Relating Prototype Theory and Label Semantics

Jonathan Lawry[1] and Yongchuan Tang[2]

[1] Department of Engineering Mathematics, University of Bristol, Bristol, United Kingdom
[2] College of Computer Science, Zhejiang University, Hangzhou, P.R. China

Abstract. An interpretation of the label semantics framework is introduced based on prototype theory. Within this interpretation it is shown that the appropriateness of an expression is characterised by an interval constraints on a parameter ε. Here ε is an uncertain distance threshold according to which an element x is sufficiently close to the prototype p_i of a label L_i for L_i to be deemed appropriate to describe x, if the distance between x and p_i is less than or equal to ε. Appropriateness measures and mass functions are then defined in terms of an underlying probability density function δ on ε.

1 Introduction

In classical logic a concept label L is defined by the set of elements from an underlying universe which satisfies L (the extension of L) and more generally in Kripke semantics [5] as a mapping from a set of possible worlds into sets of elements (an interpretation of L). Such an approach fails to capture certain aspects of our intuitive understanding of concepts in natural language, in particular the role of similarity in establishing the meaning of concept labels. Furthermore, a possible worlds model seems to overlook our natural focus on understanding reality as represented by one particular possible world (see [3] for discussion).

Prototype theory (Rosch [9]) is an alternative approach to concept representation according to which decisions regarding the applicability of a concept label to a particular instance are made on the basis of the similarity of that instance to a (set of) prototypical element(s) for that concept. Prototypes may not correspond to actual perceptions of objects or experiences but instead may identify a particular point or region of conceptual space [3] which is in some way representative of the concept. From this perspective the human ability to rank elements in terms of the degree to which they satisfy a concept L can be explained in terms of a comparison of their relative similarity (or distance) from the prototype(s) for L.

Prototype theory has been proposed as the basis for a possible interpretation of membership functions in fuzzy set theory ([1, 2]), where the membership of an element x in a concept L is taken to be a scaled version of the similarity between x and the prototype(s) for L [10]. This rather intuitive approach has the drawback that the prototype similarity interpretation of membership does not naturally result in a truth-functional calculus when concepts are combined (See Lawry [7] chapter 2 for a discussion). Consequently a prototype based model of membership does not seem to capture the underlying calculus of fuzzy set theory.

D. Dubois et al. (Eds.): Soft Methods for Hand. Var. and Imprecision, ASC 48, pp. 35–42, 2008.
springerlink.com © Springer-Verlag Berlin Heidelberg 2008

Label semantics (Lawry [6, 7, 8]) is an uncertainty theory for vague concepts which encodes the meaning of linguistic labels according to how they are used by a population of communicating agents to convey information. From this perspective, the focus is on the decision making process an intelligent agent must go through in order to identify which labels or expressions can actually be used to describe an object or value. In other words, in order to make an assertion describing an object in terms of some set of linguistic labels, an agent must first identify which of these labels are appropriate or assertible in this context. Given the way that individuals learn language through an ongoing process of interaction with the other communicating agents and with the environment, then we can expect there to be considerable uncertainty associated with any decisions of this kind. In label semantics we quantify this uncertainty in terms of appropriateness measures, linked to an associated mass function through a calculus which, while not truth-function, can be functional in a weaker sense (See Lawry [6] and [7]). In the sequel we will propose a prototype theory interpretation of label semantics which relates both appropriateness measures and mass functions to distance from prototypes and naturally captures the label semantics calculus.

2 An Overview of Label Semantics

The underlying philosophy of label semantics [8] is very close to the epistemic view of vagueness as expounded by Timothy Williamson [12]. Williamson assumes that for the extension of a vague concept there is a precise but unknown dividing boundary between it and the extension of the negation of that concept. However, while there are marked similarities between the epistemic theory and the label semantics view, there are also some subtle differences. For instance, the epistemic view would seem to assume the existence of some objectively correct, but unknown, definition of a vague concept. Instead of this we argue that individuals when faced with decision problems regarding assertions find it useful as part of a decision making strategy to assume that there is a clear dividing line between those labels which are and those which are not appropriate to describe a given instance. We refer to this strategic assumption across a population of communicating agents as the *epistemic stance*, a concise statement of which is as follows:

> *Each individual agent in the population assumes the existence of a set of labelling conventions, valid across the whole population, governing what linguistic labels and expressions can be appropriately used to describe particular instances.*

The idea is that the learning processes of individual agents, all sharing the fundamental aim of understanding how words can be appropriately used to communicate information, will eventually converge to some degree on a set of shared conventions. The very process of convergence then to some extent vindicates the epistemic stance from the perspective of individual agents.

Label semantics proposes two fundamental and inter-related measures of the appropriateness of labels as descriptions of an object or value. We begin by assuming that for all agents there is a fixed shared vocabulary in the form of a finite set of basic labels

LA for describing elements from the underlying universe Ω. A countably infinite set of expressions LE can then be generated through recursive applications of logical connectives to the basic labels in LA. The measure of appropriateness of an expression $\theta \in LE$ as a description of instance x is denoted by $\mu_\theta(x)$ and quantifies the agent's subjective probability that θ can be appropriately used to describe x. From an alternative perspective, when faced with describing instance x, an agent may consider each label in LA and attempt to identify the subset of labels that are appropriate to use. This is a totally meaningful endeavour for agents who adopt the epistemic stance. Let this complete set of appropriate labels for x be denote by $\mathscr{D}_x$. In the face of their uncertainty regarding labelling conventions agents will also be uncertain as to the composition of $\mathscr{D}_x$, and we represent this uncertainty with a probability mass function $m_x : 2^{LA} \rightarrow [0,1]$ defined on subsets of labels. We now provide formal definitions for the set of expressions LE and for mass functions m_x, following which we will propose a link between the two measures $\mu_\theta(x)$ and m_x for expression $\theta \in LE$.

