8 Tensorial Generalization of Uniaxial Creep Laws to Multiaxial States of Stress

In this chapter a method is developed in order to find tensorial constitutive and evolutional equations based upon empirical uniaxial constitutive laws found in experimental investigations. For engineering applications it is very important to generalize *uniaxial* relations to *multiaxial* states of stress. This can be achieved by applying *interpolation methods for tensor functions* , as pointed out in detail in this chapter. It is illustrated that the scalar coefficients in tensorial constitutive equations can be expressed as functions of the irreducible invariants of the argument tensors and of the empirical constitutive laws found in uniaxial tests.

Some examples should be discussed. For instance, the NORTON-BAILEY creep law and a uniaxial damage relation are generalized to *tensorial constitutive equations*.

8.1 Polynomial Representation of Tensor Functions

Let

$$Y_{ij} = f_{ij}(\boldsymbol{X}) = \varphi_0 \delta_{ij} + \varphi_1 X_{ij} + \varphi_2 X_{ij}^{(2)} \tag{8.1}$$

be an isotropic tensor function where φ_0, φ_1, φ_2 are scalar-valued functions of the integrity basis, the elements of which are the irreducible invariants of the argument tensor $\boldsymbol{X}$. Furthermore, they depend on experimental data.

First, it is possible to express the scalar functions through the principal values $X_I, \ldots, X_{III}$ and $Y_I, \ldots, Y_{III}$ if we solve the system of linear equations

$$\left.\begin{aligned} Y_I &= \varphi_0 + \varphi_1 X_I + \varphi_2 X_I^2 \,, \\ Y_{II} &= \varphi_0 + \varphi_1 X_{II} + \varphi_2 X_{II}^2 \,, \\ Y_{III} &= \varphi_0 + \varphi_1 X_{III} + \varphi_2 X_{III}^2 \,. \end{aligned}\right\} \tag{8.2}$$

The solution can be written in the form

$$\varphi_0 = \sum_{\alpha=I}^{III} P_\alpha X_{(\alpha+I)} X_{(\alpha+II)} Y_{(\alpha)} \,, \tag{8.3a}$$

$$\varphi_1 = \sum_{\alpha=I}^{III} P_\alpha \left(X_{(\alpha+I)} + X_{(\alpha+II)} \right) Y_{(\alpha)} \,, \tag{8.3b}$$

$$\varphi_2 = \sum_{\alpha=I}^{III} P_\alpha Y_{(\alpha)} \,, \tag{8.3c}$$

where the abbreviation

$$P_\alpha := \prod_{\substack{\beta=I \\ \beta\neq\alpha}}^{III} 1 / (X_\alpha - X_\beta) \tag{8.4}$$

is introduced. A similar representation was used by SOBOTKA (1984) based upon the SYLVESTER *theorem* (SEDOV, 1966).

Because of the products P_α, the expressions (8.3a-c) can only be used if all principal values are different. Therefore, in the following an *interpolation method* is used in order to determine the scalar coefficients, even if two principal values coincide.

8.2 Interpolation Methods for Tensor Functions

In extending the LAGRANGE interpolation method to a tensor-valued function, we consider the *principal values* of the argument tensor as *interpolating points* and find the tensorial representation

$$Y_{ij} = f_{ij}(\boldsymbol{X}) = \sum_{\alpha=1}^{III} {}^\alpha L_{ij} Y_\alpha + R_{ij}(\boldsymbol{X}) \tag{8.5}$$

with the tensor polynomials

$${}^\alpha L_{ij} := P_\alpha \left(X_{ik} - X_{(\alpha+I)} \delta_{ik} \right) \left(X_{kj} - X_{(\alpha+III)} \delta_{kj} \right) \,. \tag{8.6}$$

Due to the HAMILTON-CAYLEY *theorem*, the tensor-valued remainder term R_{ij} in (8.5) is always equal to the zero tensor (BETTEN 1984; 1987b). As an alternate approach, we find, by extending the NEWTON formula, the tensorial representation

$$\begin{aligned} Y_{ij} &= a_0 \delta_{ij} + a_1 \left(X_{ij} - X_I \delta_{ij} \right) \\ &\quad + a_2 \left(X_{ik} - X_I \delta_{ik} \right) \left(X_{kj} - X_{II} \delta_{kj} \right) \,, \end{aligned} \tag{8.7}$$

