2 Tensor Notation

It will be convenient in this monograph to use the compact notation often referred to as *indicial* or *index notation* . It allows a strong reduction in the number of terms in an equation and is commonly used in the current literature when stress, strain, and constitutive equations are discussed. Therefore, a basic knowledge of the index notation is helpful in studying continuum mechanics, especially constitutive modelling of materials. With such a notation, the various stress-strain relationships for materials under multi-axial states of stress can be expressed in a compact form. Thus, greater attention can be paid to physical principles rather than to the equations themselves. A short outline of this notation should therefore be given in the following. In comparison, some expressions or equations shall also be written in *symbolic* or *matrix notation* , employing whichever is more convenient for the derivation or analysis at hand, but taking care to establish the interrelationship between the two distinct notations.

2.1 Cartesian Tensors

We consider vectors and tensors in three-dimensional EUCLIDean space. For simplicity, rectangular Cartesian coordinates x_i, $i = 1, 2, 3$, are used throughout. Results may, if desired, be expressed in terms of curvlinear coordinate systems by standard techniques of tensor analysis (BETTEN, 1987c), as has been pointed out in Section 2.2 and used in Chapter 5.

In a rectangular Cartesian coordinate system, a *vector* $\boldsymbol{V}$ can be decomposed in the following three components

$$\boldsymbol{V} = (V_1, V_2, V_3) = V_1\boldsymbol{e}_1 + V_2\boldsymbol{e}_2 + V_3\boldsymbol{e}_3 \ , \tag{2.1}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are unit base vectors:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \ . \tag{2.2}$$

In 2.2 the Symbol δ_{ij} is known as KRONECKER's delta. Thus, the set of unit vectors, $\{\boldsymbol{e}_i\}$, constitutes an *orthonormal basis* .

A very useful notational device in the manipulation of matrix, vector, and tensor expressions is the *summation convention* introduced by EINSTEIN (1916):

> Whenever an index occurs twice in the same term, summation over the values 1, 2, and 3 of that index is automatically assumed, and the summation sign is omitted.

Thus, the decomposition (2.1) can be written in the more compact form

$$\boldsymbol{V} = V_i \boldsymbol{e}_i \equiv V_k \boldsymbol{e}_k \ . \tag{2.1*}$$

The repeated index i or k in (2.1*) is often called *summation index* or *dummy index* because the choice of the letter for this index is immaterial. However, we have to notice that an index must not appear more than twice in the same term of an expression or equation. Otherwise, there is a mistake. An expression such as $A_{ijk}B_{kk}$ would be meaningless.

Consider a sum in which one of the repeated indices is on the KRONECKER delta, for example,

$$\delta_{ik}A_{ij} = \delta_{1k}A_{1j} + \delta_{2k}A_{2j} + \delta_{3k}A_{3j} \ .$$

Only one term in this sum does not vanish, namely the term in which
$i = k = 1, 2, 3$. Consequently, the sum reduces to

$$\delta_{ik}A_{ij} = A_{kj} \ .$$

A similar example is: $\delta_{ij}V_j = V_i$. Notice that the summation, involving one index of the KRONECKER delta and one of another factor, has the effect of *substituting* the free index of the delta for the repeated index of the other factor. For this reason the KRONECKER delta could be called *substitution tensor* (BETTEN, 1987c).

Another example of the summation convention is the scalar product (dot product, inner product) of two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$. It can be written as

$$\boldsymbol{U} \cdot \boldsymbol{V} = U_1V_1 + U_2V_2 + U_3V_3 \equiv U_kV_k \ . \tag{2.3}$$

Note that the expression

$$(A_{ii})^2 \equiv (A_{11} + A_{22} + A_{33})^2$$

is different from the sum

$$A_{ii}^2 \equiv A_{11}^2 + A_{22}^2 + A_{33}^2 \ ,$$

where the first one is the square of the sum A_{ii}, while the second one is the sum of the squares.

The vector product (cross product) has the following form:

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \equiv \varepsilon_{ijk} \boldsymbol{e}_i A_j B_k \ . \tag{2.4a}$$

Its components are given by

$$C_i = \varepsilon_{ijk} A_j B_k \ . \tag{2.4b}$$

In (2.4a,b) the *alternating symbol* ε_{ijk} (also known as permutation symbol or third-order alternating tensor) is used. It is defined as:

$$\varepsilon_{ijk} \begin{cases} = +1 & \text{if } ijk \text{ represents an even permutation of 123 ;} \\ = 0 & \text{if any two of } ijk \text{ indices are equal ;} \\ = -1 & \text{if } ijk \text{ represents an odd permutation of 123 .} \end{cases} \tag{2.5}$$

It follows from this definition that ε_{ijk} has the symmetry properties

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{jik} = -\varepsilon_{kji} \ . \tag{2.6}$$

The triple scalar product can also be calculated by using the alternating symbol:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \varepsilon_{ijk} A_i B_j C_k \ . \tag{2.7}$$

The following relation between the KRONECKER delta and the alternating tensor is very important and useful (BETTEN, 1987c):

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \equiv 3!\, \delta_{i[p]}\delta_{j[q]}\delta_{k[r]} \ , \tag{2.8}$$

where on the right-hand side the operation of alternation is used. This process is indicated by placing square brackets around those indices to which it applies, that is, the three indices pqr are permutated in all possible ways. Thus, we obtain 3! terms. The terms corresponding to even permutations are given a plus sign, those which correspond to odd permutations a minus sign, and they are then added and divided by 3!.

From (2.8) we immediately obtain the *contraction*

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \ , \tag{2.9a}$$

for instance, i.e., the tensor of rank six in (2.8) is reduced to the fourth-order tensor (2.9a). Other *contractions* are

$$\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij} \quad \text{and} \quad \varepsilon_{pqr}\varepsilon_{pqr} = 6 \ , \tag{2.9b,c}$$

for instance.

Now let us consider a coordinate transformation, i.e., we introduce a new rectangular right-handed Cartesian coordinate system and new base vectors $\boldsymbol{e}_i^*$, $i = 1, 2, 3$. The new system may be regarded as having been derived from the old by a rigid rotation of the triad of coordinate axes about the same origin. Let a vector $\boldsymbol{V}$ have components V_i in the original coordinate system and components V_i^* in the new system. Thus, one can write:

$$\boldsymbol{V} = V_i\boldsymbol{e}_i = V_i^*\boldsymbol{e}_i^* \ . \tag{2.10}$$

