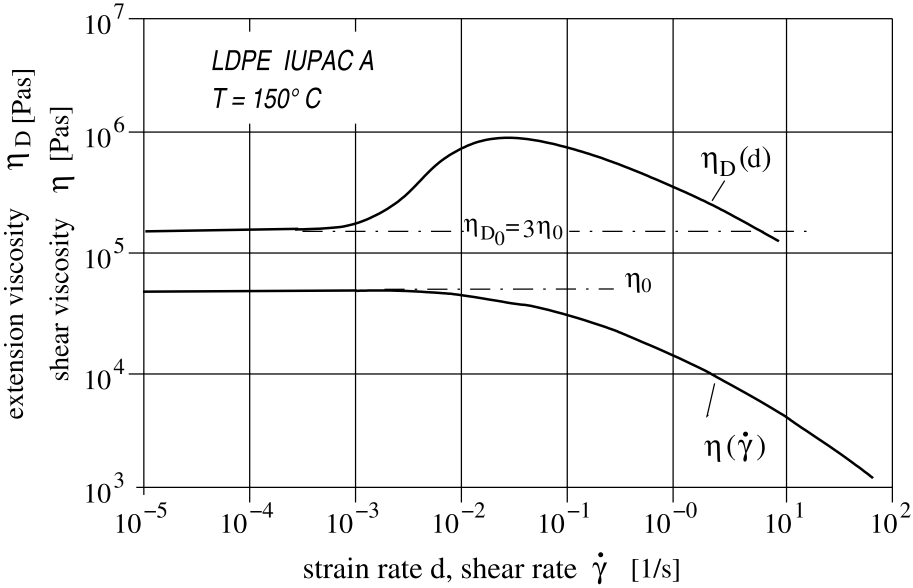

Fig. 9.1 Viscosities in strain and shear as functions of deformation rates

and MARRUCCI (1974), WALTERS (1975), BIRD et al. (1977), MIDDLEMAN (1977), LAUN (1978), SCHOWALTER (1978), and EBERT (1980), to name just a few.

Inserting the constitutive equation (9.5a) of a NEWTON*ian fluid* into CAUCHY*'s equation of motion* (3.38) yield the NAVIER-STOKES *equations* for compressible fluids as illustrated in more detail in the following.

The partial derivative $\sigma_{ji,j} \equiv \partial\sigma_{ji}/\partial x_j$ of the constitutive equation (9.5a) can be expressed in the form

$$\frac{\partial \sigma_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} + \xi\frac{\partial d_{kk}}{\partial x_j}\delta_{ij} + 2\eta\frac{\partial d_{ji}}{\partial x_j}$$

or, by utilizing the *substitution rule* $A_j\delta_{ij} = A_i$, as

$$\frac{\partial \sigma_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \xi\frac{\partial d_{kk}}{\partial x_i} + 2\eta\frac{\partial d_{ji}}{\partial x_j} \ . \tag{9.31a}$$

Inserting the partial derivative of the rate-of-deformation tensor,

$$d_{ji,j} = \frac{1}{2}\left(v_{j,ij} + v_{i,jj}\right) \quad \text{and} \quad d_{kk,i} = v_{k,ki} \equiv v_{j,ij} \ ,$$

we arrive at

$$\sigma_{ji,j} = -p_{,i} + (\xi + \eta)v_{k,ki} + \eta v_{i,jj} \; . \tag{9.31b}$$

With this result we finally arrive from CAUCHY's equation of motion (3.38) at the NAVIER-STOKES-*equations* for *compressible fluids*

$$\boxed{-p_{,i} + (\xi + \eta)v_{k,ki} + \eta\, v_{i,jj} + f_i = \rho\, \dot{v}_i} \tag{9.32a}$$

or in symbolic notation

$$\boxed{-\operatorname{grad} p + (\xi + \eta) \operatorname{grad} \operatorname{div} \boldsymbol{v} + \eta\, \Delta\boldsymbol{v} + \boldsymbol{f} = \rho\, \dot{\boldsymbol{v}}} \; . \tag{9.32b}$$

Assuming the STOKES *condition* (9.14), the factor $(\xi + \eta)$ in (9.32a,b) can be substituted by $\eta/3$.