Further terms in (8.7) are not possible because of the HAMILTON-CAYLEY *theorem*. The coefficients in (8.7) can be found by inserting the principal values:

$$a_0 = Y_I \; , \qquad a_1 = (Y_I - Y_{II}) / (X_I - X_{II}) \; , \tag{8.8a,b}$$

$$a_2 = [a_1 - (Y_{III} - Y_I) / (X_{III} - X_I)] / (X_{II} - X_{III}) \; . \tag{8.8c}$$

The interpolation formula (8.7) can be written as an isotropic tensor function (8.1) if we define

$$\varphi_0 \equiv a_0 - a_1 X_I + a_2 X_I X_{II} \; , \tag{8.9a}$$

$$\varphi_I \equiv a_1 - a_2 (X_I + X_{II}) \; , \qquad \varphi_2 \equiv a_2 \; . \tag{8.9b,c}$$

In the case of *coincident points*, we need the derivatives of the tensor function (8.1):

$$f'_{ij} := \partial Y_{ip} / \partial X_{pj} = \varphi_1 \delta_{ij} + 2\varphi_2 X_{ij} \; , \tag{8.10a}$$

$$f''_{ij} := \partial f'_{iq} / X_{qj} = 2\varphi_2 \delta_{ij} \; . \tag{8.10b}$$

For example, in the case of $X_I \neq X_{II} = X_{III}$, we find from (8.7) and (8.10a) the coefficients

$$a_0 = Y_I \; , \qquad a_1 = (Y_I - Y_{II}) / (X_I - X_{II}) \; , \tag{8.11a,b}$$

$$a_2 = \left(a_1 - f'_{II}\right) / (X_I - X_{II}) \; , \tag{8.11c}$$

if we substitute

$$Y_I = f_{II} (X_{11} \equiv X_I) \; ,$$
$$Y_{II} = f_{22} (X_{22} \equiv X_{II}) \; ,$$
$$f'_{22} (X_{22} \equiv X_{II}) \equiv f'_{II} \; .$$

Finally, if all principal values coincide, we calculate

$$a_0 = f_I \; , \qquad a_1 = f'_I \; , \qquad a_2 = f''_I / 2 \; . \tag{8.12a,b,c}$$

However, in this special case the argument tensor is a spherical one, $X_{ij} = X_I \delta_{ij}$, and therefore the formula (8.7) reduces to the trivial result: $Y_{ij} = f_{ij} = f_I \delta_{ij}$. Note that the interpolation formula for a scalar function $y = f(x)$ approaches the TAYLOR expansion for $f(x)$ at x_0 if we make x_α, $\alpha = 1, 2, \ldots, n$, coincide at x_0. An interpolation method for tensor functions with two argument tensors can be developed in a similar way (BETTEN 1987b; 1987c).

The interpolation method for tensor functions is a very useful and powerful tool. Besides many applications in tensor algebra or tensor analysis discussed by BETTEN (1987b; 1987c), engineering applications are also very important.

In the theory of finite deformation the tensorial HENCKY measure of strain and strain rate plays a central role see FITZGERALD (1980) and BETTEN (1987b; 2001a) because it can be decomposed into a sum of an isochoric distorsion and a volume change. The problem to represent the logarithmic function

$$\boldsymbol{Y} = \ln \boldsymbol{X} \quad \text{or} \quad Y_{ij} = \{\ln \boldsymbol{X}\}_{ij} \tag{8.13}$$

as an isotropic tensor function (8.1) is solved by determining the scalar functions $\varphi_0, \varphi_1, \varphi_2$. This can be done by using the interpolation method described before by BETTEN (1987b; 2001a).

Other examples are $\boldsymbol{Y} = \exp \boldsymbol{X}$ or $\boldsymbol{Y} = \sin \boldsymbol{X}$ etc., which can be treated in the same way. These functions play a central role, for instance, in problems concerning *vibro creep* (JAKOWLUK, 1993).