We denote by a_{ij} the cosine of the angle between $\boldsymbol{e}_i^*$ and $\boldsymbol{e}_j$, so that

$$a_{ij} \equiv \cos(\boldsymbol{e}_i^*, \boldsymbol{e}_j) = \boldsymbol{e}_i^* \cdot \boldsymbol{e}_j \ , \tag{2.11}$$

i.e., a_{ij} are the direction cosines of $\boldsymbol{e}_i^*$ relative to the first coordinate system, or, equivalently, a_{ij} are the components of the new base vectors $\boldsymbol{e}_i^*$, in the first system. Thus

$$\boldsymbol{e}_i^* = a_{ij}\boldsymbol{e}_j \ . \tag{2.12}$$

It is geometrically evident that the nine quantities a_{ij} are not independent. Since $\boldsymbol{e}_i^*$ are mutually perpendicular unit vectors,

$$\boldsymbol{e}_i^* \cdot \boldsymbol{e}_j^* = \delta_{ij} \ , \tag{2.13}$$

we arrive at

$$\boldsymbol{e}_i^* \cdot \boldsymbol{e}_j^* = a_{ip}\boldsymbol{e}_p \cdot a_{jr}\boldsymbol{e}_r = a_{ip}a_{jr}\boldsymbol{e}_p \cdot \boldsymbol{e}_r = a_{ip}a_{jr}\delta_{pr} = a_{ip}a_{jp}$$

by considering (2.2) and (2.12). Hence

$$a_{ip}a_{jp} = \delta_{ij} \quad \text{or in matrix notation} \quad \boldsymbol{a}\boldsymbol{a}^t = \boldsymbol{\delta} \ , \tag{2.14a}$$

where $\boldsymbol{a}^t$ is the transpose of the matrix $\boldsymbol{a}$. Because of the symmetry $\delta_{ij} = \delta_{ji}$, the result (2.14a) represents a set of six relations between the nine quantities a_{ij}. Similarly to (2.14a), we find:

$$a_{pi}a_{pj} = \delta_{ij} \quad \text{or in matrix notation} \quad \boldsymbol{a}^t\boldsymbol{a} = \boldsymbol{\delta} \ . \tag{2.14b}$$

It follows immediately from (2.14a,b) that $|a_{ij}| = \pm 1$ and, furthermore, that the transpose $\boldsymbol{a}^t$ is identical to the inverse $\boldsymbol{a}^{-1}$. Thus, the matrix $\boldsymbol{a}$ is *orthogonal*, and the reciprocal relation to (2.12) is

$$\boldsymbol{e}_i = a_{ji}\boldsymbol{e}_j^* \ . \tag{2.15}$$

Inserting (2.15) or (2.12) into (2.10), we arrive at

$$V_i^* = a_{ij}V_j \quad \text{or} \quad V_i = a_{ji}V_j^* \ , \tag{2.16a,b}$$

respectively. In particular, if $\boldsymbol{V}$ is the position vector $\boldsymbol{x}$ of the point P relative to the origin, then

$$x_i^* = a_{ij}x_j \quad \text{and} \quad x_i = a_{ji}x_j^* \ , \tag{2.17a,b}$$

where x_i^* and x_i are the coordinates of the point P in the new and original coordinate systerns, respectively.

Now, let us consider a vector function

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}) \quad \text{or in index notation:} \quad Y_i = f_i(X_p) \ , \tag{2.18a,b}$$

where $\boldsymbol{X}$ is the argument vector which is transformed to another vector $\boldsymbol{Y}$. The simplest form is a *linear transformation*

$$Y_i = T_{ij}X_j \ , \tag{2.19}$$

where T_{ij} are the cartesian components of a *second-order tensor* $\boldsymbol{T}$, also called *second-rank tensor* , i.e., a second-order tensor can be interpreted as a linear operator which transforms a vector $\boldsymbol{X}$ into an image vector $\boldsymbol{Y}$.

In extension of the law (2.16a,b), the components of a second-order tensor $\boldsymbol{T}$ transform according to the rule

$$T_{ij}^* = a_{ip}a_{jq}T_{pq} \quad \text{or} \quad T_{ij} = a_{pi}a_{qj}T_{pq}^* \ , \tag{2.20a,b}$$

which can be expressed in matrix notation:

$$\boldsymbol{T}^* = \boldsymbol{a}\boldsymbol{T}\boldsymbol{a}^t \quad \text{or} \quad \boldsymbol{T} = \boldsymbol{a}^t\boldsymbol{T}^*\boldsymbol{a} \ . \tag{2.21a,b}$$

Second-rank tensors play a central role in continuum mechanics, for instance, strain and stress tensors are second-order tensors. It is sometimes useful in continuum mechanics, especially in the theory of plasticity or in creep mechanics, to decompose a tensor into the sum of its *deviator* and a *spherical tensor* as follows:

$$T_{ij} = T'_{ij} + T_{kk}\delta_{ij}/3 \ . \tag{2.22}$$

For instance, the stress deviator

$$\sigma'_{ij} := \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 \tag{2.23}$$

is responsible for the change of shape (*distortion*), while the hydrostatic stress $\sigma_{kk}\delta_{ij}/3$ produces volume change without change of shape in an *isotropic* continuum, i.e., in a material with the same material properties in all directions. Clearly, a uniform all-around pressure should merely decrease the volume of a sphere of material with the same strength in all directions. However, if the sphere were weaker in one direction, that diameter would be changed more than others. Thus, hydrostatic pressure can produce a change of shape in *anisotropic* materials.

The deviator (2.23) is often called a *traceless tensor* , since its trace $\operatorname{tr}\boldsymbol{\sigma}' \equiv \sigma'_{kk}$ is identical to zero.

A second-order tensor has three *irreducible invariants*

$$J_1 \equiv \delta_{ij}T_{ji} = T_{jj} \equiv T_{kk} \ , \tag{2.24a}$$

$$J_2 \equiv -T_{i[i]}T_{j[j]} = (T_{ij}T_{ji} - T_{ii}T_{jj})/2 \ , \tag{2.24b}$$

$$J_3 \equiv T_{i[i]}T_{j[j]}T_{k[k]} = \det(T_{ij}) \equiv |T_{ij}| \ , \tag{2.24c}$$

which are scalar quantities appearing in the *characteristic equation*

$$\det(\lambda\delta_{ij} - T_{ij}) = \lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0 \ . \tag{2.25}$$

In (2.24b,c) the operation of alternation is used and indicated by placing square brackets around those indices to which it applies. This process is already illustrated in the context with (2.8).

We read from (2.24a,b,c): The first (linear) invariant J_1 is the trace of $\boldsymbol{T}$, the second (quadratic) invariant J_2 is defined as the *negative* sum of the three *principal minors* of order 2, while the third (cubic) invariant is given by the determinant of the tensor. A deviator has only two non-vanishing invariants:

$$J_2' = T_{ij}'T_{ji}'/2 \; , \qquad\qquad J_3' = \det(T_{ij}') \; . \tag{2.26a,b}$$

Remark: Because of the definition (2.24b) the second invariant (2.26a) of the deviator is always *positive*. Therefore, J_2 is defined as the *negative* sum of the principal minors in this text.

