7 Damage Mechanics

In sections 4.3.2 and 6.4 *constitutive equations* involving *damage* and *initial anisotropy* have been formulated in detail. This Chapter is concerned with the construction of *damage tensors* or tensors of continuity (Section 7.1).

Then, the *multiaxial state of stress* in a damaged continuum will be analyzed in detail (Section 7.2).

Finally, some *damage effective stress* concepts are proposed and discussed in Section 7.3.

7.1 Damage Tensors and Tensors of Continuity

As has been already mentioned in Chapter 4, *damage* has in general an *anisotropic* character even if the material was originally isotropic. This matter results from the microscopic nature of damage. The fissure orientation and length cause anisotropic macroscopic behavior. Therefore, damage in an isotropic or initial anisotropic material that is in a state of multiaxial stress can only be described by taking a *damage tensor* into account.

There are some different ways to construct tensors suitable for analysing the damage state in material. In the following *second-rank* and *fourth-order damage tensors* are systematically developed.

In three-dimensional space a parallelogram formed by the vectors A_i and B_i can be represented by

$$S_i = \varepsilon_{ijk} \, A_j B_k \tag{7.1a}$$

or in the dual form

$$S_{ij} = \varepsilon_{ijk} \, S_k \quad \Leftrightarrow \quad S_i = \tfrac{1}{2}\varepsilon_{ijk} \, S_{jk} \ , \tag{7.1b}$$

where ε_{ijk} is the third-order alternating tensor ($\varepsilon_{ijk} = 1$, or -1 if i, j, k are even or odd permutations of 1, 2, 3, respectively, otherwise the components ε_{ijk} are equal to zero) according to (2.5). From (7.1a,b) we immediately find

$$S_{ij} = 2!A_{[i}B_{j]} = \begin{vmatrix} A_i & A_j \\ B_i & B_j \end{vmatrix} \ . \tag{7.2}$$

Because of the decomposition (7.2) as an alternating product of two vectors the *bivector* $\boldsymbol{S}$ is called *simple* and has the following three nonvanishing essential components

$$S_{12} = A_1 B_2 - A_2 B_1 \ , \quad S_{23} = A_2 B_3 - A_3 B_2 \ , \tag{7.3a,b}$$

$$S_{31} = A_3 B_1 - A_1 B_3 \ . \tag{7.3c}$$

In rectilinear components in three-dimensional space, we see that the absolute values of the components (7.3) are the projections of the area of the parallelogram, considered above, on the coordinate planes. Thus S_{ij}, according to (7.2), represents an area vector in three-dimensional space and has an orientation fixed by (7.1a).

According to (7.1b) a surface element dS with an unit normal n_i, i.e. $dS_i = n_i dS$, is expressed by

$$dS_{ij} = \varepsilon_{ijk} dS_k \quad \Leftrightarrow \quad dS_i = \tfrac{1}{2}\varepsilon_{ijk} dS_{jk} \tag{7.4.a}$$

and

$$n_{ij} = \varepsilon_{ijk} n_k \quad \Leftrightarrow \quad n_i = \tfrac{1}{2}\varepsilon_{ijk} n_{jk} \ . \tag{7.4b}$$

The components of the *bivector* $\boldsymbol{n}$ are the direction cosines n_1, n_2, n_3:

$$n_{ij} = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} \ . \tag{7.5}$$

The principal invariants of (7.5), defined as

$$J_1 \equiv n_{ii} \ , \qquad -J_2 \equiv n_{i[i]} n_{j[j]} \ , \qquad J_3 \equiv n_{i[i]} n_{j[j]} n_{k[k]} \ , \tag{7.6a,b,c}$$

take the following values:

$$J_1(\boldsymbol{n}) = 0 \ , \quad -J_2(\boldsymbol{n}) = n_1^2 + n_2^2 + n_3^2 = 1 \ , \quad J_3(\boldsymbol{n}) = 0 \ , \tag{7.7a,b,c}$$

i.e. the only nonvanishing invariant is determined by the length of the unit normal vector n_i. In (7.6a,b,c) the same notation is used as in (2.24a,b,c).

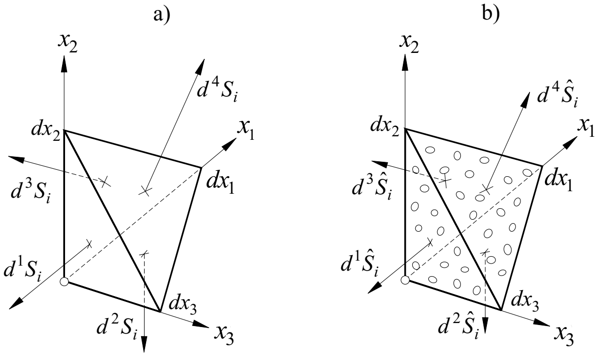

Fig. 7.1 CAUCHY's tetrahedron **a**) in an undamaged state, **b**) in a damaged state

Imagine that at a point ○ in a continuous medium a set of rectangular coordinate axes is drawn and a differential tetrahedron is bounded by parts of the three coordinate planes through ○ and a fourth plane not passing through ○, as shown in Fig. 7.1a. Such a tetrahedron can be characterized by a system of *bivectors*,

$$\begin{aligned}
d^1S_i &= -\tfrac{1}{2}\varepsilon_{ijk}(dx_2)_j(dx_3)_k \ , \\
d^2S_i &= -\tfrac{1}{2}\varepsilon_{ijk}(dx_3)_j(dx_1)_k \ , \\
d^3S_i &= -\tfrac{1}{2}\varepsilon_{ijk}(dx_1)_j(dx_2)_k \ , \\
d^4S_i &= -\tfrac{1}{2}\varepsilon_{ijk}\left[(dx_1)_j - (dx_3)_j\right]\left[(dx_2)_k - (dx_3)_k\right] \ ,
\end{aligned} \tag{7.8}$$

where the sum is the zero vector:

$$\boxed{d^1S_i + d^2S_i + d^3S_i + d^4S_i = 0_i} \ . \tag{7.9}$$

In a damaged continuum we define a "*net cross section*" $\hat{S} \equiv \psi S$ where $\psi \leq 1$ describes the "*continuity*" of the material, as mentioned in Section 4.3.1. Then, by analogy of (7.8), a tetrahedron in a *damaged continuum* (Fig. 7.1b) can be characterized by the following system of *bivectors*:

$$\begin{aligned}
d^1\hat{S}_i &= -\tfrac{1}{2}\alpha_{ijk}(dx_2)_j(dx_3)_k \equiv \alpha\, d^1S_i \ , \\
d^2\hat{S}_i &= -\tfrac{1}{2}\beta_{ijk}(dx_3)_j(dx_1)_k \equiv \beta\, d^2S_i \ , \\
d^3\hat{S}_i &= -\tfrac{1}{2}\gamma_{ijk}(dx_1)_j(dx_2)_k \equiv \gamma\, d^3S_i \ , \\
d^4\hat{S}_i &= -\tfrac{1}{2}\kappa_{ijk}\left[(dx_1)_j - (dx_3)_j\right]\left[(dx_2)_k - (dx_3)_k\right] \equiv \kappa\, d^4S_i \ ,
\end{aligned} \tag{7.10}$$

where $\alpha_{ijk} \equiv \alpha\varepsilon_{ijk}$, $\beta_{ijk} \equiv \beta\varepsilon_{ijk}$, etc. are total skew-symmetric tensors of order three, which have the essential components $\alpha_{123} \equiv \alpha$, $\beta_{123} \equiv \beta$, etc., respectively.