8.3 Tensoral Generalization of NORTON-BAILEY's Creep Law

The following example is concerned with the generalization of NORTON-BAILEY*'s power law* (Section 4.2)

$$d/d_0 = (\sigma/\sigma_0)^n \quad \text{or} \quad d = K\sigma^n \tag{8.14a,b}$$

to multi-axial states of stress where d is the strain rate, σ the uniaxial true stress, and d_0, σ_0, n, K are constants. To solve this problem, we use an isotropic tensor function

$$d_{ij} = f_{ij}(\boldsymbol{\sigma}) = \varphi_0^*\delta_{ij} + \varphi_1^*\sigma_{ij} + \varphi_2^*\sigma_{ij}^{(2)} \tag{8.15}$$

and determine the scalar coefficients $\varphi_0^*, \ldots, \varphi_2^*$ as functions of experimental data (K, n) in (8.14b) and of the integrity basis, the elements of which are the irreducible invariants of the CAUCHY stress tensor $\boldsymbol{\sigma}$.

Alternatively, we can represent the constitutive equation in the form

$$d_{ij} = f_{ij}\left(\boldsymbol{\sigma}'\right) = \varphi_0\delta_{ij} + \varphi_1\sigma_{ij}' + \varphi_2\sigma_{ij}'^{(2)} \,, \tag{8.16}$$

where $\sigma_{ij}' := \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ are the cartesian components of the *stress deviator* $\boldsymbol{\sigma'}$. For the special case of incompressible behavior $(d_{kk} \equiv 0)$, we find from (8.16) the condition

$$3\varphi_0 + \varphi_2 \sigma_{kk}^{\prime(2)} = 0 \quad \Rightarrow \quad \varphi_0 = -2\varphi_2 J_2' / 3 \tag{8.17}$$

with the quadratic invariant $J_2' \equiv \sigma_{ik}' \sigma_{ki}'/2$ of the stress deviator, so that the constitutive equation (8.16) is reduced to the simple form

$$d_{ij} = \varphi_1 \sigma_{ij}' + \varphi_2 \sigma_{ij}'' \tag{8.18}$$

containing the traceless tensors

$$\sigma_{ij}' \equiv \partial J_2' / \partial \sigma_{ij} \quad \text{and} \quad \sigma_{ij}'' \equiv \partial J_3' / \partial \sigma_{ij} \tag{8.19a,b}$$

with the cubic invariant $J_3' \equiv \sigma_{ij}' \sigma_{jk}' \sigma_{ki}'/3$ of the stress deviator. The uniaxial equivalent state of stress (index V) is characterized through the tensor variables

$$(\sigma_{ij})_V = \operatorname{diag}\{\sigma,\, 0,\, 0\} \ , \tag{8.20a}$$

$$\left(\sigma_{ij}'\right)_V = \operatorname{diag}\{2\sigma/3,\, -\sigma/3,\, -\sigma/3\} \ , \tag{8.20b}$$

$$(d_{ij})_V = \operatorname{diag}\{d,\, -\nu d,\, \nu d\} \ , \tag{8.20c}$$

where ν is the transverse contraction ratio.

In the following the diagonal elements in (8.20) are considered as interpolating points where two points coincide. Since the two coincident points in (8.20a) are zero, it may be more convenient to determine the coefficients $\varphi_0, \ldots, \varphi_2$ in the constitutive equation (8.16) instead of (8.15). Thus, we use the uniaxial creep law

$$d = (3/2)^n \, K (\sigma')^n \tag{8.21}$$

instead of (8.14b). Because of (8.20b), i.e. $X_{II} = X_{III} \equiv -\sigma/3$, and (8.21), we find from (8.11a) the coefficient

$$a_0 = Y_1 \equiv (3/2)^n \, K (\sigma')^n = K\sigma^n \ . \tag{8.22a}$$

Furthermore, because of (8.20b), (8.22a), and $Y_{II} = -\nu d = -\nu K \sigma^n$, we find from (8.11b) the coefficient

$$a_1 = (1+\nu)\, K\sigma^{n-1} \ . \tag{8.22b}$$

The derivative f_{II}' at the coincident points $X_{II} = X_{III}$ can be determined in the following way. From (8.21) we derive

$$f' \equiv \partial d / \partial \sigma' = n\, (3/2)^n \left(\sigma'\right)^{n-1} = nd/\sigma' \ , \tag{8.23a}$$