The invariants (2.24a,b,c) can be expressed through the principal values T_I, T_{II}, T_{III} of the Tensor $\boldsymbol{T}$, i.e., the *elementary symmetric functions* of the three arguments $T_I, \ldots, T_{III}$ are related to the *irreducible invariants* (2.24a,b,c) as follows:

$$T_I + T_{II} + T_{III} = J_1 \; , \tag{2.27a}$$

$$T_I T_{II} + T_{II} T_{III} + T_{III} T_I = -J_2 \; , \tag{2.27b}$$

$$T_I T_{II} T_{III} = J_3 \; . \tag{2.27c}$$

After some manipulation one can arrive from (2.22), (2.24), and (2.26) at the relations

$$J_2' = J_2 + \frac{1}{3}J_1^2 \; , \qquad J_3' = J_3 + \frac{1}{3}J_1 J_2 + \frac{2}{27}J_1^3 \; . \tag{2.28a,b}$$

In the theory of invariants the HAMILTON-CAYLEY *theorem* plays an important role. It states that

$$T_{ij}^{(3)} - J_1 T_{ij}^{(2)} - J_2 T_{ij} - J_3 \delta_{ij} = 0_{ij} \; , \tag{2.29}$$

where $T_{ij}^{(3)} \equiv T_{ip}T_{pr}T_{rj}$ and $T_{ij}^{(2)} \equiv T_{ip}T_{pj}$ are, respectively, the third and the second power of the tensor $\boldsymbol{T}$. Thus, every second-order tensor (*linear operator*) satisfies its own *characteristic equation* (2.25). BETTEN (1987c; 2001c) has proposed extended characteristic polynomials in order to find irreducible invariants for *fourth-order tensors* (Section 4.3.2).

By analogy with (2.19), a *fourth-order tensor* $\boldsymbol{A}$, having 81 components A_{ijkl}, can be interpreted as a linear operator:

$$Y_{ij} = A_{ijkl}X_{kl} \tag{2.30}$$

where X_{kl} and Y_{ij} are the cartesian components of the second-rank tensors $\boldsymbol{X}$ and $\boldsymbol{Y}$. For example, the constitutive equation

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{2.31}$$

describes the mechanical behavior of an anisotropic linear-elastic material, where σ_{ij} are the components of CAUCHY's stress tensor, ε_{ij} are the components of the infinitesimal strain tensor, and E_{ijkl} are the components (elatic constants) of the fourth-order material tensor characterising the *anisotropy* of the material.

2.2 General Bases

In the foregoing Section we have introduced an *orthonormal basis* $\{\boldsymbol{e}_i\}$ characterized by (2.2), i.e., we have restricted ourselves to rectangular cartesian coordinates. This is the simplest way to formulate the basic equations of continuum mechanics and the constitutive or evolutional equations of various materials. However, solving particular problems, it may be preferable to work in terms of more suitable coordinate systems and their associated bases.

In particular, cylindrical polar coordinates are useful for configurations which are symmetric about an axis, e.g., thick-walled tubes in Chapter 5. Another example is the system of spherical polar coordinates, which should be preferred when there is some symmetry about a point. Thus, it is useful to express the basic equations of continuum mechanics and the constitutive laws of several materials in terms of general (most curvlinear) coordinates. Thus, in the following some fundamentals of curvlinear tensor calculus should be discussed.

Let

$$x_i = x_i(\xi^p) \quad \Leftrightarrow \quad \xi^i = \xi^i(x_p) \tag{2.32}$$

be an admissible transformation of coordinates with JACOBIans

$$J \equiv |\partial x_i / \partial \xi^j| \quad \text{and} \quad K \equiv |\partial \xi^i / \partial x_j| \ ,$$

which does not vanish at any point of the considered region, then $JK = 1$. Further important properties of admissible coordinate transformations are discussed, for instance, by SOKOLNIKOFF (1964) and BETTEN (1974) in more detail.

The coordinates x_i in (2.32) referred to a right-handed orthogonal cartesian system of axes define a three-dimensional EUCLID*ean space* , while the ξ^i are *curvlinear coordinates* . Because of the admissible transformation (2.32), each set of values of x_i corresponds a unique set of values of ξ^i, and vice versa. The values ξ^i therefore determine points in the defined three-dimensional EUCLIDean space. Hence we may represent our space by the variables ξ^i instead by the cartesian system x_i, but the space remains, of course, EUCLIDean.

In this Section, we consider symbols characterized by one or several indices which may be either *subscripts* or *superscripts*, such as A_i, A^i, B_{ij}, B^{ij}, B^i_j, etc., where indices as superscripts are not taken as powers. Sometimes it is necessary to indicate the order of the indices when subscripts and superscripts occur together. In that case, for example, we write $A^i_{\bullet j}$ where the dot before j indicates that j is the second index while i is the first one.

As explained later, the values A_i and A^i can be considered as the *covariant* and the *contravariant components* , respectively, of the vector $\boldsymbol{A}$. However, the position of the indices on the "kernel" letters x and ξ in the *transformation of coordinates* (2.32) has nothing to do with *covariance* or *contravariance* and is therefore immaterial. In this context we refer to the following remarks of other authors:

- ❒ FUNG (1965, p38): The differential $d\theta^i$ is a contravariant vector, the set of variables θ^i itself does not transform like a vector. Hence, in this instance, the position of the index of θ^i must be regarded as without significance.
- ❒ GREEN/ZERNA (1968, pp5/6): The differentials $d\theta^i$ transform according to the law for contravariant tensors, so that the position of the upper index is justified. The variables θ^i themselves are in general neither contravariant nor covariant and the position of their index must be recognized as an exception. In future the index in non-tensors will be placed either above or below according to convenience. For example, we shall use either θ^i or θ_i.
- ❒ GREEN/ADKINS (1970, p1): The position of the index on coordinates x_i, y_i and θ^i is immaterial and it is convenient to use either upper or lower indices. The differential involving general curvilinear coordinates will always be denoted by $d\theta^i$ since $d\theta_i$ has a different meaning and is not a differential. For rectangular coordinates, however, we use either dx_i, dx^i for differentials, since $dx_i = dx^i$.
- ❒ MALVERN (1969, p603): **Warning**: Although the differentials dx^m are tensor components, the curvilinear coordinates x^m are not, since the coordinate transformations are general functional transformations and not the linear homogeneous transformations required for tensor components.

According to the above remarks, it is immaterial, if we write ξ^i or ξ_i. Since the differentials $d\xi^i$ transform corresponding to the law for contravariant tensors, the position of the upper index, ξ^i, is justified. Thus, the differential dT of a stationary scalar field $T = T(\xi^i)$ should be written as $dT = (\partial T/\partial \xi^i) d\xi^i$.

The position of a point P can be determined by x_i or, alternatively, by ξ^i as illustrated in Fig.2.1.

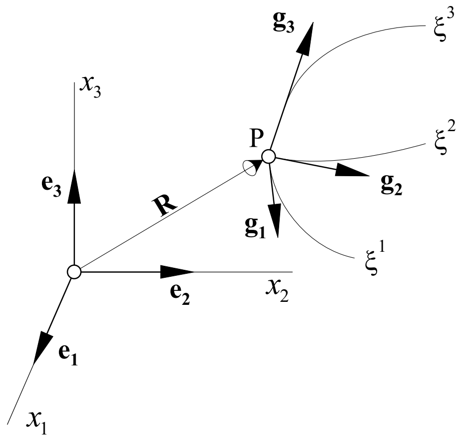

Fig. 2.1 Orthonormal and covariant base vectors

The position vector $\boldsymbol{R}$ of any point $P(x_i)$ can be decomposed in the form

$$\boldsymbol{R} = x_k \boldsymbol{e}_k \ . \tag{2.33}$$

Since the orthonormal base vectors $\boldsymbol{e}_i$ are independent of the position of the point $P(x_i)$, we deduce from (2.33) that

$$\partial \boldsymbol{R}/\partial x_i = \boldsymbol{e}_k (\partial x_k/\partial x_i) = \boldsymbol{e}_k \delta_{ki} = \boldsymbol{e}_i \ , \tag{2.34}$$

i.e., the orthonormal base vectors can be expressed by partial derivatives of the position vector $\boldsymbol{R}$ with respect to the rectangular cartesian coordinates x_i.