¿From Fig. 7.1a,b we find that only

$$dS_1 = -n_1\, dS\ , \quad d^1\hat{S}_1 = \alpha\, d^1S_1\ , \quad d^2S_2 = -n_2\, dS\ , \quad \text{etc.}$$

are non vanishing components of the *bivector* systems (7.8) and (7.10). Then the sum of (7.10) yields the vector

$$\Sigma_i \equiv d^1\hat{S}_i + \ldots + d^4\hat{S}_i = \begin{pmatrix} (\kappa-\alpha)n_1 \\ (\kappa-\beta)n_2 \\ (\kappa-\gamma)n_3 \end{pmatrix} dS\ , \tag{7.11}$$

which is not the zero vector, unless in the isotropic damage case ($\alpha = \beta = \gamma = \kappa$) or in the undamaged case ($\alpha = \beta = \gamma = \kappa = 1$) according to (7.9).

Furthermore, because of $d^1\hat{S}_1 \neq 0$, $d^1\hat{S}_2 = d^1\hat{S}_3 = 0$ etc., the damage state of the continuum at a point is characterized by the *bivectors*

$$\alpha_{1ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix}\ , \quad \beta_{2ij} = \begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}\ , \tag{7.12a,b}$$

$$\gamma_{3ij} = \begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{7.12c}$$

In the following we will examine if the *bivector*

$$\psi_{ij} = \alpha_{1ij} + \beta_{2ij} + \gamma_{3ij} = \begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix} \tag{7.13a}$$

$$\psi_{ij} = \alpha\varepsilon_{1ij} + \beta\varepsilon_{2ij} + \gamma\varepsilon_{3ij} \tag{7.13b}$$

could be a suitable *tensor of continuity*. Then the damage tensor $\boldsymbol{\omega}$ would be of the form

$$\omega_{ij} = \delta_{(k)k}\varepsilon_{kij} - \psi_{ij} = \begin{pmatrix} 0 & 1-\gamma & -(1-\beta) \\ -(1-\gamma) & 0 & 1-\alpha \\ 1-\beta & -(1-\alpha) & 0 \end{pmatrix} \tag{7.14a}$$

(no sum on the bracketed index k) or

$$\omega_{ij} = (1-\alpha)\varepsilon_{1ij} + (1-\beta)\varepsilon_{2ij} + (1-\gamma)\varepsilon_{3ij} \ . \tag{7.14b}$$

If a tensor is symmetric or antisymmetric, respectively, in one cartesian coordinate system, it is symmetric or antisymmetric in all such systems; thus symmetry and antisymmetry are really tensor properties. Therefore, the skew-symmetric tensor (7.13a) has only three essential components in any cartesian system, for instance, α, β, γ in relation to the system x_i or α^*, β^*, γ^* with respect to the system x_i^*.

The only nonvanishing invariants of the *bivectors* (7.13) and (7.14) are determined by their lengths:

$$-J_2(\boldsymbol{\psi}) \equiv -\tfrac{1}{2}\,\mathrm{tr}\,\boldsymbol{\psi}^2 \equiv -\tfrac{1}{2}\psi_{ij}\psi_{ji} = \alpha^2 + \beta^2 + \gamma^2 \ , \tag{7.15}$$

$$-J_2(\boldsymbol{\omega}) = (1-\alpha)^2 + (1-\beta)^2 + (1-\gamma)^2 \ . \tag{7.16}$$

In the undamaged state ($\alpha = \beta = \gamma = 1$) we have

$$-J_2(\boldsymbol{\psi}) = 3 \ , \quad -J_2(\boldsymbol{\omega}) = 0 \ ,$$

and

$$\psi_{ij} \rightarrow \eta_{ij} \equiv \varepsilon_{1ij} + \varepsilon_{2ij} + \varepsilon_{3ij} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} . \tag{7.17}$$

We see that the undamaged state does not yield an isotropic tensor, because the components of $\boldsymbol{\eta}$ in (7.17) transform under the change of the coordinate system.

Thus the bivector $\boldsymbol{\psi}$ defined by (7.13a,b) is not suitable to describe the state of *continuity* of a *damaged continuum*, and we have to find another tensor composed by the bivectors (7.12a,b,c). As shown below, a suitable *tensor of continuity* may be defined by

$$\boxed{\psi_{ijk} \equiv \psi_{i[jk]}} \quad \text{with} \quad \begin{cases} \psi_{1jk} \equiv \alpha_{1jk} = \alpha\varepsilon_{1jk} \\ \psi_{2jk} \equiv \beta_{2jk} = \beta\varepsilon_{2jk} \\ \psi_{3jk} \equiv \gamma_{3jk} = \gamma\varepsilon_{3jk} \end{cases} . \tag{7.18}$$

This tensor is skew-symmetric only with respect to the two bracketed indizes $[jk]$ and possesses the three essential components (α, β, γ), as illustrated in Fig. 7.2.

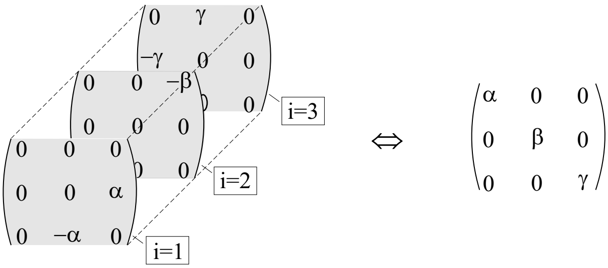

Fig. 7.2 Third-order tensor of continuity and its dual form

In the isotropic damage state ($\alpha = \beta = \gamma = \kappa$) the tensor (7.18) is total skew-symmetric, and the undamaged continuum ($\alpha = \beta = \gamma = 1$) is characterized by the third-order alternating tensor ε_{ijk}. Supplementary to (7.18) we introduce the "*damage tensor*"

$$\boxed{\omega_{ijk} \equiv \varepsilon_{ijk} - \psi_{ijk}} \quad \text{where} \quad \begin{cases} \omega_{1jk} = (1-\alpha)\varepsilon_{1jk} \\ \omega_{2jk} = (1-\beta)\varepsilon_{2jk} \\ \omega_{3jk} = (1-\gamma)\varepsilon_{3jk} \end{cases} . \tag{7.19}$$

By analogy of (7.1b) or (7.4a,b) the dual relations

$$\psi_{ijk} \equiv \psi_{i[jk]} = \varepsilon_{jkr}\psi_{ir} \quad \Leftrightarrow \quad \psi_{ir} = \tfrac{1}{2}\varepsilon_{rjk}\psi_{ijk} , \tag{7.20}$$

$$\omega_{ijk} \equiv \omega_{i[jk]} = \varepsilon_{jkr}\omega_{ir} \quad \Leftrightarrow \quad \omega_{ir} = \tfrac{1}{2}\varepsilon_{rjk}\omega_{ijk} \tag{7.21}$$

are valid.

Contrary to (7.13) and (7.14) the *dual tensor of continuity* ψ_{ij} according to (7.20) and the *dual damage tensor* ω_{ij} according to (7.21) have the diagonal forms

$$\boxed{\psi_{ij} = \mathrm{diag}\,\{\alpha,\ \beta,\ \gamma\}} \tag{7.22}$$

and

$$\omega_{ij} = \mathrm{diag}\,\{(1-\alpha),\ (1-\beta),\ (1-\gamma)\} , \tag{7.23}$$

respectively. For the undamaged continuum ($\psi_{ijk} \to \varepsilon_{ijk}$) the dual tensor of continuity ψ_{ij} is equal to KRONECKER's tensor δ_{ij}, as we can see from (7.20) or immediately from (7.22). The relations (7.20) and (7.22) are illustrated in Fig. 7.2.