In the special case of *incompressibility* the NAVIER-STOKES-*equations* (9.32a,b) governing the motion of *viscous fluids* take the following forms:

$$\boxed{-p_{,i} + \eta\, v_{i,jj} + f_i = \rho\, \dot{v}_i} \tag{9.33a}$$

and

$$\boxed{-\operatorname{grad} p + \eta\, \Delta\boldsymbol{v} + \boldsymbol{f} = \rho\, \dot{\boldsymbol{v}}} \; . \tag{9.33b}$$

The mechanical interpretation of each term in (9.33a,b) can be obtained as follows. The first term on the left-hand side represents the pressure gradient, the second one expresses the viscous frictional force, and the third term represents the body force. Taking into account the operator (3.5), the right-hand side of (9.33a,b) can be split into two parts,

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) \quad \text{symbolic} \quad \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) \; , \tag{9.34a,b}$$

where the first term represents the *inertia force* arising because of the *local rate*, while the second one characterizes the *convective rate* of change of *linear momentum*. Note, all terms listed above are computed per unit volume of the fluid and are acting on each fluid particle. Thus, the NAVIER-STOKES-*equations* (9.33a,b) or (9.32a,b) state that the pressure gradient force, the viscous force, the body force, and the inertia force acting on a fluid particle are in balance.

9.2 Nonlinear Viscous Fluids

For *non-*NEWTON*ian fluids* we first assume a constitutive equation of the form

$$\sigma_{ij} = \sigma_{ij}(L_{pq}, \rho, T) \ , \tag{9.35}$$

where $L_{ij} := \partial v_i / \partial x_j \equiv v_{i,j}$ are the cartesian components of the velocity-gradient tensor $\boldsymbol{L}$. This tensor can be split, according to

$$L_{ij} = d_{ij} + w_{ij} \ ,$$

into the symmetric *rate-of-deformation tensor* $\boldsymbol{d}$ and the skew-symmetric *spin* or *vorticity tensor* $\boldsymbol{w}$.

Constitutive equations must be invariant under changes of frame of reference, i.e., two observers, even if in relative motion with respect to each other, observe the same stress in a loaded material. The *principle of material frame-indifference* is also called the *principle of material objectivity*. As has been pointed out in more detail by BETTEN (2001a), the spin tensor is **not** objective, while the rate-of-deformation tensor is an objective tensor. This can be proved in the following way.

Let us consider a rigid-body motion, which can be split into a time dependent *rotation*, characterized by the orthogonal tensor $\boldsymbol{Q}(t)$, and into *translation*, characterized by the time dependent vector $\boldsymbol{c}(t)$, so that this motion is described by the transformation

$$\bar{x}_i(a_p, t) = Q_{ij}(t) x_j(a_p, t) + c_i(t) \ . \tag{9.36}$$

Hence, by differentiating with respect to time t, we arrive at the result

$$\bar{v}_i \equiv \dot{\bar{x}}_i = Q_{ip} v_p + \dot{Q}_{ip} x_p + \dot{c}_i \neq Q_{ip} v_p \ , \tag{9.37}$$

from which we read that the transformation law of a vector, $\bar{v}_i = Q_{ip} v_p$, is not satisfied, i.e., the velocity vector is **not** objective.

With the result (9.37) we first find for the velocity-gradient tensor $\boldsymbol{L}$

$$\left.\begin{aligned} \bar{L}_{ij} &\equiv \partial \bar{v}_i / \partial \bar{x}_j = (\partial \bar{v}_i / \partial x_q)\,(\partial x_q / \partial \bar{x}_j) \\ \bar{L}_{ij} &= \left(Q_{ip} v_{p,q} + \dot{Q}_{iq}\right)(\partial x_q / \partial \bar{x}_j) \ . \end{aligned}\right\} \tag{9.38}$$

By transvection with Q_{ik} and considering the orthogonal relation

$$Q_{ik} Q_{ij} = \delta_{kj} \ ,$$

we find from the motion (9.36) its inversion:

$$x_k = Q_{ik}(\bar{x}_i - c_i) \quad \text{or} \quad x_i = Q_{ji}(\bar{x}_j - c_j) \tag{9.39}$$

hence

$$\partial x_q / \partial \bar{x}_j = Q_{jq} \ , \tag{9.40}$$

so that (9.38) becomes

$$\bar{L}_{ij} \equiv \bar{v}_{i,j} = Q_{ip}Q_{jq}v_{p,q} + \dot{Q}_{iq}Q_{jq} \neq Q_{ip}Q_{jq}v_{p,q} \ , \tag{9.41}$$

i.e., the *velocity gradient tensor* $\boldsymbol{L}$ in (9.35) is **not** objective.