Analogous to (2.34), we find

$$\partial \boldsymbol{R}/\partial \xi^i = (\partial \boldsymbol{R}/\partial x_p) \left(\partial x_p/\partial \xi^i\right) = \boldsymbol{e}_p \left(\partial x_p/\partial \xi^i\right) = \boldsymbol{g}_i \ , \tag{2.35}$$

i.e., the geometrical meaning of the vector $\partial \boldsymbol{R}/\partial \xi^i$ is simple: it is a base vector directed tangentially to the ξ^i-coordinate curve. From (2.35) we observe that the *covariant base vectors* $\boldsymbol{g}_i$ are no longer independent of the coordinates ξ^i, in contrast to the orthonormal base vectors. Furthermore, they need not be mutually perpendicular or of unit length.

The inverse form to (2.35) is given by

$$\boldsymbol{e}_i \equiv \partial \boldsymbol{R}/\partial x_i = (\partial \boldsymbol{R}/\partial \xi^p)\,(\partial \xi^p/\partial x_i) = \boldsymbol{g}_p\,(\partial \xi^p/\partial x_i) \ . \tag{2.36}$$

Besides the *covariant base vectors* defined in (2.35), (2.36), a set of *contravariant base vectors* $\boldsymbol{g}^i$ is obtained from the constant unit vectors $\boldsymbol{e}_i \equiv \boldsymbol{e}^i$ as follows

$$\boldsymbol{g}^i = \left(\partial \xi^i/\partial x_p\right) \boldsymbol{e}^p \quad \Leftrightarrow \quad \boldsymbol{e}^i = (\partial x_i/\partial \xi^p)\, \boldsymbol{g}^p \ . \tag{2.37}$$

This set of contravariant base vectors, $\boldsymbol{g}^i$, are often called the *dual* or *reciprocal basis* of the *covariant basis* $\boldsymbol{g}_i$, and they are denoted by superscripts. In the special case of rectangular cartesian coordinates, the covariant and contravariant base vectors are identical ($\boldsymbol{e}_i \equiv \boldsymbol{e}^i$).

From (2.35), (2.37) and considering the orthonormal condition we arrive at the relation

$$\boxed{\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_{ij} \equiv \delta_i^j} \ . \tag{2.38}$$

between the two bases. For example, the contravariant base vector $\boldsymbol{g}^1$ is orthogonal to the two covariant vectors $\boldsymbol{g}_2$ and $\boldsymbol{g}_3$. Since these vectors directed tangentially to the ξ^2- and ξ^3-curves, the contravariant base vector $\boldsymbol{g}^1$ is perpendicular to the ξ^1-surface (Fig.2.2).

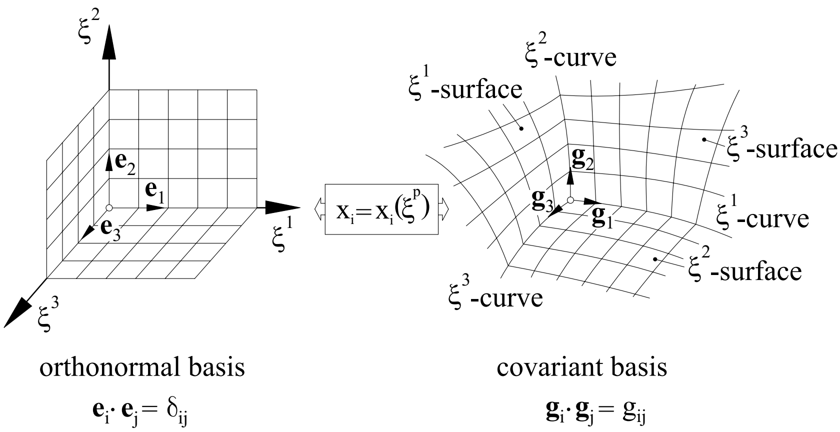

Fig. 2.2 Coordinate surfaces and base vectors

In addition to the relation (2.38) one can form the following scalar products:

$$\boxed{\boldsymbol{g}_i \cdot \boldsymbol{g}_j \equiv g_{ij}}\ , \qquad \boxed{\boldsymbol{g}^i \cdot \boldsymbol{g}^j \equiv g^{ij}}\ . \tag{2.39a,b}$$

These quantities, g_{ij} and g^{ij}, are called the *covariant* and *contravariant metric tensors* , respectively, and, because of (2.38), the KRONECKER *tensor* δ^i_j can be interpreted as a *mixed metric tensor* . The metric tensors (2.38), (2.39a,b) are symmetric since the scalar products of two vectors are commutative.

Inserting the base vectors (2.35) or (2.37) into (2.39a,b), respectively, we can express the *covariant* or *contravariant metric tensors* as

$$\boxed{g_{ij} = \left(\partial x_k/\partial \xi^i\right)\left(\partial x_k/\partial \xi^j\right)}\ \boxed{g^{ij} = \left(\partial \xi^i/\partial x_k\right)\left(\partial \xi^j/\partial x_k\right)}\ , \tag{2.40a,b}$$

from which we immediately arrive at the reciprocal relation

$$\boxed{g_{ik} g^{jk} = \delta^j_i}\ . \tag{2.41}$$

This result represents a system of linear equations from which the contravariant metric tensor can be calculated accordingly

$$g^{ij} = G^{ij}/g \quad \text{with} \quad G^{ij} \equiv (-1)^{i+j} U(g_{ij})\ , \tag{2.42}$$

when the covariant metric tensor is given, where G^{ij} is the *cofactor* of the element g_{ij} in the determinant $g \equiv |g_{ij}|$. From the reciprocal relation (2.41) we deduce the determinant of the contravariant metric tensor as $|g^{ij}| = 1/g$.

The magnitudes of the covariant and contravariant base vectors follow directly from (2.39a,b):

$$|\boldsymbol{g}_i| = \sqrt{\boldsymbol{g}_i \cdot \boldsymbol{g}_{(i)}} = \sqrt{g_{i(i)}}\ , \quad |\boldsymbol{g}^i| = \sqrt{\boldsymbol{g}^i \cdot \boldsymbol{g}^{(i)}} = \sqrt{g^{i(i)}}\ , \tag{2.43a,b}$$

where the index is not summed, as indicated by parentheses.