Especially, from Fig. 7.2 we can see the skew-symmetric character of the third order tensor of continuity indicated in (7.20) and its three essential components α, β, γ. These values are fractions which represent the net cross-sectional elements perpendicular to the coordinate axes x_1, x_2, x_3 (Fig. 7.1b) and which can be measured in tests on specimens cut along three mutually perpendicular directions x_1, x_2, x_3.

According to (7.4a) a damaged surface element $d\hat{S}$ can be expressed in the dual form

$$d\hat{S}_{ij} = \varepsilon_{ijk} d\hat{S}_k \quad \Leftrightarrow \quad d\hat{S}_i = \tfrac{1}{2}\varepsilon_{ijk} d\hat{S}_{jk} \ , \tag{7.24}$$

and using the tensor of continuity (7.18) we find

$$d\hat{S}_{ij} = \psi_{ijk} dS_k \quad \Leftrightarrow \quad d\hat{S}_i = \tfrac{1}{2}\psi_{ijk} dS_{jk} \ . \tag{7.25}$$

Note that the bivector $d\hat{S}_{ij}$ or dS_{jk} in (7.25) must have the same indices with respect to which the tensor (7.18) is skew-symmetric. Combining (7.4a) and (7.25) we have the linear transformations

$$d\hat{S}_{ij} = \tfrac{1}{2}\psi_{ijpq} dS_{pq} \ , \quad d\hat{S}_i = \psi_{ir} dS_r \ , \tag{7.26a,b}$$

where ψ_{ir} is the tensor (7.20), (7.22), while ψ_{ijpq} is a fourth-order non-symmetric tensor defined as

$$\psi_{ijpq} \equiv \psi_{kij}\varepsilon_{kpq} \ , \tag{7.27a}$$

which, by using (7.20), can be expressed through

$$\psi_{ijpq} = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,\psi_{rr} - (\psi_{ip}\delta_{jq} - \psi_{iq}\delta_{jp}) - (\delta_{ip}\psi_{jq} - \delta_{iq}\psi_{jp}) \ . \tag{7.27b}$$

This tensor has the antisymmetric properties

$$\psi_{ijpq} = -\psi_{jipq} = -\psi_{ijqp} = \psi_{jiqp} \ , \tag{7.28}$$

and is symmetric only with respect to the index pairs, i.e.

$$\psi_{ijpq} = \psi_{pqij} \ . \tag{7.29}$$

More briefly, the properties of (7.28) and (7.29) can be indicated by

$$\psi_{ijpq} = \psi_{([ij][pq])} \ . \tag{7.30}$$

The essential components of the tensor (7.27) are given by

$$\psi_{ijpq} = \begin{cases} \alpha, \beta, \gamma, & \text{if } ij \text{ is an even permutation of } pq \\ -\alpha, -\beta, -\gamma, & \text{if } ij \text{ is an odd permutation of } pq, \\ 0, & \text{otherwise} \end{cases} \tag{7.31a}$$

which means

$$\begin{gathered}\psi_{2323}=\psi_{3232}\equiv\alpha\ ,\quad \psi_{3131}=\psi_{1313}\equiv\beta\ ,\\ \psi_{1212}=\psi_{2121}\equiv\gamma\ ,\\ \psi_{3223}=\psi_{2332}\equiv-\alpha\ ,\quad \psi_{1331}=\psi_{3113}\equiv-\beta\ ,\\ \psi_{2112}=\psi_{1221}\equiv-\gamma\ .\end{gathered} \tag{7.31b}$$

In the isotropic damage state ($\alpha=\beta=\gamma=\kappa$) the tensor (7.27) is proportional to KRONECKER's generalized delta

$$\delta_{ijpq}\equiv\varepsilon_{kij}\varepsilon_{kpq}=\begin{vmatrix}\delta_{kk} & \delta_{kp} & \delta_{kq}\\ \delta_{ik} & \delta_{ip} & \delta_{iq}\\ \delta_{jk} & \delta_{jp} & \delta_{jq}\end{vmatrix}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}\ ,$$

and is identical to that one in the undamaged continuum characterized by ($\alpha=\beta=\gamma=1$).

In order to construct the tensor of continuity (7.22) we can use the following way. In addition to Fig. 7.1 let us consider a fictitious undamaged

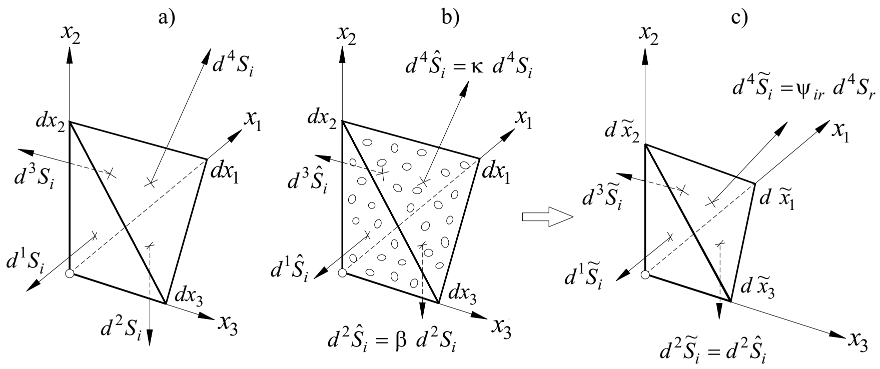

Fig. 7.3 CAUCHY's tetrahedron **a**) undamaged configuration, **b**) anisotropic damaged configuration, **c**) fictitious undamaged configuration

configuration as illustrated in Fig. 7.3c, which is, similar to (7.8) and (7.10), characterized by the following system of bivectors:

$$\begin{aligned}
d^1\tilde{S}_i &= -\tfrac{1}{2}\varepsilon_{ijk}(d\tilde{x}_2)_j(d\tilde{x}_3)_k\equiv d^1\hat{S}_i\ ,\\
d^2\tilde{S}_i &= -\tfrac{1}{2}\varepsilon_{ijk}(d\tilde{x}_3)_j(d\tilde{x}_1)_k\equiv d^2\hat{S}_i\ ,\\
d^3\tilde{S}_i &= -\tfrac{1}{2}\varepsilon_{ijk}(d\tilde{x}_1)_j(d\tilde{x}_2)_k\equiv d^3\hat{S}_i\ ,\\
d^4\tilde{S}_i &= -\tfrac{1}{2}\varepsilon_{ijk}\left[(d\tilde{x}_1)_j-(d\tilde{x}_3)_j\right]\left[(d\tilde{x}_2)_k-(d\tilde{x}_3)_k\right]\ ,
\end{aligned} \tag{7.32}$$

where, by analogy of (7.9), the vector sum is equal to the zero vector:

$$\boxed{d^1\tilde{S}_i + d^2\tilde{S}_i + d^3\tilde{S}_i + d^4\tilde{S}_i = 0_i} \ . \tag{7.33}$$