For the *rate-of-deformation tensor* $\boldsymbol{d}$, we find:

$$\begin{aligned} \bar{d}_{ij} &= (\partial \bar{v}_i / \partial \bar{x}_j + \partial \bar{v}_j / \partial \bar{x}_i) / \, 2 \\ &= Q_{ip}Q_{jq}d_{pq} + \left(\dot{Q}_{iq}Q_{jq} + Q_{iq}\dot{Q}_{jq}\right) \Big/ \, 2 \ , \end{aligned} \tag{9.42}$$

where

$$\dot{Q}_{iq}Q_{jq} + Q_{iq}\dot{Q}_{jq} = (Q_{iq}Q_{jq})^{\cdot} \equiv \dot{\delta}_{ij} \equiv 0_{ij} \ , \tag{9.43}$$

so that the *objectivity* of $\boldsymbol{d}$ is proved:

$$\boxed{\bar{d}_{ij} = Q_{ip}Q_{jq}d_{pq}} \ . \tag{9.44}$$

Consequently, equation (9.35) must be modified according to

$$\sigma_{ij} = \sigma_{ij}(d_{pq}, \rho, T) \ . \tag{9.45}$$

Furthermore, because of the principle of material objectivity, the components of the stress tensor (σ_{ij}) must be independent of superposed rigid-body motion, so that the requirement

$$\sigma_{ij} \left(Q_{pr}Q_{qs}d_{rs}, \rho, T \right) = Q_{ip}Q_{jq}\sigma_{pq} \tag{9.46}$$

is satisfied, where Q_{ij} are the cartesian components of an orthogonal tensor. A tensor-valued function with the property (9.46) is called an *isotropic tensor function* of the argumenttensor $\boldsymbol{d}$. The most general tensor polynomial function which fulfills (9.46) is of the form

$$\boxed{\sigma_{ij} = -p\delta_{ij} + \alpha d_{ij} + \beta d_{ij}^{(2)}} \ , \tag{9.47}$$

where p, α, and β are functions of ρ, T and the three irreducible invariants of the argumenttensor $\boldsymbol{d}$. The representation (9.47) is complete because of the

HAMILTON-CAYLEY *theorem*, which states that a tensor satisfies its own characteristic equation (BETTEN, 1987c).

Materials which obey the constitutive equation (9.47) are called REINER-RIVLIN *fluids* . They belong to the class of the *non*-NEWTON*ian fluids*.

As an example, let us consider a simple shear flow characterized by the velocity field

$$\boldsymbol{v} = (\dot{\gamma} x_2, 0, 0)^T \ , \tag{9.48}$$

for which the nonvanishing components of the rate-of-deformation tensor $\boldsymbol{d}$ are given by $d_{12} = d_{21} = \dot{\gamma}/2$, while the square $\boldsymbol{d}^2$ has the diagonal form with components $d_{11}^{(2)} = d_{22}^{(2)} = \dot{\gamma}^2/4$ and $d_{33}^{(2)} = 0$. The invariants are $I_1 = I_3 = 0$ and $I_2 = \operatorname{tr} \boldsymbol{d}^2 = \dot{\gamma}^2/2$. With these values we calculate from (9.47) the following nonvanishing stress components:

$$\sigma_{12} = \alpha \left(\dot{\gamma}^2\right) \dot{\gamma}/2 \equiv \eta \left(\dot{\gamma}^2\right) \dot{\gamma} \ , \tag{9.49a}$$

$$\sigma_{11} = \sigma_{22} = -p + \beta \left(\dot{\gamma}^2\right) \dot{\gamma}^2/4 \ , \qquad \sigma_{33} = -p \ . \tag{9.49b,c}$$

We see, in contrast to a NEWTONian fluid, the shear viscosity in (9.49a) is an even function of the shear rate $\dot{\gamma}$, i.e., it is a function of the square $\dot{\gamma}^2$.