$$f_{II}' = n d_{II} \, / \sigma_{II}' \ . \tag{8.23b}$$

From (8.20a,b) we read $\sigma' = -\sigma/3$ and $d_{II} = -\nu d_1 \equiv -\nu d = -\nu K\sigma^n$, so that (8.23b) can be written as

$$f'_{II} = 3\nu n K\sigma^{n-1} \; . \tag{8.23c}$$

Considering (8.20b) and (8.22b), we calculate from (8.11c) the coefficient

$$a_2 = (1 + \nu - 3\nu n)\, K\sigma^{n-2} \; . \tag{8.22c}$$

Inserting (8.22a,b,c) in (8.9a,b,c), we finally determine the scalar functions

$$\varphi_0 = \frac{1}{9}\,(1 - 8\nu + 6\nu n)\, K\sigma^n \; , \tag{8.24a}$$

$$\varphi_1 = \frac{2}{3}\left(1 + \nu + \tfrac{3}{2}\nu n\right) K\sigma^{n-1} \; , \tag{8.24b}$$

$$\varphi_2 = (1 + \nu - 3\nu n)\, K\sigma^{n-2} \; . \tag{8.24c}$$

Assuming the incompressibility (8.18) and neglecting tensorial nonlinearity ($\varphi_2 = 0 \Rightarrow a_2 = 0$, $\varphi_0 = 0$, and $\varphi_1 = a_1$) we find from (8.16) the simplified constitutive equation

$$d_{ij} = a_1\sigma'_{ij} \quad \text{or} \quad d_{ij} = \frac{3}{2}K\sigma^{n-1}\sigma'_{ij} \; , \tag{8.25a,b}$$

if we use (8.22b) with $\nu = 1/2$. The result (8.25b) is identical to a constitutive equation proposed by LECKIE and HAYHURST (1977). If we insert the MISES equivalent stress $\sigma = \sqrt{3J'_2}$ into (8.25b), we can find the constitutive equation

$$d_{ij} = \frac{3}{2}K\left(3J'_2\right)^{(n-1)/2}\sigma'_{ij} \tag{8.25c}$$

used by ODQUIST and HULT (1962).

The equivalent stress σ in (8.24a,b,c) can be determined as a function of the stress invariants if we use the *hypothesis of the equivalent dissipation rate*:

$$\dot{D} := \sigma_{ij}d_{ji} \overset{!}{=} \sigma d \; , \tag{8.26}$$

where $\dot{D}$ is called the *rate of dissipation* of creep energy. The result is

$$\sigma^3 + A\sigma^2 + B\sigma + C = 0 \; , \tag{8.27}$$

where the abbreviations

$$A \equiv -(1 - 8\nu + 6\nu n)\, J_1/9 \; , \tag{8.28a}$$

$$B \equiv -4\,(1 + \nu + 3\nu n/2)\; J'_2/3 \; , \tag{8.28b}$$

$$C \equiv -(1 + \nu - 3\nu n)\left(3J'_3 + 2J_1J'_2/3\right) \tag{8.28c}$$

have been used. Thus, the scalar coefficients (8.24a,b,c) are functions of the *irreducible invariants*

$$J_1 \equiv \sigma_{kk} \ , \qquad J_2' \equiv \sigma_{ik}'\sigma_{ki}'/\,2 \ , \qquad J_3' \equiv \sigma_{ij}'\sigma_{jk}'\sigma_{ki}'/\,3 \qquad \text{(8.29a,b,c)}$$

and of *experimental data* $(K,\, n,\, \nu)$:

$$\varphi_\alpha = \varphi_\alpha\left(J_1,\, J_2',\, J_3';\, K,\, n,\, \nu\right) \ , \qquad \alpha = 0,\, 1,\, 2\ . \qquad (8.30)$$

This statement is compatible with the representation theory of tensor-valued functions (4.80) in which the coefficients φ_α are scalar-valued functions of the *integrity basis* (8.29a,b,c).