An increment $d\boldsymbol{R}$ of the position vector $\boldsymbol{R}$ in Fig.2.1 can be decomposed in the following ways

$$d\boldsymbol{R} = \boldsymbol{e}_i\, dx_i = \boldsymbol{g}_i\, d\xi^i = \boldsymbol{g}^i\, d\xi_i\ . \tag{2.44}$$

Forming the scalar product

$$\boldsymbol{g}^j \cdot \boldsymbol{e}_i \, dx_i = \boldsymbol{g}^j \cdot \boldsymbol{g}_i \, d\xi^i = \delta^j_i \, d\xi^i = d\xi^j \ ,$$

and then inserting (2.37), we find the transformation

$$d\xi^j = \left(\partial\xi^j / \partial x_p\right) \boldsymbol{e}^p \cdot \boldsymbol{e}_i \, dx_i = \left(\partial\xi^j / \partial x_p\right) dx_p$$

or

$$d\xi^i = \left(\partial\xi^i / \partial x_p\right) dx_p \ , \tag{2.45a}$$

i.e., the $d\xi^i$ in (2.44) transforms *contravariant* (matrix $\partial\xi^i/\partial x_p$) and can be identified with the usual total differential of the variable ξ^i, so that the use of upper index is justified.

In a similar way one can find the *covariant* transformation

$$d\xi_i = \left(\partial x_p / \partial\xi^i\right) dx_p \ , \tag{2.45b}$$

which essentially differs from (2.45a) and cannot be interpreted as the total differential.

Using (2.44) with (2.39a,b) the square of the line element ds can be written in the form

$$ds^2 = d\boldsymbol{R} \cdot d\boldsymbol{R} = dx_k \, dx_k = g_{ij} \, d\xi^i \, d\xi^j = g^{ij} \, d\xi_i \, d\xi_j \ . \tag{2.46}$$

Hence, the reason for the term *metric tensor* g_{ij} is account for. In addition to (2.46), the mixed form $ds^2 = d\xi_k d\xi^k$ is also possible. This form follows immediately from (2.46) because of the *rule* of *raising* ($g^{ij}A_j = A^i$) and *lowering* ($g_{ij}A^j = A_i$) the indices.

Considering the decompositions of two vectors

$$\boldsymbol{A} = A^k \boldsymbol{g}_k = A_k \boldsymbol{g}^k \ , \qquad \boldsymbol{B} = B^k \boldsymbol{g}_k = B_k \boldsymbol{g}^k \ , \tag{2.47a,b}$$

we then can represent the scalar product in the following forms

$$\boldsymbol{A} \cdot \boldsymbol{B} = g_{ij} A^i B^j = A^k B_k = A_k B^k = g^{ij} A_i B_j \ , \tag{2.48}$$

from which we can determine the length of a vector $\boldsymbol{B} \equiv \boldsymbol{A}$ as

$$|\boldsymbol{A}| \equiv A = \sqrt{g_{ij} A^i A^j} = \sqrt{A^k A_k} = \sqrt{A_k A^k} = \sqrt{g^{ij} A_i A_j} \ . \tag{2.49}$$

Alternatively, the scalar product (2.48) can be expressed by $AB \cos\alpha$, so that the angle between two vectors can be calculated from the following formula:

$$\cos\alpha = g_{ij} A^i B^j / (AB) \ . \tag{2.50}$$

In particular, the angle α_{12} between the ξ^1-curve and ξ^2-curve, i.e., between the covariant base vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ in Fig.2.2 can be determined in the following way

$$\left.\begin{array}{r}\underbrace{\boldsymbol{g}_1\cdot\boldsymbol{g}_2}\equiv g_{12}\\ |\boldsymbol{g}_1||\boldsymbol{g}_2|\cos\alpha_{12}\end{array}\right\}\quad\Rightarrow\quad\cos\alpha_{12}=\frac{g_{12}}{\sqrt{g_{11}g_{12}}}\;, \tag{2.51a}$$

where the relations (2.39a) and (2.43a) have been used. Cyclic permutations yield

$$\cos\alpha_{23}=\frac{g_{23}}{\sqrt{g_{22}g_{33}}}\quad\text{and}\quad\cos\alpha_{31}=\frac{g_{31}}{\sqrt{g_{33}g_{11}}}\;. \tag{2.51b,c}$$

From the result (2.51a,b,c) we deduce the following theorem:

> **A necessary and sufficient condition that a given curvlinear coordinate system be *orthogonal* is that the g_{ij} vanish for $i \neq j$ at every point in a region considered, i.e., the matrix (g_{ij}) has the diagonal form.**

According to (2.47a), there are two decompositions of a vector $\boldsymbol{A}$: The *contravariant components* A^k are the components of $\boldsymbol{A}$ in the directions of the *covariant base vectors* $\mathbf{g}_k$, while the *covariant components* A_k are the components of $\boldsymbol{A}$ corresponding with the *contravariant base* vectors $\boldsymbol{g}^k$, as illustrated in Fig.2.3.

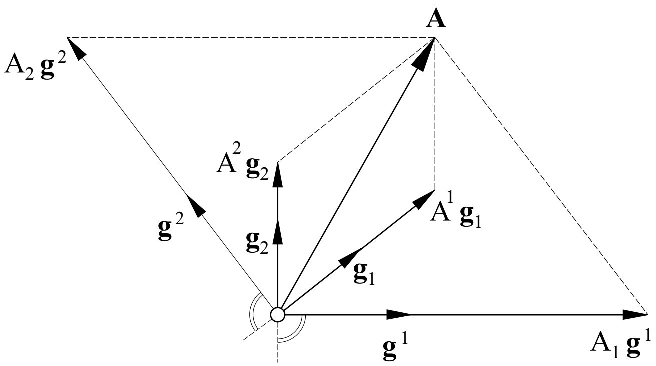

Fig. 2.3 Decomposition of a vector in covariant and contravariant components

A relation between the covariant and contravariant components, A_i and A^i, can be achieved by forming the scalar products of (2.47a), respectively, with the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ according to

$$\boxed{A_i = g_{ik} A^k} \quad \text{and} \quad \boxed{A^i = g^{ik} A_k} \, . \tag{2.52a,b}$$

These results express the *rule of lowering and raising the indices* , respectively. This operation can also be applied to the base vectors in order to find relations between the *covariant* and *contravariant bases* :

$$\boxed{\boldsymbol{g}_i = g_{ik} \boldsymbol{g}^k} \quad \text{and} \quad \boxed{\boldsymbol{g}^i = g^{ik} \boldsymbol{g}_k} \, . \tag{2.53a,b}$$

Comparing the decompositions (2.47a) with the decompositions

$$\boldsymbol{A} = \bar{A}_k \boldsymbol{e}^k \equiv \bar{A}^k \boldsymbol{e}_k \tag{2.54}$$

with respect to the orthonormal basis $\boldsymbol{e}_k \equiv \boldsymbol{e}^k$, and considering (2.36), (2.37), one arrives at the folllowing relations between the covariant or contravariant components, A_i or A^i, and the cartesian components $\bar{A}_k \equiv \bar{A}^k$ of the vector $\boldsymbol{A}$:

$$\boxed{A_i = \left(\partial x_p \,/ \partial \xi^i\right) \bar{A}_p} \, , \qquad \boxed{A^i = \left(\partial \xi^i \,/ \partial x_p\right) \bar{A}^p} \, . \tag{2.55a,b}$$

We see in (2.55a,b) the same transformation matrices $(\partial x_p / \partial \xi^i)$ and $(\partial \xi^i / \partial x_p)$, respectively, as in (2.35) and (2.37).