From (9.41) and (9.44) we read that the velocity gradient tensor $\boldsymbol{L}$ is **not** objective, while the rate-of-deformation tensor $\boldsymbol{d}$ fulfills the requirement of *material objectivity*, for instance. Further examples are discussed in the following.

For the *spin* or *vorticity tensor* $\boldsymbol{w}$, which is the skew-symmetric part of the velocity gradient tensor $\boldsymbol{L}$, we obtain

$$\bar{w}_{ij} = Q_{ip} Q_{jq} w_{pq} + \dot{Q}_{iq} Q_{jq} \neq Q_{ip} Q_{jq} w_{pq} \ , \tag{9.50}$$

i.e., the spin tensor is **not** objective. In arriving at the result (9.50) we have taken into consideration the relation (9.43).

Because of (3.29) the CAUCHY stress tensor is objective, i.e.,

$$\bar{\sigma}_{ij} = Q_{ip} Q_{jq} \sigma_{pq} \ . \tag{9.51}$$

Thus we find

$$\dot{\bar{\sigma}}_{ij} = Q_{ip} Q_{jq} \dot{\sigma}_{pq} + \left(\dot{Q}_{ip} Q_{jq} + Q_{ip} \dot{Q}_{jq}\right) \sigma_{pq} \neq Q_{ip} Q_{jq} \dot{\sigma}_{pq} \ , \tag{9.52}$$

hence, the material time derivative of CAUCHY's stress tensor is **not** objective. Whereas the JAUMANN *stress rate*

$$\overset{\circ}{\sigma}_{ij} = \dot{\sigma}_{ij} - w_{ik} \sigma_{kj} + \sigma_{ik} w_{kj} \tag{9.53}$$

fulfills the requirement of *objectivity*, since we arrive from

$$\overset{\circ}{\bar{\sigma}}_{ij} = \dot{\bar{\sigma}}_{ij} - \bar{w}_{ik}\bar{\sigma}_{kj} + \bar{\sigma}_{ik}\bar{w}_{kj} \tag{9.54}$$

at the relation

$$\overset{\circ}{\bar{\sigma}}_{ij} = Q_{ip}Q_{jq}\overset{\circ}{\sigma}_{pq} + \left(\dot{Q}_{ip}Q_{jq} + Q_{ip}Q_{jr}Q_{kq}\dot{Q}_{kr}\right)\sigma_{pq} \ , \tag{9.55}$$

where the second term on the right-hand side is equal to zero because of the orthogonal relation $Q_{kq}Q_{kr} = \delta_{qr}$ and (9.43), hence

$$\overset{\circ}{\bar{\sigma}}_{ij} = Q_{ip}Q_{jq}\overset{\circ}{\sigma}_{pq} \ . \tag{9.56}$$

The *convective stress rate*

$$\overset{\vartriangle}{\sigma}_{ij} = \overset{\circ}{\sigma}_{ij} + d_{ik}\sigma_{kj} + \sigma_{ik}d_{kj} = \dot{\sigma}_{ij} + \sigma_{ik}L_{kj} + L_{ki}\sigma_{kj} \tag{9.57}$$

is obtained by adding the objective expression $d_{ik}\sigma_{kj} + \sigma_{ik}d_{kj}$ to the JAUMANN stress rate. Thus, the *convective stress rate* is *objective*.

By analogy of (3.8a), we define a *deformation gradient* according to $\bar{F}_{ij} := \partial\bar{x}_i/\partial a_j$ and arrive by differentiation of (9.36) and application of the chain rule at the following result

$$\bar{F}_{ij} = \frac{\partial\bar{x}_i}{\partial a_j} = \frac{\partial\bar{x}_i}{\partial x_p}\frac{\partial x_p}{\partial a_j} = Q_{ip}F_{pj} \ , \tag{9.58}$$

from which we can follow that the *deformation gradient* does **not** fulfill the requirement of *objectivity* (BETTEN, 2001a).