In the case of incompressible behavior ($\nu = 1/2$), the first invariant J_1 has no influence. The cubic equation (8.27) then takes the reduced form

$$\sigma^3 + B^*\sigma^2 + C^* = 0 \qquad (8.27^*)$$

with the abbreviations

$$B^* \equiv -\,(2+n)\,J_2' \quad \text{and} \quad C^* \equiv \frac{9}{2}\,(n-1)\,J_3' \qquad (8.28^*\text{b,c})$$

depending on the irreducible invariants (8.29b,c) of the stress deviator.

Some authors (BROWN et al. 1986) are losing faith in NORTON-BAILEY*'s law* since they feel that their new θ *projection concept* provides a far more comprehensive description of creep behavior for design. In this new approach, normal creep curves are envisaged as the sum of a decaying primary and an ascending tertiary stage, i.e., the *secondary stage* is merely the period of ostensibly constant rate observed when the decay in the creep rate during the primary stage is offset by the gradual acceleration caused by tertiary processes. This concept neglects the secondary component and may be valid for some special materials, e.g. $\frac{1}{2}Cr\frac{1}{2}Mo\frac{1}{4}V$, as has been discussed in detail by BROWN et al.(1986). However, an extended secondary creep stage can be observed for many materials. Thus, in spite of the discussion by BROWN et al. (1986), it is very important that NORTON-BAILEY*'s law* be generalized to multi-axial states of stress. This can be achieved by applying a *tensorial interpolation method* as has been illustrated above.

8.4 Tensorial Generalization of a Creep Law including Damage

Involving the damage state in the tertiary creep stage (section 4.3.1) the uniaxial relation

$$d/d_0 = (\sigma/\sigma_0)^n D^m \quad \text{with} \quad D := 1/(1-\omega) \tag{8.31}$$

should be generalized to multi-axial states of stress where ω is the damage parameter (material deterioration) introduced by KACHANOV (1958) and also used by RABOTNOV (1969).

To generalize (8.31), we consider the tensor-valued function

$$d_{ij} = \begin{cases} f_{ij}(\boldsymbol{\sigma},\, \boldsymbol{D}) \\ \frac{1}{2} \sum\limits_{\nu,,\mu=0}^{2} \psi_{[\nu,\,\mu]} \left(\sigma_{ik}^{(\nu)} D_{kj}^{(\mu)} + D_{ik}^{(\mu)} \sigma_{kj}^{(\nu)} \right) \end{cases} , \tag{8.32}$$

where ν and μ are exponents of the CAUCHY stress tensor $\boldsymbol{\sigma}$ and the second-rank tensor $\boldsymbol{D}$ with the components

$$D_{ij} = (\delta_{ij} - \omega_{ij})^{(-1)}$$

given by the damage tensor $\boldsymbol{\omega}$.

Now, the main problem is to determine the scalar coefficients $\psi_{[\nu,\,\mu]}$ as functions of the integrity basis containing 10 *irreducible invariants* (BETTEN, 1987b; 1987c) and experimental data. To solve this problem, we suggest the following method which may be useful for practical applications as has been discussed by BETTEN, (1988b; 2001c) .

A representation with the same tensor generators as contained in the function (8.32) can be found by seperating the two variables $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ in the following way:

$$d_{ij} = f_{ij}(\boldsymbol{\sigma},\, \boldsymbol{D}) = \frac{1}{2} (X_{ik} Y_{kj} + Y_{ik} X_{kj}) \ , \tag{8.33}$$

where the isotropic tensor functions

$$\left.\begin{aligned} X_{ij} = X_{ij}(\boldsymbol{\sigma}) &= \varphi_0^* \delta_{ij} + \varphi_1^* \sigma_{ij} + \varphi_2^* \sigma_{ij}^{(2)} \\ \varphi_\nu^* = \varphi_\nu^*(\mathrm{tr}\, \boldsymbol{\sigma}^\lambda) &= \varphi_\nu^* \left(\sigma_I,\, \sigma_{II},\, \sigma_{III} \right) \end{aligned}\right\} , \tag{8.34}$$

$$\left.\begin{aligned} Y_{ij} = Y_{ij}(\boldsymbol{D}) &= \Phi_0 \delta_{ij} + \Phi_1 D_{ij} + \varphi_2 D_{ij}^{(2)} \\ \Phi_\mu = \Phi_\mu(\mathrm{tr}\, \boldsymbol{D}^\lambda) &= \Phi_\mu \left(D_I,\, D_{II},\, D_{III} \right) \end{aligned}\right\} \tag{8.35}$$

($\mu,\, \nu = 0,\, 1,\, 2$ and $\lambda = 1,\, 2,\, 3$) are used.