The three area vectors $d^1\tilde{S}_i, \ldots, d^3\tilde{S}_i$ in (7.32) are identical to the corresponding vectors in (7.10) of the damaged configuration. The fourth vector $d^4\tilde{S}_i$ in (7.32), having the same magnitude as $d^4\hat{S}_i$ in (7.10), differ from the vector d^4S_i in (7.8) not only in length, but also in its direction. Therefore, the vectors $d^4\tilde{S}_i$ and d^4S_i are connected by a linear operator $\boldsymbol{\psi}$ of rank two (second order tensor):

$$\boxed{d^4\tilde{S}_i = \psi_{ir} d^4 S_r} \ . \tag{7.34}$$

Comparing the three systems of bivectors (7.8), (7.10), (7.32) and using the equations (7.9), (7.33) in connection with the transformation (7.34), we find the relation:

$$\begin{aligned} &\psi_{ir}\varepsilon_{rjk}\left[(dx_2)_j(dx_3)_k + (dx_3)_j(dx_1)_k + (dx_1)_j(dx_2)_k\right] \\ &\quad = \alpha_{ijk}(dx_2)_j(dx_3)_k + \beta_{ijk}(dx_3)_j(dx_1)_k + \gamma_{ijk}(dx_1)_j(dx_2)_k \ , \end{aligned} \tag{7.35}$$

where the transvection $\psi_{ir}\varepsilon_{rjk}$ leads to the third-order tensor of continuity:

$$\psi_{ir}\varepsilon_{rjk} \equiv \psi_{ijk} = \psi_{i[jk]} \ , \tag{7.36}$$

which is skew-symmetric with respect to the bracketed index pair $[jk]$. The result (7.36) is contained in (7.18) and (7.20).

Because of $\alpha_{ijk} \equiv \alpha\varepsilon_{ijk}$, etc. the terms on the right-hand side of (7.35) are vectors with magnitudes

$$\left| d^1\tilde{S}_i \right| = \tfrac{1}{2}\alpha_{1jk}(dx_2)_j(dx_3)_k \ , \quad \text{etc.}$$

and with the directions of the basis vectors ${}^1e_i, {}^2e_i, {}^3e_i$ of the cartesian coordinate system. Therefore, in connection with (7.36), relation (7.35) can be written in the following form:

$$\psi_{ijk}\left[(dx_2)_j(dx_3)_k + \ldots\right] = {}^1e_i\alpha_{1jk}(dx_2)_j(dx_3)_k + \ldots \ , \tag{7.37}$$

from which we immediately read the decomposition:

$$\psi_{ijk} = {}^1e_i\alpha_{1jk} + {}^2e_i\beta_{2jk} + {}^3e_i\gamma_{3jk} \ , \tag{7.38a}$$

or because of $\alpha_{1jk} \equiv \alpha\varepsilon_{1jk}$, etc.:

$$\psi_{ijk} = \alpha\, {}^1e_i\varepsilon_{1jk} + \beta\, {}^2e_i\varepsilon_{2jk} + \gamma\, {}^3e_i\varepsilon_{3jk} \; . \tag{7.38b}$$

By analogy of (7.1b) we find the dual relation from (7.36):

$$\psi_{ijk} = \psi_{i[jk]} = \varepsilon_{jkr}\psi_{ir} \quad \Leftrightarrow \quad \psi_{ir} = \tfrac{1}{2}\varepsilon_{rjk}\psi_{ijk} \; , \tag{7.39}$$

and finally the diagonal form:

$$\psi_{ir} = \tfrac{1}{2}\psi_{ipq}\varepsilon_{jpq} = \mathrm{diag}\{\alpha, \beta, \gamma\} \tag{7.40a}$$

in accordance with (7.22). Inserting the decomposition (7.38b) into (7.40a) and replacing $\delta_{1j} \equiv {}^1e_j$, etc., we see that the second rank tensor of continuity can be decomposed in terms of dyadics formed from the basis vectors:

$$\psi_{ij} = \alpha\left({}^1\boldsymbol{e} \otimes {}^1\boldsymbol{e}\right)_{ij} + \beta\left({}^2\boldsymbol{e} \otimes {}^2\boldsymbol{e}\right)_{ij} + \gamma\left({}^3\boldsymbol{e} \otimes {}^3\boldsymbol{e}\right)_{ij} \; . \tag{7.40b}$$

The relations (7.39) and (7.40a) are illustrated in Fig. 7.2. Especially, from Fig. 7.2 we can see the skew-symmetric character of a third-order tensor of continuity indicated in (7.36) and its three essential components (α, β, γ). These values are fractions which represent the net cross-sectional elements perpendicular to the coordinate axes x_1, x_2, x_3 (Fig. 7.3b) and which can be measured in tests on specimens cut along three mutually perpendicular directions x_1, x_2, x_3. Such experiments are carried out by BETTEN and his coworkers as discussed in Chapter 13.

The damage may sometimes develop *isotropically*, as observed by JOHNSON (1960) for R.R. 59 Al alloy. In this special case ($\alpha = \beta = \gamma \equiv \psi$), the second rank tensor of continuity (7.39) is a *spherical tensor*:

$$\psi_{ijk} = \psi\varepsilon_{jkr}\delta_{ir} = \psi\varepsilon_{ijk} \quad \Leftrightarrow \quad \psi_{ir} = \tfrac{1}{2}\psi\varepsilon_{rjk}\varepsilon_{ijk} = \psi\delta_{ir} \tag{7.41}$$

and, contrary to (7.39), the third order tensor of continuity is now totally skew-symmetric ($\psi_{ijk} \equiv \psi_{[ijk]}$).

Instead of the continuity tensor $\boldsymbol{\psi}$ according to (7.39) we can use the damage tensor $\boldsymbol{\omega}$ defined by (7.19), (7.23) and characterized by the dual relation (7.21). In view of polynominal representations of constitutive equations it is convenient to use the tensor (4.84), as discussed in Section 4.3.2 in more detail.

7.2 Stresses in a Damaged Continuum

In the undamaged continuum (Fig. 7.4a) CAUCHY's formula

$$p_i = \sigma_{ji} n_j \tag{7.42}$$

is derived from equilibrium, where p_i and n_i are the components of the stress vector $\boldsymbol{p}$ and the unit vector normal $\boldsymbol{n}$, respectively. In the same way we get to the corresponding relation for a *damaged continuum*,

$$\hat{p}_i \psi(n) = \psi_{jk} \hat{\sigma}_{ki} n_j \ , \tag{7.43}$$

where ψ_{jk} are the components of the continuity tensor $\boldsymbol{\psi}$ according to (7.22).

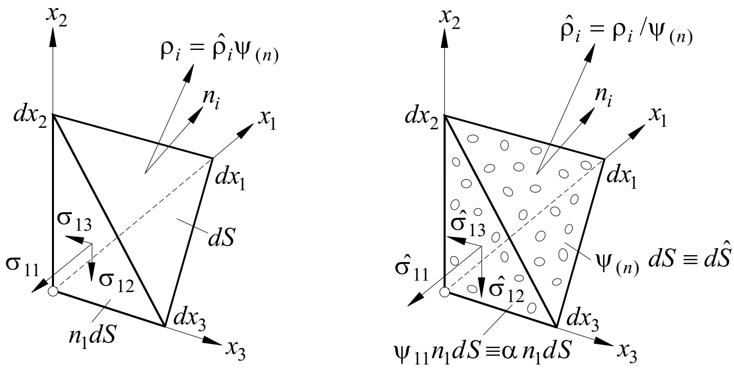

Fig. 7.4 Stress tensor regarding **a**) an undamaged, **b**) a damaged continuum

The surface elements dS and $d\hat{S}$ in Fig. 7.4 are subjected to the *same* force vector:

$$dP_i = p_i \, dS \equiv \hat{p}_i \, d\hat{S} = d\hat{P}_i \ . \tag{7.44}$$

Thus, considering (7.42) and (7.43), we finally find the actual *net-stress tensor* $\hat{\boldsymbol{\sigma}}$ as a transformation from CAUCHY's tensor:

$$\sigma_{ij} = \psi_{ir} \hat{\sigma}_{rj} = \sigma_{ji} \quad \Leftrightarrow \quad \hat{\sigma}_{ij} = \psi_{ir}^{(-1)} \sigma_{rj} \neq \hat{\sigma}_{ji} \ . \tag{7.45}$$

By suitable transvections we find $\sigma_{ij} \hat{\sigma}_{jk}^{(-1)} = \psi_{ik}$ and $\hat{\sigma}_{ij} \sigma_{jk}^{(-1)} = \psi_{ik}^{(-1)}$.