The LAGRANGE *strain tensor* (3.14) is defined as

$$\lambda_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) \ . \tag{9.59}$$

Considering a rigid-body motion, we have the following relations

$$\left.\begin{aligned} \bar{\lambda}_{ij} &= \frac{1}{2}\left(\bar{F}_{ki}\bar{F}_{kj} - \bar{\delta}_{ij}\right) \\ \bar{F}_{ki} &= Q_{kp}F_{pi} \\ \bar{F}_{kj} &= Q_{kq}F_{qj} \\ \bar{\delta}_{ij} &= Q_{ip}Q_{jq}\delta_{pq} \end{aligned}\right\} \quad \Rightarrow \quad \bar{\lambda}_{ij} = \frac{1}{2}\left(Q_{kp}Q_{kq}F_{pi}F_{qj} - Q_{ip}Q_{jq}\delta_{pq}\right) \ . \tag{9.60}$$

Inserting the orthogonal relation $Q_{kp}Q_{kq} = \delta_{pq}$ and applying the substitution rule, one obtains from (9.60) the result

$$\bar{\lambda}_{ij} = \frac{1}{2}\left(\delta_{pq}F_{pi}F_{qj} - \delta_{ij}\right) = \frac{1}{2}\left(F_{ri}F_{rj} - \delta_{ij}\right) \equiv \lambda_{ij} \; , \tag{9.61}$$

which states that the components of the LAGRANGE strain tensor are **not** effected by a superimposed rigid-body motion, i.e., the principle of material objectivity is fulfilled.

The material time derivative of the LAGRANGE strain tensor (9.59) can be expressed by

$$\dot{\boldsymbol{\lambda}} = \boldsymbol{F}^T \boldsymbol{d}\, \boldsymbol{F} \quad \text{or by} \quad \dot{\lambda}_{ij} = F^T_{ip} d_{pq} F_{qj} = F_{pi}F_{qj}d_{pq} \; . \tag{9.62a,b}$$

Thus, a superimposed rigid-body motion yields

$$\dot{\bar{\lambda}}_{ij} = \bar{F}_{pj}\bar{F}_{qi}\bar{d}_{pq} \; . \tag{9.63}$$

Taking (9.44) and (9.58) into account, equation (9.63) can be written in the form

$$\dot{\bar{\lambda}}_{ij} = Q_{pk}F_{ki}Q_{ql}F_{lj}Q_{pr}Q_{qs}d_{rs} \; . \tag{9.64}$$

Since $\boldsymbol{Q}$ is an orthogonal tensor, we arrive from (9.64) at the relation

$$\dot{\bar{\lambda}}_{ij} = \delta_{kr}\delta_{ls}F_{ki}F_{lj}d_{rs} = F_{ri}F_{sj}d_{rs} \; . \tag{9.65}$$

Comparing (9.65) with (9.62b), we finally obtain the result

$$\boxed{\dot{\bar{\lambda}}_{ij} \equiv \dot{\lambda}_{ij}} \; , \tag{9.66}$$

stating that the *material time derivative of the* LAGRANGE *strain tensor* is objective.

The EULER *strain tensor* in (3.19) is defined as

$$\eta_{ip} = \frac{1}{2}\left(\delta_{ip} - F^{(-1)}_{ki}F^{(-1)}_{kp}\right) \; , \tag{9.67}$$

hence

$$\bar{\eta}_{ip} = \frac{1}{2}\left(\bar{\delta}_{ip} - \bar{F}^{(-1)}_{ki}\bar{F}^{(-1)}_{kp}\right) \; , \tag{9.68}$$

where

$$\bar{F}^{(-1)}_{ij} := \partial a_i/\partial \bar{x}_j = \left(\partial a_i/\partial x_p\right)\left(\partial x_p/\partial \bar{x}_j\right) = F^{(-1)}_{ip}\left(\partial x_p/\partial \bar{x}_j\right) \; . \tag{9.69}$$

From (9.36) we read

$$x_i = Q^{(-1)}_{ij}\left(\bar{x}_j - c_j\right) = Q_{ji}\left(\bar{x}_j - c_j\right) \quad \Rightarrow \quad \partial x_p/\partial \bar{x}_j = Q_{jp} = Q^{(-1)}_{pj} \; ,$$

so that (9.69) reduces to the relation

$$\bar{F}_{ij}^{(-1)} = F_{ip}^{(-1)} Q_{pj}^{(-1)} , \tag{9.70}$$

which is the inverse form of (9.58). Note that the inverse of a matrix product π is the matrix product formed by writing down the inverses of the factors of π in reverse order, for instance

$$(\boldsymbol{ABC}\ldots\boldsymbol{Z})^{-1} = \boldsymbol{Z}^{-1}\ldots\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} . \tag{9.71}$$

This rule can also be applied to the transpose of a matrix product.