In the *tensor analysis* the *Nabla operator* ∇, sometimes called *del operator* , plays a fundamental role. It is a differential operator, and can be decomposed with respect to the orthonormal basis:

$$\nabla = \boldsymbol{e}_i \nabla_i \equiv \boldsymbol{e}_i \frac{\partial}{\partial x_i} \, . \tag{2.56a}$$

Substituting $\boldsymbol{e}_i \equiv \boldsymbol{e}^i$ in (2.56a) by (2.37), and utilizing the chain rule

$$\frac{\partial}{\partial x_i} = \frac{\partial \xi^p}{\partial x_i} \frac{\partial}{\partial \xi^p} \, ,$$

we find the decomposition of the *Nabla operator* with respect to the contravariant basis:

$$\boxed{\nabla = \boldsymbol{g}^k \frac{\partial}{\partial \xi^k}} \, . \tag{2.56b}$$

The *divergence* of a vector field $\boldsymbol{A}$ is defined as the scalar product of the Nabla operator (2.56b) and the vector (2.47a):

$$\boxed{\operatorname{div} \boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \partial A^i / \partial \xi^i + A^i \Gamma^j_{.ij} \equiv A^i|_i} \, . \tag{2.57}$$

The functions $T^k_{.ij}$ in (2.57) are called the CHRISTOFFEL *symbols of the second kind* . They are the coefficients in the following decompositions

$$\partial \boldsymbol{g}_i / \partial \xi^j \equiv \Gamma^k_{.ij} \boldsymbol{g}_k \equiv \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \boldsymbol{g}_k = \Gamma_{ijk} \boldsymbol{g}^k \ , \tag{2.58a}$$

$$\partial \boldsymbol{g}^i / \partial \xi^j \equiv -\Gamma^i_{.kj} \boldsymbol{g}^k \equiv - \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \boldsymbol{g}^k \ . \tag{2.58b}$$

Because of the definition (2.58a), and considering (2.35), (2.36), the CHRISTOFFEL *symbols of the second kind* can be expressed in the form

$$\Gamma^k_{.ij} = \frac{\partial \xi^k}{\partial x_p} \frac{\partial^2 x_p}{\partial \xi^i \partial \xi^j} \ . \tag{2.59a}$$

They are symmetric with respect to the lower indices (i, j) and can be related to the metric tensors in the following way

$$\Gamma^k_{.ij} \equiv \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \frac{1}{2} g^{kl} \left(\partial g_{il} \left/ \partial \xi^j \right. + \partial g_{jl} \left/ \partial \xi^i \right. - \partial g_{ij} \left/ \partial \xi^l \right. \right) \ . \tag{2.59b}$$

In addition, the CHRISTOFFEL *symbols of the first kind* are given by

$$\Gamma_{kij} \equiv [ij, k] = [ji, k] = \frac{1}{2} \left(\partial g_{ik} \left/ \partial \xi^j \right. + \partial g_{jk} \left/ \partial \xi^i \right. - \partial g_{ij} \left/ \partial \xi^k \right. \right) \ . \tag{2.60}$$

Comparing the two sets (2.59b) and (2.60), we read $g^{kl} \Gamma_{lij} \equiv \Gamma^k_{.ij}$ in agreement with the rule of raising the indices.

Evidently there are $n = 3$ distinct CHRISTOFFEL symbols of each kind for each independent g_{ij}, and, since the number of independent g_{ij}'s is

$$n(n+1)/2 \, ,$$

the number of independent CHRISTOFFEL symbols is

$$n^2(n+1)/2 \, .$$

Note that the CHRISTOFFEL symbols, in general, are not tensors. This is valid also for the partial derivatives $\partial A^i / \partial \xi^j$ and $\partial A_i / \partial \xi^j$ with respect to curvlinear coordinates.

However, the derivative

$$\partial \boldsymbol{A}/\partial \xi^j = A^i|_j \boldsymbol{g}_i = A_i|_j \boldsymbol{g}^i \tag{2.61}$$

is a tensor, where the expressions

$$A^i|_j \equiv \partial A^i/\partial \xi^j + A^k \Gamma^i_{.kj} \quad \text{and} \quad A_i|_j \equiv \partial A_i/\partial \xi^j - A_k \Gamma^k_{.ij} \qquad \text{(2.62a,b)}$$

are the *covariant derivatives* of the contravariant and covariant vector components, respectively. These derivatives transform like the components of a second-rank-tensor. The trace of (2.62a) immediately yields the divergence (2.57).

Let the vector $\boldsymbol{A}$ in (2.61) be the gradient of a scalar field. We then have

$$\partial \boldsymbol{A} \,/\partial \xi^j \,= \partial(\nabla \Phi)\,/\partial \xi^j \,= T_{ij} \boldsymbol{g}^i \,, \tag{2.63}$$

where

$$T_{ij} = T_{ji} = \frac{\partial^2 \Phi}{\partial \xi^i \partial \xi^j} - \Gamma^k_{.ij} \frac{\partial \Phi}{\partial \xi^k} \equiv \Phi_{,i}|_{\,j} \tag{2.64}$$

is a symmetric second-rank covariant tensor.

The LAPLACE *operator* Δ is defined as $\Delta \Phi = \operatorname{div} \operatorname{grad} \Phi$. Using (2.56b), (2.57), (2.58), (2.39b), and the abbreviation (2.64), the following relation is obtained:

$$\Delta \Phi = \nabla \cdot \nabla \Phi = \left(\boldsymbol{g}^i \frac{\partial}{\partial \xi^i} \right) \cdot \left(\boldsymbol{g}^k \frac{\partial \Phi}{\partial \xi^k} \right) = g^{ij} T_{ij} \equiv g^{ij} \; \Phi_{,i}|_{\,j} \,. \tag{2.65}$$

The *gradient of a vector* $\boldsymbol{A}$ is formed by the *dyadic product* of the Nabla operator (2.56b) and the field vector (2.47a) in connection with (2.62a,b):

$$\boldsymbol{T} \equiv \nabla \otimes \boldsymbol{A} = \left(\boldsymbol{g}^j \frac{\partial}{\partial \xi^j} \right) \otimes \left(A^i \boldsymbol{g}_i \right) \,=\, A^i\big|_{\,j} \boldsymbol{g}^j \otimes \boldsymbol{g}_i \equiv T^i_j \boldsymbol{g}^j \otimes \boldsymbol{g}_i \tag{2.66a}$$

$$\boldsymbol{T} \equiv \nabla \otimes \boldsymbol{A} = \left(\boldsymbol{g}^j \frac{\partial}{\partial \xi^j} \right) \otimes \left(A_i \boldsymbol{g}^i \right) \,=\, A_i\big|_{\,j} \boldsymbol{g}^j \otimes \boldsymbol{g}^i \equiv T_{ij} \boldsymbol{g}^j \otimes \boldsymbol{g}^i \,. \tag{2.66b}$$

In contrast to (2.64), the tensor $T_{ij} \equiv A_i|_j$ in (2.66b) is **not** symmetric. From the representations (2.66a,b) we read that the mixed components T^i_j and the covariant components T_{ij} of *dyadic* $\boldsymbol{T} \equiv \nabla \otimes \boldsymbol{A}$ are identical with the covariant derivatives (2.62a,b), respectively.