As indicated in (7.45), the actual net-stress tensor $\hat{\boldsymbol{\sigma}}$ is *non-symmetric*, unless we have isotropic damage expressed by $\psi_{ir} = \psi \delta_{ir}$.

Because of the symmetry $\sigma_{ij} = (\sigma_{ij} + \sigma_{ji})/2$ of CAUCHY*'s stress tensor* $\boldsymbol{\sigma}$ we find the representations

$$\sigma_{ij} = \tfrac{1}{2} \left(\psi_{ip} \delta_{jq} + \delta_{iq} \psi_{jp} \right) \hat{\sigma}_{pq} \equiv \varphi_{ijpq} \hat{\sigma}_{pq} \, , \tag{7.46a}$$

$$\hat{\sigma}_{ij} = \tfrac{1}{2}\left(\psi^{(-1)}_{ip}\delta_{jq} + \psi^{(-1)}_{iq}\delta_{jp}\right)\sigma_{pq} \equiv \varPhi_{ijpq}\sigma_{pq} \tag{7.46b}$$

from (7.45). We see that the fourth-order tensors $\boldsymbol{\varphi}$ and $\boldsymbol{\varPhi}$ defined as (7.46a,b) are only symmetric with respect to two indices:

$$\varphi_{ijpq} = \varphi_{jipq} \,, \qquad \varPhi_{ijpq} = \varPhi_{ijqp} \,, \tag{7.47a,b}$$

that is, the *actual net-stress tensor* $\hat{\boldsymbol{\sigma}}$ is non-symmetric in the anisotropic damage case:

$$\frac{\hat{\sigma}_{12}}{\hat{\sigma}_{21}} = \frac{\beta}{\alpha} \,, \qquad \frac{\hat{\sigma}_{23}}{\hat{\sigma}_{32}} = \frac{\gamma}{\beta} \,, \qquad \frac{\hat{\sigma}_{31}}{\hat{\sigma}_{13}} = \frac{\alpha}{\gamma} \,. \tag{7.48}$$

This fact is a disadvantage, and it is awkward to use the actual net-stress tensor $\hat{\boldsymbol{\sigma}}$ in constitutive equations with a symmetric strain rate tensor $\boldsymbol{d}$. Therefore, we introduce a *transformed net-stress tensor* $\boldsymbol{t}$ defined by the operation

$$t_{ij} = \tfrac{1}{2}\left(\hat{\sigma}_{ik}\psi^{(-1)}_{kj} + \psi^{(-1)}_{ki}\hat{\sigma}_{jk}\right) \,, \tag{7.49}$$

which is *symmetric*. Inserting (7.46b) into (7.49) we have

$$\boxed{t_{ij} = C^{(-1)}_{ijpq}\sigma_{pq}} \,, \tag{7.50}$$

where

$$C^{(-1)}_{ijpq} = \tfrac{1}{2}\left(\psi^{(-1)}_{ip}\psi^{(-1)}_{jq} + \psi^{(-1)}_{iq}\psi^{(-1)}_{jp}\right) \tag{7.51}$$

is a symmetric fourth-order tensor

$$C^{(-1)}_{ijpq} = C^{(-1)}_{jipq} = C^{(-1)}_{ijqp} = C^{(-1)}_{pqij} \,, \tag{7.52}$$

which is identical to the tensor D_{ijpq} in (4.94). In the undamaged ($\boldsymbol{\psi} \to \boldsymbol{\delta}$) and total damaged state ($\boldsymbol{\psi} \to \boldsymbol{0}$) we have

$$C^{(-1)}_{ijpq} \to E_{ijpq} \quad \Rightarrow \quad t_{ij} \to \sigma_{ij} \tag{7.53}$$

and

$$C^{(-1)}_{ijpq} \to \infty_{ijpq} \quad \Rightarrow \quad t_{ij} \to \infty_{ij}(\text{singular}) \,, \tag{7.54}$$

respectively, where

$$E_{ijpq} = \tfrac{1}{2}\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) \tag{7.55}$$

is the zero power tensor of rank four.

The inverse form of (7.50) is given by

$$\boxed{\sigma_{ij} = C_{ijpq} t_{pq}} \ , \tag{7.56}$$

where

$$C_{ijpq} = \tfrac{1}{2} \left(\psi_{ip} \psi_{jq} + \psi_{iq} \psi_{jp} \right) \tag{7.57}$$

is a symmetric fourth-order tensor of continuity,

$$C_{ijpq} = C_{jipq} = C_{ijqp} = C_{pqij} \ , \tag{7.58}$$

which is connected with the tensor (7.51) by the relation

$$C_{ijpq} C^{(-1)}_{pqk\ell} = C^{(-1)}_{ijpq} C_{pqk\ell} = C^{(0)}_{ijk\ell} \equiv E_{ijk\ell} \ . \tag{7.59}$$

Because of the symmetry properties (7.52) and (7.58) the fourth-order tensor of continuity (7.57) and its inversion (7.51) can be represented by 6×6 square matrices, which, because of (7.40a,b), have the diagonal forms:

$$C_{ijk\ell} = \operatorname{diag} \left\{ C_{1111}, \, C_{2222}, \, C_{3333}, \, C_{1212}, \, C_{2323}, \, C_{3131} \right\} \ , \tag{7.60a}$$

$$C_{ijk\ell} = \operatorname{diag} \left\{ \alpha^2, \, \beta^2, \, \gamma^2, \, \tfrac{1}{2}\alpha\beta, \, \tfrac{1}{2}\beta\gamma, \, \tfrac{1}{2}\gamma\alpha \right\} \ , \tag{7.60b}$$

and

$$C^{(-1)}_{ijk\ell} = \operatorname{diag} \left\{ C^{(-1)}_{1111}, \, C^{(-1)}_{2222}, \, C^{(-1)}_{3333}, \, C^{(-1)}_{1212}, \, C^{(-1)}_{2323}, \, C^{(-1)}_{3131} \right\} \ , \tag{7.61a}$$

$$C^{(-1)}_{ijk\ell} = \operatorname{diag} \left\{ \frac{1}{\alpha^2}, \, \frac{1}{\beta^2}, \, \frac{1}{\gamma^2}, \, \frac{2}{\alpha\beta}, \, \frac{2}{\beta\gamma}, \, \frac{2}{\gamma\alpha} \right\} \ , \tag{7.61b}$$

that is, the components of the *pseudo-net stress tensor* $\boldsymbol{t}$, according to (7.50), are given in the following manner:

$$\begin{aligned} t_{11} &= \frac{1}{\alpha^2}\sigma_{11} \ , & t_{12} &= \frac{1}{\alpha\beta}\sigma_{12} \ , & t_{13} &= \frac{1}{\alpha\gamma}\sigma_{13} \ , \\ t_{21} &= t_{12} \ , & t_{22} &= \frac{1}{\beta^2}\sigma_{22} \ , & t_{23} &= \frac{1}{\beta\gamma}\sigma_{23} \ , \\ t_{31} &= t_{13} \ , & t_{32} &= t_{23} \ , & t_{33} &= \frac{1}{\gamma^2}\sigma_{33} \ . \end{aligned} \tag{7.62}$$

The results (7.50) and (7.56) can also be found in the following way. Using the linear transformations

$$t_{ij} = \tfrac{1}{2}\left(\delta_{ir}\psi_{js}^{(-1)} + \psi_{is}^{(-1)}\delta_{jr}\right)\hat{\sigma}_{rs} \tag{7.63}$$

and

$$\hat{\sigma}_{pq} = \tfrac{1}{2}\left(\delta_{pr}\psi_{qs} + \delta_{ps}\psi_{qr}\right)t_{rs}\ , \tag{7.64}$$

which connect a fictitious symmetric tensor $\boldsymbol{t}$ with the actual non-symmetric net stress tensor $\hat{\boldsymbol{\sigma}}$, we immediately find (7.50) by inserting (7.46b) into (7.63) and (7.56) by inserting (7.64) into (7.46a), respectively.