Thus, we find the representation (8.32) with the scalar coefficients

$$\psi_{[\nu,\,\mu]} = \varphi_\nu^* \Phi_\mu \ , \qquad \mu,\, \nu = 0,\, 1,\, 2 \ , \tag{8.36}$$

where the scalars φ_ν^* are determined by BETTEN (1986a; 1986b):

$$\varphi_0^* = \varphi_0 - J_1\varphi_1/3 + J_1^2\varphi_2/9 \,, \tag{8.37a}$$

$$\varphi_1^* = \varphi_1 - 2J_1\varphi_2/3 \,, \qquad \varphi_2^* \equiv \varphi_2 \,. \tag{8.37b,c}$$

The coefficients Φ_μ can be found by solving the following system of linear equations:

$$\left.\begin{aligned} \Phi_0 + D_I\Phi_1 \ + D_I^2\Phi_2 \ &= (D_I)^{m_I} \,, \\ \Phi_0 + D_{II}\Phi_1 \ + D_{II}^2\Phi_2 \ &= (D_{II})^{m_{II}} \,, \\ \Phi_0 + D_{III}\Phi_1 + D_{III}^2\Phi_2 &= (D_{III})^{m_{III}} \,. \end{aligned}\right\} \tag{8.38}$$

The exponents $m_I, \ldots, m_{III}$ in (8.38) are determined by using the creep law (8.31) in tests on specimens cut along the mutually perpendicular directions x_1, x_2, x_3.

Because of

$$D_{ij} := (\delta_{ij} - \omega_{ij})^{(-1)} \equiv \psi_{ij}^{(-1)} \quad \text{and} \quad \psi_{ij} = \mathrm{diag}\{\alpha,\, \beta,\, \gamma\}$$

according to (4.84) and (7.40a), respectively, the principal values in (8.38) can be expressed through

$$D_I = 1/\alpha \,, \qquad D_{II} \equiv 1/\beta \,, \qquad D_{III} \equiv 1/\gamma \,, \tag{8.39}$$

where the essential components $\alpha,\ \beta,\ \gamma$ are fractions that represent the net cross-sectional elements of CAUCHY's tetrahedron perpendicular to the coordinate axes (BETTEN, 1983a). In the case of two equal parameters, for instance $\alpha \neq \beta = \gamma$, the scalars $\Phi_\mu,\ \mu = 0,\ 1,\ 2$, in (8.38) can be determined by using the interpolation method described above in (8.1) to (8.12).

Instead of (8.34), we can use the isotropic tensor function

$$X_{ij} = X_{ij}\left(\boldsymbol{\sigma}'\right) = \varphi_0\delta_{ij} + \varphi_1\sigma'_{ij} + \varphi_2\sigma'^{(2)}_{ij} \tag{8.40}$$

and find the representation

$$d_{ij} = \frac{1}{2}\sum_{\nu,\,\mu=0}^{2} \varphi_\nu\Phi_\mu \left(\sigma'^{(\nu)}_{ik} D^{(\mu)}_{kj} + D^{(\mu)}_{ik}\sigma'^{(\nu)}_{kj}\right) \,, \tag{8.41}$$

where the scalar coefficients φ_ν are determined in the functions (8.24a,b,c) and the Φ_μ are taken from (8.38).

The scalar coefficients $\psi_{[\nu,\,\mu]} \equiv \varphi_\nu\Phi_\mu$ in the representation (8.41) must be functions of the integrity basis

$$\left.\begin{array}{l} J_1 \equiv \sigma_{kk}\,, \qquad J_2' \equiv \frac{\sigma_{ij}'\sigma_{ji}'}{2}\,, \qquad J_3' \equiv \frac{\sigma_{ij}'\sigma_{jk}'\sigma_{ki}'}{3}\,, \\ L_1 \equiv D_{kk}\,, \quad L_2 \equiv D_{kk}^{(2)}\,, \qquad L_3 \equiv D_{kk}^{(3)}\,, \\ \Omega_1' \equiv \sigma_{ij}' D_{ji}\,, \Omega_2' \equiv \sigma_{ij}'^{(2)} D_{ji}\,, \Omega_3' \equiv \sigma_{ij}' D_{ji}^{(2)}\,,\ \Omega_4' \equiv \sigma_{ij}'^{(2)} D_{ji}^{(2)} \end{array}\right\} \tag{8.42}$$