Inserting the inverse (9.70) into (9.68), we obtain

$$\bar{\eta}_{ip} = \frac{1}{2}\left(\bar{\delta}_{ip} - F_{kr}^{(-1)} Q_{ri}^{(-1)} F_{ks}^{(-1)} Q_{sp}^{(-1)}\right) . \tag{9.72}$$

Because $\boldsymbol{Q}$ is an *orthogonal tensor*, i.e., the inverse of $\boldsymbol{Q}$ is identical to the transpose of $\boldsymbol{Q}$, and since

$$\bar{\delta}_{ip} = Q_{ir} Q_{ps} \delta_{rs} \tag{9.73}$$

the relation (9.72) reduces to

$$\bar{\eta}_{ip} = \frac{1}{2} Q_{ir} Q_{ps}\left(\delta_{rs} - F_{kr}^{(-1)} F_{ks}^{(-1)}\right) . \tag{9.74}$$

Considering the definition (9.67), we can write (9.74) in the following form

$$\boxed{\bar{\eta}_{ip} = Q_{ir} Q_{ps} \eta_{rs}} , \tag{9.75}$$

showing that the EULER*ian strain tensor* is an *objective tensor*.

The material time derivative of the EULERian strain tensor (9.67) can be expressed by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{d} - \boldsymbol{\eta}\boldsymbol{L} - \boldsymbol{L}^T\boldsymbol{\eta} \quad \text{or by} \quad \dot{\eta}_{ij} = d_{ij} - \eta_{ip} L_{pj} - \eta_{jp} L_{pi} . \tag{9.76a,b}$$

Thus, a superimposed rigid-body motion yields

$$\dot{\bar{\eta}}_{ij} = \bar{d}_{ij} - \bar{\eta}_{ip} \bar{L}_{pj} - \bar{\eta}_{jp} \bar{L}_{pi} . \tag{9.77}$$

Inserting (9.41) into (9.77) and considering (9.44) and (9.75), we obtain the result

$$\boxed{\dot{\bar{\eta}}_{ij} = Q_{ik} Q_{jl} \dot{\eta}_{kl} - \left(Q_{ik} Q_{jl} + Q_{il} Q_{jk}\right) Q_{ps} \dot{Q}_{pl} \eta_{ks}} , \tag{9.78}$$

stating that the *material time derivative of the* EULER*ian strain tensor* is **not** an objective tensor.

The OLDROYD *time derivative* of a symmetric second rank tensor $\boldsymbol{T}$ is defined according to

$$\overset{\triangledown}{T}_{ij} := \dot{T}_{ij} + T_{ip}L_{pj} + T_{jp}L_{pi} \ . \tag{9.79}$$

Applying this derivative to the EULERian strain tensor, $T_{ij} \equiv \eta_{ij}$, and taking into account the relation (9.76b), we immediately obtain the identity

$$\boxed{\overset{\triangledown}{\eta}_{ij} \equiv d_{ij}} \ , \tag{9.80}$$

i.e., the OLDROYD *time derivative of the* EULER*ian strain tensor* can be interpreted as the rate-of-deformation tensor. Hence, the requirement of material objectivity is fulfilled.

The *second* PIOLA-KIRCHHOFF *stress tensor* (3.42) is defined as

$$\tilde{T}_{ij} = \frac{\rho_0}{\rho} F_{ip}^{(-1)} F_{jq}^{(-1)} \sigma_{pq} = \tilde{T}_{ji} \ , \tag{9.81}$$

hence

$$\bar{\tilde{T}}_{ij} = \frac{\rho_0}{\rho} \bar{F}_{ip}^{(-1)} \bar{F}_{jq}^{(-1)} \bar{\sigma}_{pq} \ . \tag{9.82}$$

