Group Decision Making with Soft AHP Based on the Random Set View of Fuzzy Sets

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Abstract. We present a variant of the Analytic Hierarchy Process intended to facilitate consensus search in group decision making. This soft methodology combines fuzzy sets and probabilistic information to provide judgements oriented by the actors' attitude towards negotiation. A Monte Carlo approach is taken to derive a preference structure distribution which should finally be studied to extract knowledge about the resolution process.

1 Introduction

The Analytic Hierarchy Process [15], AHP, is a multicriteria decision making technique that provides in an absolute scale the priorities corresponding to the alternatives being compared in a context with multiple scenarios, multiple actors and multiple criteria.

Its methodology consists on three stages: (i) modelling, (ii) valuation and (iii) priorization and synthesis. In the first stage, a hierarchy of the relevant aspects of the problem is constructed. In the second stage, the preferences of the actors involved in the resolution process are elicited by means of reciprocal pairwise comparison matrices using judgements based on the fundamental scale $\{1/9, 1/8, \ldots, 1, \ldots, 8, 9\}$ proposed by Saaty [15]. Finally, in the third stage, local, global and total priorities are obtained.

In its initial formulation (Conventional AHP), the valuation process is a deterministic one. However, most real applications require considering actors' uncertainty when comparing tangible and intangible aspects. There are a number of procedures to deal with the uncertainty inherent in the judgement elicitation process. Interval judgements [12], reciprocal random distributions [6] and fuzzy numbers [10, 4, 5] are some of the most extended procedures.

Using fuzzy judgements to elicit the actors' preferences, we present a new approach to include the actors' attitude in the negotiation process in AHP-group decision making. This complements, in the fuzzy setting, the recent Bayesian approach to Stochastic AHP in [2]. To incorporate their attitude, we associate a probability distribution to the α-level parameter. Jointly considered, the fuzzy judgements and the α-level distributions are built into a soft AHP allowing us to deal with the AHP-group decision making problem in a more realistic and effective way than traditional approaches [16, 14, 9].

The two traditional approaches followed in AHP-group decision making are aggregation of individual judgements (AIJ) and aggregation of individual priorities (AIP). Other approaches can be found in [7, 13]. In the AIJ procedure, a judgement matrix for

D. Dubois et al. (Eds.): Soft Methods for Hand. Var. and Imprecision, ASC 48, pp. 427–434, 2008.
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the group is constructed from the individual judgements and group priorities are calculated from it. The three most commonly used methods to determine the entries in the group judgement matrix are consensus, voting and aggregation of judgements. In AIP, group priorities are computed from individual priorities using an aggregation method. In both procedures, the most widely used aggregation technique is the weighted geometric mean.

The paper is structured as follows. After this brief Introduction, Section 2 describes in an intuitive way the new approach based on the well-known relationships between fuzzy sets and random sets. Section 3 explains how individual judgements for the negotiation are elicited. Section 4 shows how to obtain the preference structure distribution on the possible rankings of the alternatives. Finally, Section 5 suggests how to exploit this distribution from a learning perspective.

2 A Soft AHP Approach

The proposed approach extends AHP methods which allow imprecise judgements in the form of real intervals. That extension to group decision problems aims at incorporating the actors' attitude towards negotiation with emphasis on consensus search. The basic notion is that actors may accept enlarging their interval judgements, moving farther from their personal judgement, in an attempt to find overlap areas of larger compatibility with the others' views. The role of the analyst is to facilitate the process and extract knowledge from the problem resolution.

Briefly, the steps are as follows.

First, actors elicit pairwise comparison matrices whose elements are fuzzy intervals. These basic judgements fix the framework for the process, establishing the less imprecise position matching to the actor's ideas, a more imprecise interval with the maximal admissible concessions and a continuum of intermediate positions. The underlying fuzzy sets semantics is that of preference: the membership function denotes how well a number qualifies as an acceptable quantification of the actor's judgement of relative importance.

Second, for a specific negotiation process, the actors decide, on the basis of subjective factors and interests, the kind of position to be adopted: tougher or more open. That attitude towards the negotiation is represented by a probability distribution on the interval $[0,1]$ of membership values. The negotiation weight distribution assesses more weight to the positions more comfortable or convenient to the actor.

Within a fixed context, e.g. in decisions repeated over time, basic judgements may remain the same while the negotiation weight distribution varies with the circumstances of each negotiation.