Because of the non-symmetric property of the actual net-stress tensor we find from (7.64) the decomposition

$$\hat{\sigma}_{pq} = \hat{\sigma}_{(pq)} + \hat{\sigma}_{[pq]}\ , \tag{7.65}$$

where the symmetric and antisymmetric parts are given by

$$\hat{\sigma}_{(pq)} = \left(t_{pr}\psi_{rq} + \psi_{pr}t_{rq}\right)/\,2 \tag{7.66}$$

and

$$\hat{\sigma}_{[pq]} = \left(t_{pr}\psi_{rq} - \psi_{pr}t_{rq}\right)/\,2\ , \tag{7.67}$$

respectively. In the special case of isotropic damage ($\psi_{ij} = \psi\delta_{ij}$) we have $\hat{\sigma}_{(pq)} = \psi t_{pq}$ and $\hat{\sigma}_{[pq]} = 0_{pq}$.

An interpretation of the introduced pseudo-net stress tensor (7.49) can be given in the following way. An alternative form of CAUCHY's formula (7.42) is

$$dP_i = \sigma_{ji}\,dS_j\ , \tag{7.68}$$

where dP_i is the actual force vector (7.44), and according to (7.26b) we can write

$$dP_i = \sigma_{ji}\psi_{jr}^{(-1)}\,d\hat{S}_r\ , \tag{7.69a}$$

or inserting (7.56) we find the relation

$$dP_i = \psi_{ip}t_{pr}\,d\hat{S}_r\ , \tag{7.69b}$$

which can be multiplied by $\psi_{ki}^{(-1)}$, so that we have

$$\psi_{ki}^{(-1)}\,dP_i = t_{kr}\,d\hat{S}_r\ , \tag{7.70a}$$

or after changing the indices:

$$\psi_{ik}^{(-1)}\,dP_k \equiv d\tilde{P}_i = t_{ji}\,d\hat{S}_j\ . \tag{7.70b}$$

Comparing (7.68) and (7.70b) we see that (7.70b) can be interpreted as CAUCHY's formula for the damaged configuration, which is subjected to the pseudo-force $d\tilde{P}_i \equiv \psi_{ik}^{(-1)}\, dP_k$ instead to the actual force dP_i.

Because of the non-symmetric properties of the "net-stress tensor" $\hat{\boldsymbol{\sigma}}$ and the operator $\boldsymbol{\varphi}$, i.e.,

$$\hat{\sigma}_{ij} = \tfrac{1}{2}\left(\hat{\sigma}_{ij} + \hat{\sigma}_{ji}\right) + \tfrac{1}{2}\left(\hat{\sigma}_{ij} - \hat{\sigma}_{ji}\right) \tag{7.71}$$

and

$$\varphi_{ijpq} = \tfrac{1}{2}\left(\varphi_{ijpq} + \varphi_{ijqp}\right) + \tfrac{1}{2}\left(\varphi_{ijpq} - \varphi_{ijqp}\right) \ , \tag{7.72}$$

respectively, we find, from (7.46a), the decompositions:

$$\begin{aligned}\sigma_{ij} = {} & \tfrac{1}{4}\left(\varphi_{ijpq} + \varphi_{ijqp}\right)\left(\hat{\sigma}_{pq} + \hat{\sigma}_{qp}\right) \\ & + \tfrac{1}{4}\left(\varphi_{ijpq} - \varphi_{ijqp}\right)\left(\hat{\sigma}_{pq} - \hat{\sigma}_{qp}\right) \ ,\end{aligned} \tag{7.73a}$$

$$\begin{aligned}\sigma_{ij} = {} & \tfrac{1}{8}\left(\psi_{ip}\delta_{jq} + \psi_{jp}\delta_{iq} + \psi_{iq}\delta_{jp} + \psi_{jq}\delta_{ip}\right)\left(\hat{\sigma}_{pq} + \hat{\sigma}_{qp}\right) \\ & + \tfrac{1}{8}\left(\psi_{ip}\delta_{jq} + \psi_{jp}\delta_{iq} - \psi_{iq}\delta_{jp} - \psi_{jq}\delta_{ip}\right)\left(\hat{\sigma}_{pq} - \hat{\sigma}_{qp}\right) \ .\end{aligned} \tag{7.73b}$$

Because of (7.47a) the right-hand sides in (7.73a) and (7.73b) are symmetric with respect to the indices i and j. Furthermore, we see the symmetry with respect to the indices p and q. This fact can be seen immediately from (7.46a). In the special case of isotropic damage, i.e., $\psi_{ij} = \psi\delta_{ij}$ or $\hat{\sigma}_{pq} = \hat{\sigma}_{qp}$, the second term of the right-hand side in (7.73b) vanishes. Then, equation (7.73b) is identical to those formulated by RABOTNOV (1969).

In a similar way, from (7.46b) we find the decomposition of the "net stress tensor" $\hat{\sigma}$ into a symmetric and an antisymmetric part:

$$\hat{\sigma}_{ij} = \tfrac{1}{2}\left(\Phi_{ijpq} + \Phi_{jipq}\right)\sigma_{pq} + \tfrac{1}{2}\left(\Phi_{ijpq} - \Phi_{jipq}\right)\sigma_{pq} \ , \tag{7.74a}$$

$$\begin{aligned}\hat{\sigma}_{ij} = {} & \tfrac{1}{4}\left(\psi_{ip}^{(-1)}\delta_{jq} + \psi_{iq}^{(-1)}\delta_{jp} + \psi_{jp}^{(-1)}\delta_{iq} + \psi_{jq}^{(-1)}\delta_{ip}\right)\sigma_{pq} \\ & + \tfrac{1}{4}\left(\psi_{ip}^{(-1)}\delta_{jq} + \psi_{iq}^{(-1)}\delta_{jp} - \psi_{jp}^{(-1)}\delta_{iq} - \psi_{jq}^{(-1)}\delta_{ip}\right)\sigma_{pq} \ .\end{aligned} \tag{7.74b}$$

The results given above may be expressed by the damage tensor $\boldsymbol{\omega}$. For instance, from (7.27a,b) in connection with (7.19) and because of the substitution $\psi_{ij} \equiv \delta_{ij} - \omega_{ij}$ we have

$$\psi_{ijpq} = \delta_{ijpq} - \omega_{kij}\varepsilon_{kpq} \equiv \left(\varepsilon_{kij} - \omega_{kij}\right)\varepsilon_{kpq} \ , \tag{7.75a}$$

$$\begin{aligned}\psi_{ijpq} = {} & \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\right)\left(1 - \omega_{rr}\right) + \left(\omega_{ip}\delta_{jq} - \omega_{iq}\delta_{jp}\right) \\ & + \left(\delta_{ip}\omega_{jq} - \delta_{iq}\omega_{jp}\right) \ .\end{aligned} \tag{7.75b}$$