In curvlinear coordinates, the constitutive equation (2.31) of the linear theory of elasticity should be expressed in the form

$$\tau^{ij} = E^{ijkl}\gamma_{kl} \ , \tag{2.67}$$

where the infinitesimal strain tensor $\boldsymbol{\gamma}$ is formed by covariant derivatives from the displacement vector $\boldsymbol{w}$ according to

$$\gamma_{ij} = \left(w_i|_{\,j} + w_j|_{\,i} \right) /2 \ \ . \tag{2.68}$$

In the absence of body forces the divergence of the stress tensor must be equal to zero. Thus, the *equations of equilibrium* are then given by

$$\tau^{ij}\,|_i = 0^j \ . \tag{2.69}$$

Note that for applications in solid mechanics the *physical components* of the tensors τ_{ij}, γ_{ij} and w_i, used in (2.67), (2.68), (2.69), have to be calculated accordingly to

$$\sigma^{ij} = \tau^{ij}\sqrt{g_{(ii)}g_{(jj)}} \ , \ \varepsilon_{ij} = \gamma_{ij}\sqrt{g^{(ii)}g^{(jj)}} \ , \ u_i = w_i\sqrt{g^{(ii)}} \ , \tag{2.70a,b,c}$$

where the bracketed indices should not be summed.

Considering a vector (2.47a) the components of which, $A^1\boldsymbol{g}_1$, $A^2\boldsymbol{g}_2$, $A^3\boldsymbol{g}_3$, form the edges of the parallelepiped whose diagonal is $\boldsymbol{A}$. Since the $\boldsymbol{g}_i$ are **not** unit vectors in general, we see that the lengths of edges of this parallelepiped, or the *physical components* of $\boldsymbol{A}$, are determined by the expressions

$$A^1\sqrt{g_{11}} \ , \qquad A^2\sqrt{g_{22}} \ , \qquad A^3\sqrt{g_{33}} \ ,$$

since $g_{11} = \boldsymbol{g}_1 \cdot \boldsymbol{g}_1, \ldots, g_{33} = \boldsymbol{g}_3 \cdot \boldsymbol{g}_3$.

As an example, let us introduce cylindrical coordinates

$$x_1 = \xi^1 \cos\xi^2 \ , \qquad x_2 = \xi^1 \sin\xi^2 \ , \qquad x_3 = \xi^3 \ , \tag{2.71}$$

where $\xi^1 \equiv r$, $\xi^2 \equiv \varphi$ and $\xi^3 \equiv z$. The covariant base vectors (2.35) are then given by

$$\left.\begin{aligned} \boldsymbol{g}_1 &= \boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi \ , \\ \boldsymbol{g}_2 &= -\boldsymbol{e}_1 r \sin\varphi + \boldsymbol{e}_2 r \cos\varphi \ , \\ \boldsymbol{g}_3 &= \boldsymbol{e}_3 \ , \end{aligned}\right\} \tag{2.72a}$$

while the contravariant base vectors (2.37) are immediately found according to

$$\boldsymbol{g}^1 = \boldsymbol{g}_1 \ , \qquad \boldsymbol{g}^2 = \boldsymbol{g}_2 \,/ r^2 \ , \qquad \boldsymbol{g}^3 = \boldsymbol{g}_3 \ . \tag{2.72b}$$

Since the cylindrical coordinates (2.71) are orthogonal, the metric tensors (2.39a,b), (2.40a,b) have the diagonal forms

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.73a,b}$$

while the nonvanishing CHRISTOFFEL *symbols* (2.59a,b), (2.60) are given by

$$\varGamma_{122} \equiv -[12,2] = \varGamma^1_{.22} = -r, \quad \varGamma_{221} \equiv [21,2] = [12,2] = r, \quad \varGamma^2_{.12} = 1/r. \tag{2.74}$$

Taking the covariant derivative (2.62b) into account, the tensor (2.68) can also be represented in the following way

$$\gamma_{ij} = \left(\partial w_i \,/ \partial \xi^j + \partial w_j \,/ \partial \xi^i \right) /2 - w_k \varGamma^k_{.ij} \ . \tag{2.75}$$

The physical components (2.70c) of the displacement vector are in cylindrical coordinates because of (2.42b) very simple:

$$u_r = w_1 \ , \qquad u_\varphi = w_2/r \ , \qquad u_z = w_3 \ , \tag{2.76}$$

and, likewise, we calculate from (2.70b) the physical components

$$\left.\begin{array}{lll} \varepsilon_r = \gamma_{11} \ , & \varepsilon_\varphi = \gamma_{22}/r^2 \ , & \varepsilon_z = \gamma_{33} \ , \\ \varepsilon_{r\varphi} = \gamma_{12}/r \ , & \varepsilon_{rz} = \gamma_{13} \ , & \varepsilon_{z\varphi} = \gamma_{32}/r \ , \end{array}\right\} \tag{2.77}$$

of the infinitesimal strain tensor, so that we finally arrive at the following components

$$\left.\begin{array}{l} \varepsilon_r = \partial u_r \,/ \partial r \ , \quad \varepsilon_\varphi = \left(\partial u_\varphi \,/ \partial \varphi + u_r \right) /r \ , \quad \varepsilon_z = \partial u_z \,/ \partial z \ , \\ \varepsilon_{r\varphi} = \left[\left(\partial u_r \,/ \partial \varphi \right) /r + \partial u_\varphi \,/ \partial r - u_\varphi \,/ r \right] /2 \ , \\ \varepsilon_{rz} = (\partial u_r/\partial z + \partial u_z/\partial r)/2 \ , \\ \varepsilon_{z\varphi} = \left[\left(\partial u_z \,/ \partial \varphi \right) /r + \partial u_\varphi \,/ \partial z \right] /2 \ , \end{array}\right\} \tag{2.78}$$

by considering (2.74), (2.75), and (2.76).