Considering (9.51) and (9.69), i.e.,

$$\bar{\sigma}_{pq} = Q_{ps} Q_{qt} \sigma_{st} \quad \text{and} \quad \bar{F}_{ip}^{(-1)} = F_{ir}^{(-1)} Q_{rp}^{(-1)} = F_{ir}^{(-1)} Q_{pr} \ ,$$

respectively, we arrive at the relations

$$\bar{\tilde{T}}_{ij} = \frac{\rho_0}{\rho} F_{ir}^{(-1)} F_{jk}^{(-1)} \underbrace{Q_{pr} Q_{ps}}_{\delta_{rs}} \underbrace{Q_{qk} Q_{qt}}_{\delta_{kt}} \sigma_{st}$$

$$\bar{\tilde{T}}_{ij} = \frac{\rho_0}{\rho} \bar{F}_{ir}^{(-1)} \bar{F}_{jk}^{(-1)} \sigma_{rk} \ . \tag{9.83}$$

Comparing (9.83) with the definition (9.81), we finally obtain the result

$$\boxed{\bar{\tilde{T}}_{ij} \equiv \tilde{T}_{ij}} \ , \tag{9.84}$$

stating that, analogous to (9.66), the *second* PIOLA-KIRCHHOFF *stress tensor* is objective.

The *first* PIOLA-KIRCHHOFF *stress tensor*

$$T_{ij} = \frac{\rho_0}{\rho} F_{ik}^{(-1)} \sigma_{kj} \neq T_{ji} \tag{9.85}$$

can be expressed by the second one (9.81) according to

$$\tilde{T}_{ij} = T_{iq} F_{jq}^{(-1)} \quad \Rightarrow \quad T_{ij} = \tilde{T}_{ik} F_{jk} \ , \tag{9.86}$$

hence

$$\left.\begin{aligned} \bar{T}_{ij} &= \bar{\tilde{T}}_{ik} \bar{F}_{jk} \\ \bar{F}_{jk} &= Q_{jr} F_{rk} \\ \bar{\tilde{T}}_{ik} &= \tilde{T}_{ik} \end{aligned}\right\} \quad \Rightarrow \quad \bar{T}_{ij} = \underbrace{\tilde{T}_{ik} F_{rk}}_{T_{ir}} Q_{jr} \ ,$$

$$\boxed{\bar{T}_{ij} = T_{ir} Q_{jr}} \ , \tag{9.87}$$

i.e., the *first* PIOLA-KIRCHHOFF *stress tensor* is **not** objective in contrast to the second one according to (9.84).

The above discussed examples illustrate that the components of objective tensors defined in the reference configuration (*material description*) do not change, if a rigid-body motion is superimposed, for instance, the components of the LAGRANGE strain tensor:

$$\boxed{\bar{\lambda}_{ij} \equiv \lambda_{ij}} \ . \tag{9.61}$$

Whereas the components of objective tensors defined in the actual configuration (*spatial description*) change according to the transformation law of the tensor, if a rigid-body motion is superimposed, for instance, the components of the EULERian strain tensor:

$$\boxed{\bar{\eta}_{ij} = Q_{ip} Q_{jr} \eta_{pr}} \ . \tag{9.75}$$

The above discussed examples are listed in Table 9.1.

Table 9.1 Objective and non-objective tensors

tensor	material objectivity	
deformation gradient	**not** fulfilled	(9.58)
velocity gradient tensor	**not** fulfilled	(9.41)
rate-of-deformation tensor	fulfilled	(9.44)
spin tensor	**not** fulfilled	(9.50)
CAUCHY stress tensor	fulfilled	(9.51)
material time derivative of CAUCHY's stress tensor	**not** fulfilled	(9.52)
JAUMANN stress rate	fulfilled	(9.56)
convective stress rate	fulfilled	(9.57)
LAGRANGE strain tensor	fulfilled	(9.61)
material time derivative of the LAGRANGE strain tensor	fulfilled	(9.66)
EULERian strain tensor	fulfilled	(9.75)
material time derivative of the EULERian strain tensor	**not** fulfilled	(9.78)
OLDROYD time derivative of the EULERian strain tensor	fulfilled	(9.80)
first PIOLA-KIRCHHOFF stress tensor	**not** fulfilled	(9.87)
second PIOLA-KIRCHHOFF stress tensor	fulfilled	(9.84)