Third, both kinds of information are fused, using the notions from random set theory and its connection to fuzzy sets, so that basic judgements are revised yielding adequate negotiation judgements. Now, the correct semantic interpretation of these 'posterior' fuzzy judgements is the possibility semantics.

Interval methods cannot capture the graduality which appears naturally in this setting. The proposed method is soft since it conjugates several approaches to imprecision and uncertainty, even with different semantic interpretations. Since it does not rely exclusively on Fuzzy Set Theory and does not lead to fuzzy priorities, we would call it a ‘Soft AHP’ method, rather than a ‘Fuzzy AHP’ method.

The next stage concern the aggregation of the actors’ views and the analysis of the possible preference structures are more supported by the imprecise judgements emitted. The preference structure distribution should be analyzed to gain understanding of the decision and negotiation process and detect patterns, something harder or not possible in methods which ultimately provide a single ranking.

3 Individual Negotiation Judgements

We start with m actors who approach the decision process willing to make a consensus decision. Each actor has a weight $\beta_k > 0$ in the decision, with $\sum_k \beta_k = 1$. For simplicity of presentation, we assume a local context, i.e. a single criterion. The input for the analysis is a matrix of fuzzy pairwise comparisons between alternatives and a probability distribution encoding each actor’s potential framework and attitude towards negotiation, respectively.

Each actor provides a matrix $A^{[k]} = [a_{ij}^{[k]}]_{i,j=1}^{n}$, *whose entries are fuzzy intervals.*

The core and support of the fuzzy interval represent the most and less restrictive positions actor k might be willing to adopt. Thus we also call it a *basic judgement*. The remaining α-cuts

$$(a_{ij}^{[k]})_\alpha = \{x \mid a_{ij}^{[k]}(x) \geq \alpha\}$$

represent intermediate positions between those two extremes. As α approaches 0, consensus becomes easier since the overlap between the actors’ positions increases.

In practice, the $a_{ij}^{[k]}$ can often be chosen to be trapezoidal fuzzy sets for half the entries of $A^{[k]}$, but not for all of them since that would lead to a violation of the reciprocity property of AHP. The other entries are determined by reciprocity between $a_{ij}^{[k]}$ and $a_{ji}^{[k]}$, so that, for each fixed $\alpha \in [0,1]$,

$$\min(a_{ij}^{[k]})_\alpha = (\max(a_{ji}^{[k]})_\alpha)^{-1}, \quad \max(a_{ij}^{[k]})_\alpha = (\min(a_{ji}^{[k]})_\alpha)^{-1}.$$

Each actor provides a probability distribution $P^{[k]}$ *on the real interval* $[0,1]$.

After basic judgements are elicited, for one particular negotiation each actor chooses a distribution on $[0,1]$ according to his specific attitude towards that negotiation. This *negotiation weight distribution* can be given in the form of a density function with support $[0,1]$ and determines the sort of position which will be given more preponderance in the negotiation. Distributions concentrated close to 1 represent tougher positions with little room for concessions, while distributions concentrated close to 0 represent very open positions primarily willing to ease consensus, even if reached farther from the actor’s ideal position.

Intuitively, the density function should be unimodal in the sense of being a quasiconvex function. It represents weighting, rather than random behaviour.

For clarity of presentation, assume that the mode $m^{[k]}$ is unique. Then, $(a_{ij}^{[k]})_{m^{[k]}}$ represents the central position of actor k. Level sets $(a_{ij}^{[k]})_\alpha$ for $\alpha < m^{[k]}$, being longer intervals, represent less demanding positions given a lesser weight by the negotiation weight distribution $P^{[k]}$. Actors may or may not choose $m^{[k]} = 1$. In general, they need not, since it is unclear that an actor's dominating attitude in judgement modelling will bring the decision process to a more satisfactory conclusion for him. Inversely, α-cuts for $\alpha > m^{[k]}$ represent more stringent positions which are given smaller weight too.

Once both $A^{[k]}$ and $P^{[k]}$ are fixed, we must merge those two pieces of information. We will do that by using consonant random sets.