Finally, the physical components of the stress tensor are deduced from (2.70a):

$$\left.\begin{aligned} \sigma_r &= \tau^{11} \,, \quad & \sigma_\varphi &= r^2\tau^{22} \,, \quad & \sigma_z &= \tau^{33} \,, \\ \sigma_{r\varphi} &= r\tau^{12} \,, \quad & \sigma_{\varphi z} &= r\tau^{23} \,, \quad & \sigma_{zr} &= \tau^{31} \,, \end{aligned}\right\} \tag{2.79}$$

so that we arrive from (2.69) by considering

$$A^{ij}|_k = \partial A^{ij} \Big/ \partial \xi^k + \Gamma^i_{.kp} A^{pj} + \Gamma^j_{.kp} A^{ip} \tag{2.80}$$

and (2.74) at the *equations of equilibrium*

$$\left.\begin{aligned} \partial\sigma_r /\partial r + (\partial\sigma_{r\varphi} /\partial\varphi)/r + \partial\sigma_{zr} /\partial z + (\sigma_r - \sigma_\varphi)/r &= 0 \,, \\ \partial\sigma_{r\varphi} /\partial r + (\partial\sigma_\varphi /\partial\varphi)/r + \partial\sigma_{\varphi z} /\partial z + 2\sigma_{r\varphi} /r &= 0 \,, \\ \partial\sigma_{zr} /\partial r + (\partial\sigma_{\varphi z} /\partial\varphi)/r + \partial\sigma_z /\partial z + \sigma_{zr} /r &= 0 \,. \end{aligned}\right\} \tag{2.81}$$

For *isotropic* materials the fourth-order elasticity tensor in (2.31) has the form

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \,, \tag{2.82a}$$

while the material tensor in (2.67) is represented by

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu\left(g^{ik} g^{jl} + g^{il} g^{jk}\right) \,, \tag{2.82b}$$

so that we find from (2.67) in connection with (2.77) and (2.79) the following constitutive equations:

$$\left.\begin{aligned} \sigma_r &= 2\mu\varepsilon_r + \lambda(\varepsilon_r + \varepsilon_\varphi + \varepsilon_z) \,, \\ \sigma_\varphi &= 2\mu\varepsilon_\varphi + \lambda(\varepsilon_r + \varepsilon_\varphi + \varepsilon_z) \,, \\ \sigma_z &= 2\mu\varepsilon_z + \lambda(\varepsilon_r + \varepsilon_\varphi + \varepsilon_z) \,, \\ \sigma_{r\varphi} &= 2\mu\varepsilon_{r\varphi} \,, \quad \sigma_{\varphi z} = 2\mu\varepsilon_{\varphi z} \,, \quad \sigma_{zr} = 2\mu\varepsilon_{zr} \,. \end{aligned}\right\} \tag{2.83}$$

According (2.65) together with (2.64), the LAPLACE *operator* is defined in general. In the special case of cylindrical coordinates with (2.73a,b), (2.74) this operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \,, \tag{2.84}$$

which occurs as a differential, for example, in the LAPLACE or POISSON equation. These partial differential equations are fundamental for many applications in continuum mechanic.

In the representation theory of tensor functions the *irreducible basic invariants*

$$S_1 = \delta_{ij}\bar{A}_{ji} \, , \qquad S_2 = \bar{A}_{ij}\bar{A}_{ji} \, , \qquad S_3 = \bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} \qquad \text{(2,85a,b,c)}$$

or, alternatively, the *irreducible main invariants*

$$J_1 \equiv S_1 \, , \quad J_2 = \left(S_2 - S_1^2\right)/2 \, , \qquad \text{(2.86a,b)}$$

$$J_3 = \frac{2S_3 - 3S_2S_1 + S_1^3}{6} \qquad \text{(2.86c)}$$

play a central role, since they form an *integrity basis* . In (2,85a,b,c) the $\bar{A}_{ij}$ are the components of the tensor $\boldsymbol{A}$ with respect to the orthonormal basis $\boldsymbol{e}_i$. The invariants (2,85a,b,c), e.g., can be expressed in terms of covariant or contravariant components of the tensor $\boldsymbol{A}$. To do this we need the following transformations

$$A_{ij} = \frac{\partial x_p}{\partial \xi^i}\frac{\partial x_q}{\partial \xi^j}\bar{A}_{pq} \quad \Leftrightarrow \quad \bar{A}_{ij} = \frac{\partial \xi^p}{\partial x_i}\frac{\partial \xi^q}{\partial x_j}A_{pq} \, , \qquad (2.87)$$

and

$$A^{ij} = \frac{\partial \xi^i}{\partial x_p}\frac{\partial \xi^j}{\partial x_q}\bar{A}_{pq} \quad \Leftrightarrow \quad \bar{A}^{ij} = \frac{\partial x_i}{\partial \xi^p}\frac{\partial x_j}{\partial \xi^q}A_{pq} \, , \qquad (2.88)$$

which are extended forms of (2.35), (2.36), and (2.37). Inserting the transformation (2.87) in (2,85a,b,c) and considering (2.40a,b), we finally find the *irreducible basic invariants* in the following forms:

$$S_1 = g^{pq}A_{pq} = g_{pq}A^{pq} = A_k^k \, , \qquad (2.89a)$$
$$S_2 = g^{ip}g^{jq}A_{ji}A_{pq} = g_{ip}g_{jq}A^{ji}A^{pq} = A_k^iA_i^k \, , \qquad (2.89b)$$
$$S_3 = g^{ip}g^{jq}g^{kr}A_{ij}A_{qk}A_{rp} = \cdots = A_j^iA_k^jA_i^k \, , \qquad (2.89c)$$

and then the *main invariants* (2.86a,b,c) also in terms of *covariant*, *contravariant* or *mixed tensor components*.

A lot of tensor operations are included in the **MAPLE tensor package**. Examples are illustrated in the following MAPLE program, where the metric tensors and the CHRISTOFFEL *symbols* have been calculated for *cylindrical* and *spherical* coordinates

$\boxed{\odot\ 2\_1.mws}$

```
> with(tensor):
> cylindrical_coord:=[r,phi,z]:
```

covariant metric tensor:

```
> g_compts:=array(symmetric,sparse,1..3,1..3):
```

```
> g_compts[1,1]:=1:  g_compts[2,2]:=r^2:
> g_compts[3,3]:=1:
> g:=create([-1,-1], eval(g_compts));
```

$$g := \mathrm{table}([compts = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, index\_char = [-1, -1]])$$

```
> D1g:=d1metric(g,cylindrical_coord):
> Gamma[kij]:=[ijk];

>     CHRISTOFFEL[first_kind][cylindrical]:=

>     Gamma[kij]=Christoffel1(D1g):
```

$$\Gamma_{kij} := [ijk]$$

The results are printed as a list on the CD-ROM. They are identical to those values in (2.74).

```
> spherical_coord:=[r,phi,theta]:
```

covariant metric tensor:

```
> g_compts:=array(symmetric,sparse,1..3,1..3):
> g_compts[1,1]:=1:  g_compts[2,2]:=r^2:
> g_compts[3,3]:=(r*sin(phi))^2:
> g:=create([-1,-1], eval(g_compts));
```

$$g := \mathrm{table}([compts = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin(\phi)^2 \end{bmatrix}, index\_char = [-1, -1]])$$

```
> D1g:=d1metric(g,spherical_coord):
> Gamma[kij]:=[ijk];

>     CHRISTOFFEL[first_kind][spherical]:=

>     Gamma[kij]=Christoffel1(D1g):
```

$$\Gamma_{kij} := [ijk]$$

The results are printed as a list on the CD-ROM. In a similar way one can find the CHRISTOFFEL *symbols* of the second kind.