Furthermore, instead of (7.46a) and (7.73b) we find

$$\sigma_{ij} = \tfrac{1}{2}\left[\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} - (\omega_{ip}\delta_{jq} + \delta_{iq}\omega_{jp})\right]\hat{\sigma}_{pq} \tag{7.76a}$$

and

$$\begin{aligned}\sigma_{ij} = {} & \tfrac{1}{2}\left(\hat{\sigma}_{ij} + \hat{\sigma}_{ji}\right) \\ & - \tfrac{1}{8}\left(\omega_{ip}\delta_{jq} + \delta_{iq}\omega_{jp} + \omega_{iq}\delta_{jp} + \delta_{ip}\omega_{jq}\right)\left(\hat{\sigma}_{pq} + \hat{\sigma}_{qp}\right) \\ & - \tfrac{1}{8}\left(\omega_{ip}\delta_{jq} + \delta_{iq}\omega_{jp} - \omega_{iq}\delta_{jp} - \delta_{ip}\omega_{jq}\right)\left(\hat{\sigma}_{pq} - \hat{\sigma}_{qp}\right) \ .\end{aligned} \tag{7.76b}$$

By using the inverse

$$\psi_{ir}^{(-1)} \equiv \frac{1}{2\det(\psi)}\varepsilon_{rqp}\varepsilon_{ik\ell}\psi_{pk}\psi_{q\ell} \tag{7.77}$$

and because of the symmetry $\sigma_{ij} = (\sigma_{ij} + \sigma_{ji})/2$, we find the following relations for the net stress tensor:

$$\begin{aligned}\hat{\sigma}_{ij} = {} & \frac{1}{2\det(\boldsymbol{\delta} - \boldsymbol{\omega})}\Big[\left(\delta_{is}\delta_{jt} + \delta_{it}\delta_{js}\right)\left(1 - \omega_{rr}\right) \\ & + \left(\omega_{is}\delta_{jt} + \omega_{it}\delta_{js}\right) + \tfrac{1}{2}\varepsilon_{ik\ell}\left(\varepsilon_{spq}\delta_{jt} + \varepsilon_{tpq}\delta_{js}\right)\omega_{pk}\omega_{q\ell}\Big]\sigma_{st} \ ,\end{aligned} \tag{7.78a}$$

$$\begin{aligned}= {} & \frac{1}{\det(\boldsymbol{\delta} - \boldsymbol{\omega})}\Big[\left(1 - \omega_{rr}\right)\sigma_{ij} + \omega_{ir}\sigma_{rj} \\ & + \tfrac{1}{2}\varepsilon_{ik\ell}\varepsilon_{spq}\omega_{pk}\omega_{q\ell}\sigma_{sj}\Big] \ ,\end{aligned} \tag{7.78b}$$

$$\begin{aligned}= {} & \frac{1}{\det(\boldsymbol{\delta} - \boldsymbol{\omega})}\Big\{\left[1 - J_1(\boldsymbol{\omega}) - J_2(\boldsymbol{\omega})\right]\sigma_{ij} \\ & + \left[1 - J_1(\boldsymbol{\omega})\right]\omega_{ir}\sigma_{rj} + \omega_{ir}^{(2)}\sigma_{rj}\Big\} \ ,\end{aligned} \tag{7.78c}$$

where

$$J_1(\boldsymbol{\omega}) \equiv \delta_{ij}\omega_{ji} \ , \qquad J_2(\boldsymbol{\omega}) \equiv \tfrac{1}{2}\left(\omega_{ij}\omega_{ji} - \omega_{ii}\omega_{jj}\right) \tag{7.79a,b}$$

are invariants of the damage tensor $\boldsymbol{\omega}$.

Finally, we consider CAUCHYs stress equations of equilibrium,

$$\sigma_{ji,j} = 0_i \ , \tag{7.80}$$

in the absence of the body forces. Then by using the transformation (7.45), we have the equilibrium equations in the net stresses:

$$\hat{\sigma}_{ri}\psi_{jr,j} + \psi_{jr}\hat{\sigma}_{ri,j} = 0_i \ . \tag{7.81}$$

The symmetry of CAUCHY's stress tensor ($\sigma_{ij} = \sigma_{ji}$) resulting from moment equilibrium yields the condition

$$\psi_{ip}\hat{\sigma}_{pj} = \psi_{jq}\hat{\sigma}_{qi} \quad \text{or} \quad \hat{\sigma}_{ij} = \psi_{iq}^{(-1)}\psi_{jp}\hat{\sigma}_{pq} \ , \tag{7.82a,b}$$

which states that the net stress tensor is non-symmetric. From (7.82b) we find the decomposition into a symmetric part and an antisymmetric one:

$$\begin{aligned}\hat{\sigma}_{ij} = \tfrac{1}{4}\left(\psi_{iq}^{(-1)}\psi_{jp} + \psi_{ip}^{(-1)}\psi_{jq}\right)(\hat{\sigma}_{pq} + \hat{\sigma}_{qp}) \\ + \tfrac{1}{4}\left(\psi_{iq}^{(-1)}\psi_{jp} - \psi_{ip}^{(-1)}\psi_{jq}\right)(\hat{\sigma}_{pq} + \hat{\sigma}_{qp}) \ .\end{aligned} \tag{7.83}$$

For the isotropic damage case ($\psi_{ij} = \psi\delta_{ij}$), the relation (7.83) is equal to the decomposition

$$\hat{\sigma}_{ij} = (\hat{\sigma}_{ji} + \hat{\sigma}_{ij})/\,2 + (\hat{\sigma}_{ji} - \hat{\sigma}_{ij})/\,2 \equiv \hat{\sigma}_{ji} \ , \tag{7.84}$$

i.e., the *net stress tensor* is symmetric in this special case only.

7.3 Damage Effective Stress Concepts

During the last two or three decades many scientists have devoted much effort to the stress analysis in a damaged material, and the notation *damage effective stress* has been introduced. In the following some various *damage effective stress concepts* should be reviewed.

In the case of damage being isotropic measured in terms of a single *scalar parameter* $\omega (0 \le \omega \le 1)$, the *effective stress tensor* $\bar{\boldsymbol{\sigma}}$ is expressed in the form

$$\boxed{\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1-\omega)} \qquad \textbf{(model I)}$$

where $\boldsymbol{\sigma}$ denotes the CAUCHY *stress tensor*. This assumption leads to simple models of mechanical behavior coupling damage and is adequate in some cases, especially under conditions of proportional loading (LEMAITRE, 1984,1992) or in some materials (JOHNSON, 1960). However, many scientists (HAYHURST, 1972; LECKIE and HAYHURST, 1974; LEE, PENG and WANG, 1985; CHOW and WANG, 1987; 1988) experimentally observed that all initially isotropic or anisotropic materials under conditions of nonproportional loading and most brittle materials even though under conditions of proportional loading develop *anisotropic damage*, for which the *damage variables* can no longer be *scalars*, but are of *tensorial nature* (LECKIE and ONAT, 1981). The damage variables are then *vectors*, *second-order* or *fourth-order tensors* (BETTEN, 2001b). ONAT (1986), ONAT and LECKIE (1988) and ADAMS et al. (1992) showed that the damage variables in isothermal mechanical behavior are irreducible tensors of even orders.