Indeed, each fuzzy basic judgement $a_{ij}^{[k]}$, together with the negotiation weight distribution $P^{[k]}$, easily provides a random set (a random interval) which is consonant, i.e. monotonic. We just have to take the level mapping $L_{ij}^{[k]}$ defined on the interval $[0,1]$, endowed with the probability measure $P^{[k]}$, and with interval values given by

$$L_{ij}^{[k]}(\alpha) = (a_{ij}^{[k]})_\alpha, \quad \alpha \in [0,1].$$

It must be stressed that different choices of basic judgements and negotiation weight distribution may encode the same information. For any increasing bijective transformation $\phi : [0,1] \to [0,1]$, the pair $([\phi \circ a_{ij}^{[k]}]_{i,j}; P^{[k]} \circ \phi^{-1})$ represents the same information as $(A^{[k]}; P^{[k]})$. Therefore, the procedure is invariant under increasing bijective transformations of the scale interval $[0,1]$, a nice property from the measurement-theoretical point of view.

In turn, all the information of that random set is contained in its one-point coverage function $\pi_{ij}^{[k]}$ given by

$$\pi_{ij}^{[k]}(x) = P^{[k]}(x \in a_{ij}^{[k]}).$$

We call $\pi_{ij}^{[k]}$ a *negotiation judgement* or *final judgement*. Observe that $\pi_{ij}^{[k]}$ can be reinterpreted as a fuzzy set, by invoking again the connection between random sets and fuzzy sets.

Let us show how negotiation judgements combine the information in $a_{ij}^{[k]}$ and $P^{[k]}$.

Denote by $F^{[k]}$ the distribution function of $P^{[k]}$. Then, one can prove that

$$\pi_{ij}^{[k]}(x) = F^{[k]}(\max\{\alpha \in [0,1] \mid x \in (a_{ij}^{[k]})_\alpha\}) = F^{[k]}(a_{ij}^{[k]}(x)).$$

If $P^{[k]}$ is given by a density function with full support, as seems reasonable, then $F^{[k]}$ is invertible and a classical theorem of Probability Theory tells us that $P^{[k]} \circ (F^{[k]})^{-1}$ is a uniform distribution in $[0,1]$. Therefore, we have

$$\pi_{ij}^{[k]} = F^{[k]} \circ a_{ij}^{[k]},$$

$$P^{[k]} \circ (F^{[k]})^{-1} \sim \mathscr{U}[0,1].$$

Taking $\phi = F^{[k]}$ above, we deduce that the pair $([\pi_{ij}^{[k]}]_{i,j}; \mathscr{U}[0,1])$ contains the same information as the original pair $(A^{[k]}; P^{[k]})$. But since the uniform distribution gives equal weight to each α, all the information is now in the $\pi_{ij}^{[k]}$.

It is possible to compare the actors' positions via $\pi_{ij}^{[k]}$, since it recasts the information in a common scale, with uniform weighting for all actors. In this representation, a fuzzy set very steep in the area surrounding the $F^{[k]}(m^{[k]})$-cut means that actor k strongly wishes to remain close to his central position, while a more flexible position would be characterized by a fast 'opening' towards larger intervals for $\alpha < F^{[k]}(m^{[k]})$.

In order to simplify the elicitation process, the analyst may predetermine the shape of the fuzzy intervals and the density function so that only a few parameters, easily interpretable, are left for actors to specify. One possible way is as follows.

The frame judgements $a_{ij}^{[k]}$ are taken to be trapezoidal, so that only the end-points of their core and support must be elicited. Note that both intervals need not have the same center. In some situations, it may be easier to elicit the support end-points indirectly by indicating the percentage of the corresponding core end-point the actor might eventually be willing to concede.

For the negotiation weight distribution $P^{[k]}$, the simplest choice is a triangular distribution, which is determined once the mode $m^{[k]}$ is specified. The value $m^{[k]}$ reflects intuitively the attitude toward negotiation, with $m^{[k]} = 1$ representing a tough attitude and $m^{[k]} = 0$ a fully open one. Trapezoidal distributions are possible as well.

Another possibility for the $P^{[k]}$ is the beta $\beta(p,q)$ family of distributions. Appropriate choices of p,q control not only the position of the center of the distribution but also its dispersion around the actor's central position.

4 The Preference Structure Distribution

Our final aim is to quantify how much support receives each possible ranking (preference structure) of the alternatives in view of the information collected so far. A way to overcome the difficulty to solve the problem analitically is to simulate by Monte Carlo methods many crisp judgement matrices which are compatible with the positions expressed by the actors.