and experimental data. To show this we can start from the hypothesis (8.26) and find similarly to (8.27) the cubic equation

$$\sigma^3 + A^*\sigma^2 + B^*\sigma + C^* = 0\,, \tag{8.43}$$

if we insert (8.33), (8.35) and (8.24a,b,c) into the hypothesis (8.26). In (8.43) the following abbreviations are used:

$$\begin{aligned} A^* \equiv -\tfrac{1}{9}(1 - 8\nu + 6\nu n)\,\big[&\Phi_0 J_1 + \Phi_1\left(\Omega_1' + \tfrac{1}{3}J_1 L_1\right) \\ &+ \Phi_2\left(\Omega' + \tfrac{1}{3}J_1 L_2\right)\big]\big/\,D^m\,, \end{aligned} \tag{8.44a}$$

$$\begin{aligned} B^* \equiv -\tfrac{2}{3}\left(1 + \nu + \tfrac{3}{2}\nu n\right)\big[&2\Phi_0 J_2' + \Phi_1\left(\Omega_2' + \tfrac{1}{3}J_1\Omega_1'\right) \\ &+ \Phi_2\left(\Omega_4' + \tfrac{1}{3}J_1\Omega_3'\right)\big]\big/\,D^m\,, \end{aligned} \tag{8.44b}$$

$$\begin{aligned} C^* \equiv -(1 + \nu - 3\nu n)\,\big[&3\Phi_0\left(J_3' + \tfrac{2}{9}J_1 J_2'\right) \\ &+ \Phi_1\left(J_2'\Omega_1' + J_3' L_1 + \tfrac{1}{3}J_1\Omega_2'\right) \\ &+ \Phi_2\left(J_2'\Omega_3' + J_3' L_2 + \tfrac{1}{3}J_1\Omega_4'\right)\big]\big/\,D^m\,, \end{aligned} \tag{8.44c}$$

$$D \equiv (D_I D_{II} D_{III})^{1/3}\,, \qquad m \equiv (m_I + m_{II} + m_{III})/\,3\,. \tag{8.44d,e}$$

We see that the elements of the integrity basis (8.42) and experimental data are contained in (8.44a-e). Thus the coefficients $\psi_{[\nu,\,\mu]} \equiv \varphi_\nu \Phi_\mu$ in (8.41) are scalar functions of the integrity basis (8.42) and experimental data

$$K,\, n,\, \nu;\, m_I,\, m_{II},\, m_{III};\, D_I,\, D_{II},\, D_{III}$$

found in creep tests on specimens cut along three mutually perpendicular directions.

In the case (4.79) of damage and initial anisotropy we can use for simplification the constitutive equation

$$d_{ij} = f_{ij}(\boldsymbol{t},\,\boldsymbol{\tau}) = \frac{1}{2}\sum_{\nu,\,\mu=0}^{2} \psi^*_{[\nu,\,\mu]}\left(t_{ik}^{(\nu)}\tau_{kj}^{(\mu)} + \tau_{ik}^{(\mu)}t_{kj}^{(\nu)}\right)\,, \tag{8.45}$$

where the linear transformations

$$t_{ij} = D_{ijpq}\sigma_{pq} = t_{ji} \quad \text{with} \quad D_{ijpq} := (D_{ip}D_{jq} + D_{iq}D_{jp})/\,2\ , \tag{8.46}$$

$$\tau_{ij} = A_{ijpq}\sigma_{pq} = \tau_{ji}\ , \tag{8.47}$$

have been introduced in section 4.3.2 according to (4.94) and (4.95), respectively. Then the scalar functions in (8.45) can be determined in a very similar way as described above.