MURAKAMI and OHNO (1981) derived an *asymmetric effective stress tensor*,

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\mathbf{1} - \boldsymbol{\omega})^{-1} \, ,$$

where $\mathbf{1}$ denotes the second-order identity tensor and $\boldsymbol{\omega}$ is a symmetric second-order damage tensor. Only the symmetric part of $\boldsymbol{\sigma}^*$, i.e.,

$$\boxed{\bar{\boldsymbol{\sigma}} = \left[\boldsymbol{\sigma}(\mathbf{1} - \boldsymbol{\omega})^{-1} + (\mathbf{1} - \boldsymbol{\omega})^{-1}\boldsymbol{\sigma}\right]\big/\, 2} \qquad \textbf{(model II)} \, ,$$

has been considered by MURAKAMI (1988) in constitutive equations.

CHOW and WANG (1987) postulated an alternative model of the effective stress tensor in the damage principal coordinate system, which was applied in elasticity, plasticity and ductile fracture (CHOW and WANG, 1987; 1988; KATTAN and VOYIADJIS, 1990; VOYIADJIS and KATTAN, 1990) . It is easy to show that this model coincides with the following *tensorial* expression:

$$\boxed{\bar{\boldsymbol{\sigma}} = (\mathbf{1} - \boldsymbol{\omega})^{-1/2}\boldsymbol{\sigma}(\mathbf{1} - \boldsymbol{\omega})^{-1/2}} \qquad \textbf{(model III)} \, .$$

In particular, if $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$ are *coaxial* in their principal directions, then they are commutative, $\boldsymbol{\sigma}\boldsymbol{\omega} = \boldsymbol{\omega}\boldsymbol{\sigma}$, and both models, II and III, reduce to:

$$\boxed{\bar{\boldsymbol{\sigma}} = (\mathbf{1} - \boldsymbol{\omega})^{-1}\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{1} - \boldsymbol{\omega})^{-1}} \qquad \textbf{(model IV)} \, .$$

This model is the tensorial generalization of those proposed by SIDOROFF (1981), LECKIE and HAYHURST (1974), and LEE et al. (1985) in the principal coordinate system.

In each of the above discussed **models I-IV** the effective stress tensor $\bar{\boldsymbol{\sigma}}$ depends *linear* on the CAUCHY stress tensor $\boldsymbol{\sigma}$. In general, the fourth-order tensor $\boldsymbol{M}$ as a *linear transformation* in the relation

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{M}[\boldsymbol{\sigma}] \quad \text{or} \quad \bar{\sigma}_{ij} = M_{ijk\ell}\sigma_{k\ell}$$

is named the *damage effective tensor* (ZHENG and BETTEN, 1996). Originally, RABOTNOV (1968) had not considered the relation $\bar{\boldsymbol{\sigma}} = \boldsymbol{M}[\boldsymbol{\sigma}]$ but defined a symmetric *net-stress tensor* $\hat{\boldsymbol{\sigma}}$ by way of a linear transformation $\boldsymbol{\sigma} = \boldsymbol{\Omega}[\hat{\boldsymbol{\sigma}}]$ or in index notation according to (4.78), where the fourth-order tensor $\boldsymbol{\Omega}$ is assumed to be symmetric.

However, BETTEN (1982b) has pointed out in more detail that the fourth-order tensor $\boldsymbol{\Omega}$ in the linear transformation $\boldsymbol{\sigma} = \boldsymbol{\Omega}[\hat{\boldsymbol{\sigma}}]$ and consequently the *net-stress tensor* $\hat{\boldsymbol{\sigma}}$ cannot be symmetric if the damage develops *anisotropically*. Instead of $\hat{\boldsymbol{\sigma}}$, BETTEN (1983b) introduced a *transformed* net-stress tensor $\boldsymbol{t}$, called *pseudo-net-stress tensor*, as an effective stress tensor,

$$\boxed{\bar{\boldsymbol{\sigma}} = (\mathbf{1} - \boldsymbol{\omega})^{-1}\boldsymbol{\sigma}(\mathbf{1} - \boldsymbol{\omega})^{-1}} \qquad (\textbf{model V}) \ ,$$

which is *symmetric* even in cases of *anisotropic damage*. This model can be expressed in index notation according to (7.50) with (7.51), if we take the relation (4.84) into account.

Because of the broad applicability and versatility of **model V** to engineering problems (BETTEN, 1986a; 1991b; 1998), this model has been developed step by step and discussed in more detail in Section 7.2.

It must be emphasized that there is no substantive difference between **models V** and **III** since both tensors $(\mathbf{1} - \boldsymbol{\omega})$ and $(\mathbf{1} - \boldsymbol{\omega})^{1/2}$ are positive-definite second-order symmetric tensors and are phenomenological measures of the anisotropic damage state. Furthermore, it has been pointed out by ZHENG and BETTEN (1996) that the difference between **models II** and **III** is negligible, if the damage is not highly developed.

Besides the concepts of *damage effective stress* (**models I**-**V**) various *damage equivalence principles* play an important role in the development of continuum damage mechanics. For instance, the *strain equivalence hypothesis* (LEMAITRE, 1985; 1992; CHABOCHE, 1988; LEMAITRE and CHABOCHE, 1990; BETTEN, 2001a; 2001c; OMERSPAHIC and MATTIASSON, 2007) states that a damaged material element under the applied stress $\boldsymbol{\sigma}$ exhibits the same strain response as the undamaged one submitted to the effective stress $\bar{\boldsymbol{\sigma}}$. Unfortunately, this hypothesis leads to asymmetric effective compliance and stiffness matrices if *anisotropic damage* develops. To remove this inconsistency, SIDOROFF (1981) proposed the *complementary energy equivalence hypothesis* by replacing the equivalence for strain response with the equivalence for *complementary energy*. We particularly stress that the concept of effective stress becomes meaningful, only if either the *strain* or *complementary equivalence hypothesis* (as well as some other additional equivalence hypotheses for yield criterion function, failure criterion function, etc.) is employed.

Assume that the damage state can be characterized in terms of a set $\Re$ of scalars, vectors, and/or tensors of different orders, which operate as internal variables. The remarkable role of the effective stress tensor concept requires the most general representation for the *damage effect tensor*, or more generally, the *effective stress tensor*:

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{M}[\boldsymbol{\sigma}] \quad \text{with} \quad \boldsymbol{M} = \boldsymbol{M}(\Re)$$

or

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \Re) \, .$$

ZHENG and BETTEN (1996) postulate a generalized *damage equivalence hypothesis*. Then, the so-called *damage isotropy principle* is established that in order to coincide with the damage equivalence hypothesis, the effective stress tensor $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \Re)$ as a second-order tensor-valued function of $\boldsymbol{\sigma}$ and $\Re$ has to be isotropic. Particularly, this property is irrespective of the initial material symmetry (isotropy or anisotropy) and the type of damage variables. As a consequence, the damage effect tensor $\boldsymbol{M}(\Re)$ is an isotropic fourth-order tensor-valued function of the damage state variables $\Re$. As *isotropic tensor functions*, the *effective stress tensor* $\bar{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \Re)$ and the *damage effect tensor* $\boldsymbol{M}(\Re)$ can be formulated in general *invariant forms* according to the theory of *representations for tensor functions* (RIVLIN, 1970; SPENCER, 1971; WANG, 1971; BOEHLER, 1979; 1987; ZHENG, 1994; BETTEN, 1986a; 1998; 2001c; 2003a). Damage material constants are then consistently introduced to these *invariant damage models*.