We begin by choosing a random value η in [0,1] according to a uniform distribution. For each k, we select the η quantile of the negotiation weight distribution $P^{[k]}$,

$$q^{[k]} = (F^{[k]})^{-1}(\eta).$$

Then we perform simulations to select crisp values

$$\xi_{ij}^{[k]} \in (a_{ij}^{[k]})_{q^{[k]}} = (\pi_{ij}^{[k]})_{\eta}.$$

It is enough to simulate only for those entries of the matrix which were directly chosen by actor k. For the rest of the matrix, reciprocity is enforced by the relationship

$$\xi_{ji}^{[k]} = (\xi_{ij}^{[k]})^{-1}.$$

A uniform distribution or another distribution, if deemed appropriate, can be used. That overcomes some problems with reciprocity appearing in many variants of Fuzzy AHP.

The latter part is analogous to known stochastic methods to solve AHP with imprecise judgements [12]. The computational complexity is the same as for those interval methods, since obtaining a crisp judgement matrix involves $(n-1)n/2$ simulations per actor and only one additional simulation is needed to fix $q^{[k]}$.

Note that simulation is applied at a different height level $q^{[k]}$ for each actor. In the long run, the Glivenko-Cantelli Theorem ensures that, for each actor, the empirical distribution approximates the weights provided by actor k, as the number of simulations increases.

Once Monte Carlo judgement matrices are obtained, well-established methods for Group Decision Making with AHP can be used to obtain the preference structure distribution. For the sake of completeness, we describe a possible continuation of the analysis until its conclusion.

Each actor k and each simulated crisp judgement matrix $[\xi_{ij}^{[k]}]_{i,j}$ provide a vector of priority values for the alternatives. There are several methods for obtaining the priorities, and several ways to aggregate individual preferences. We suggest the methods based on the geometric mean, for their good properties in the group decision setting. Barzilai and Golany [3] proved that AIJ using the weighted geometric mean method (WGGM) followed by derivation of priorities by the rowwise geometric mean method (RGGM) yields the same result than derivation of priorities by RGGM followed by AIP by WGGM. Moreover, Escobar et al. [8] showed that AIJ has good properties with respect to consistency, in that the aggregate judgement matrix tends to decrease the inconsistency levels of the less consistent actors.

For instance, in the AIJ method we calculate the matrix Ξ^G of aggregate group judgements

$$\xi_{ij}^G = \prod_{k=1}^{m} (\xi_{ij}^{[k]})^{\beta_k}, \quad i, j = 1, \ldots, n.$$

Then, priorities for the alternatives are derived as

$$\omega_i^G = \prod_{k=1}^{m} (\xi_{ij}^G)^{1/k}, \quad i = 1, \ldots, n.$$

Alternatives are ranked according to the values ω_i^G. There are $n!$ possible rankings or preference structures, which can be identified with permutations of n elements.

After sufficiently many simulations, we end up with an empirical distribution on preference structures. For each possible preference structure $\mathscr{R}$, it gives us the proportion $\lambda_{\mathscr{R}}$ of samples leading to that ranking.

5 Exploiting the Model

From the standpoint that we should seek to extract knowledge from the resolution of the decision problem, the preference structure distribution contains rich information which should be explored in search for patterns, see [7].

Visual methods for representing the group information, e.g. [17], provide a starting point for exploring the preference structures. Individual preferences can be compared to group preferences to detect similarities and patterns. Graphical and statistical tools such as clustering, fuzzy clustering and multidimensional scaling are appropriate for this stage of the analysis, see e.g. [11]. Our research group (GDMZ) is currently working in this area with application to large e-democracy and e-cognocracy decision problems.

A reasonable approach to synthesizing the information in the preference structure distribution goes by applying voting methods well-studied in Social Choice Theory. In order to take into account the information contained in individual rankings, methods using the whole ranking seem more appropriate. An example is the Borda count method, other methods are available.

With the Borda method, the best alternative in a preference structure is given n points, the second best $n-1$ points, and so on. Each preference structure has its weight given in the preference structure distribution, resulting

$$v_i = \sum_{\mathscr{R}} \lambda_{\mathscr{R}}[(n+1) - \mathscr{R}(i)], \quad i = 1, \ldots, n.$$

Alternatives can be ranked or chosen according to the values v_i, which result from aggregation over all $n!$ preference structures.

An alternative to AIJ and AIP allowing interval judgements is the AIPS (aggregation of individual preference structures) method in [7]. In that paper, preference structures are calculated for each actor, then aggregated, allowing to compare each actor's preference structure to the group's.

Acknowledgement. Research partially supported by the Spanish *Ministerio de Educación y Ciencia* (TSI2005–02511 and MTM2005–02254), and the *Gobierno de Aragón* (PM2007–034).

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