Further applications concerning the tensorial generalization of uniaxial relations in continuum mechanics have been considered by BETTEN (1989; 2001c). For example, the plastic behaviour of solids loaded under uni-axial stress σ may be expressed by the stress-strain-relations

$$\sigma/\sigma_F = [\tanh\,(E\varepsilon/\sigma_F)^n]^{(1/n)}\ , \tag{8.48a}$$

$$\sigma/\sigma_F = (E\varepsilon/\sigma_F)\,/\,[1 + (E\varepsilon/\sigma_F)^n]^{(1/n)}\ , \tag{8.48b}$$

proposed by BETTEN (1975b), where σ_F ist the yield stress in a uni-axial tension test, and E represents the modulus of elasticity - often called "YOUNG's modulus"(1807); however, this modulus was already used by EULER (1760). The exponent n regulates the elastic-plastic transition. For instance, an elastic-perfectly plastic behaviour is characterized by $n \to \infty$.

It has been shown by BETTEN (1975c) that independently of the parameter n the *limit carying capacity* coincides wth that for a percectly plastic body $(n \to \infty)$. Hence a new aspect of the uniqueness of the *limit load* may be formulated as we can read in the book of ZYCZKOWSKI (1981, page 210):

> *Uniqueness understood as the independence of that load of the assumed stress-strain diagram belonging to the class of asymptotically perfect plasticity. Such independence may be observed in many cases.*

For engineering applications, it is very important to generalize the relations (8.48a,b) to multiaxial states of stress. This can be achieved by using an isotropic tensor function (8.1).

Similar to (8.48a,b) we can assume the following *creep functions*

$$\kappa(t) = [\tanh\,(t^n)]^{(1/n)}\ , \tag{8.49a}$$

$$\kappa(t) = t/\,(1 + t^n)^{(1/n)}\ , \tag{8.49b}$$

which are compared with the creep function (11.8) of the KELVIN *solid* (Fig. 11.17) by using the following MAPLE program.

```
> kappa(t)[KELVIN]:=1-exp(-G*t/eta);
```

⊙ 8_1.mws

$$\kappa(t)_{KELVIN} := 1 - e^{\left(-\frac{G\,t}{\eta}\right)}$$

```
> kappa(t)[tan_hyper]:=(tanh(t^n))^(1/n);
```

$$\kappa(t)_{tan\_hyper} := \tanh(t^n)^{\left(\frac{1}{n}\right)}$$

```
> kappa(t)[root]:=t/(1+t^n)^(1/n);
```

$$\kappa(t)_{root} := \frac{t}{(1+t^n)^{\left(\frac{1}{n}\right)}}$$

```
> alias(H=Heaviside, th=thickness):
> plot1:=plot({1,H(t-5),1-exp(-t)},
  t=0..5.001, th=1,color=black):
> plot2:=plot({tanh(t),(tanh(t^n))^(1/n)},
  t=0..5.001, th=4,color=black, style=point,
  symbolsize=12,symbol=cross):
> plot3:=plot({t/(1+t), t/(1+t^n)^(1/n)},
  t=0..5.001, th=2,color=black, style=point,
  symbolsize=12,symbol=circle):
> plots[display]({plot1,plot2,plot3});
```

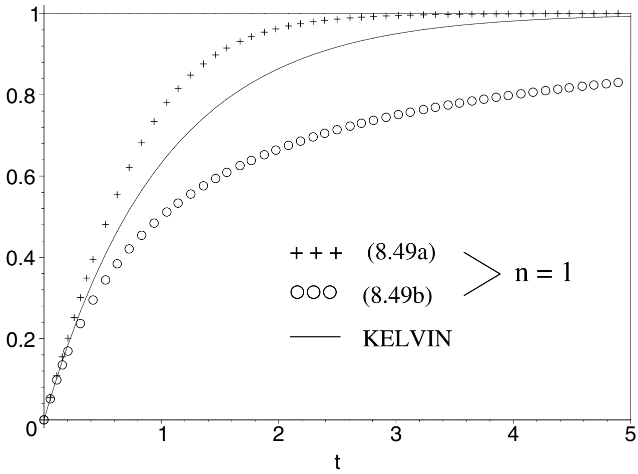

Fig. 8.1 Creep functions

Another example is the tensorial generalization of the RAMBERG-OSGOOD relation, also discussed by BETTEN (1989; 2001c) including own experiments on *aluminium alloy*.