Definition 1 (Label Expressions)
The set of label expressions LE generated from LA, is defined recursively as follows: If $L \in LA$ *then* $L \in LE$*; If* $\theta, \varphi \in LE$ *then* $\neg\theta, \theta \wedge \varphi, \theta \vee \varphi \in LE$.

Definition 2 (Mass Function on Labels)
$\forall x \in \Omega$ *a mass function on labels is a function* $m_x : 2^{LA} \rightarrow [0,1]$ *such that* $\sum_{S \subseteq LA} m_x(S) = 1$.

Note that there is no requirement for the mass associated with the empty set to be zero. Instead, $m_x(\emptyset)$ quantifies the agent's belief that none of the labels are appropriate to describe x. We might observe that this phenomena occurs frequently in natural language, especially when labelling perceptions generated along some continuum. For example, we occasionally encounter colours for which none of our available colour descriptors seem appropriate. Hence, the value $m_x(\emptyset)$ is an indicator of the describability of x in terms of the labels LA.

The link between the mass function m_x and the appropriateness measures $\mu_\theta(x)$ is motivated by the intuition that the assertion 'x is θ' directly provides information dependent on θ, as to what are the possible values for $\mathscr{D}_x$. For example, the assertion 'x is *blue*' would mean that *blue* is an appropriate label for x, from which we can infer that $blue \in \mathscr{D}_x$. Similarly, the assertion 'x is *green and not blue*' would mean that *green* is an appropriate label for x while *blue* is not, so that we can infer $green \in \mathscr{D}_x$ and $blue \notin \mathscr{D}_x$. Another way of expressing this information is to say that $\mathscr{D}_x$ must be a member of the set of sets of labels which contain *green* but do not contain *blue* i.e. $\mathscr{D}_x \in \{S \subseteq LA : green \in S,\ blue \notin S\}$. More generally, we can define a functional mapping λ from LE into $2^{2^{LA}}$ (i.e. the set containing all possible sets of label sets) for which the assertion 'x is θ' enables us to infer that $\mathscr{D}_x \in \lambda(\theta)$. This mapping is defined recursively as follows:

Definition 3 (λ-mapping)
$\lambda : LE \rightarrow 2^{\mathscr{F}}$ *is defined recursively as follows:* $\forall L \in LA$, $\forall \theta,\ \varphi \in LE$; $\lambda(L) = \{S \in \mathscr{F} : L \in S\}$; $\lambda(\theta \wedge \varphi) = \lambda(\theta) \cap \lambda(\varphi)$; $\lambda(\theta \vee \varphi) = \lambda(\theta) \cup \lambda(\varphi)$; $\lambda(\neg\theta) = \lambda(\theta)^c$.

The λ-mapping then provides us with a means of evaluating the appropriateness measure of an expression θ directly from m_x, as corresponding to the subjective belief that $\mathscr{D}_x \in \lambda(\theta)$ so that:

Definition 4 (Appropriateness Measures)
For any expression $\theta \in LE$ and $x \in \Omega$, the appropriateness measure $\mu_\theta(x)$ can be determined from the mass function m_x according to:

$$\forall \theta \in LE \quad \mu_\theta(x) = \sum_{S \in \lambda(\theta)} m_x(S)$$

From this relationship the following list of general properties hold for expressions θ and φ in LE [6]:

Theorem 1 (Lawry [6, 7])

- *If $\theta \models \varphi$ then $\forall x \in \Omega\ \mu_\theta(x) \leq \mu_\varphi(x)$*
- *If $\theta \equiv \varphi$ then $\forall x \in \Omega\ \mu_\theta(x) = \mu_\varphi(x)$*
- *If θ is a tautology then $\forall x \in \Omega\ \mu_\theta(x) = 1$*
- *If θ is a contradiction then $\forall x \in \Omega\ \mu_\theta(x) = 0$*
- $\forall x \in \Omega\ \mu_{\neg\theta}(x) = 1 - \mu_\theta(x)$

Notice, here that the laws of excluded middle, non-contradiction and idempotence are all preserved.

In practice an agent's estimation of both m_x and $\mu_\theta(x)$ should depend on their experience of language use involving examples similar to x. Clearly the form of this knowledge is likely to be both varied and complex. However, one natural type of assessment for an agent to make would be to order or rank label in terms of their estimated appropriateness for x. This order information could then be combined with estimates of appropriateness measure values for the basic labels (i.e. elements of LA) in order to provide estimates of values for compound expressions (i.e. elements of LE).

Definition 5 (Ordering on Labels)
For $x \in \Omega$ let $\preceq_x$ be an ordering on LA such that for $L, L' \in LA$, $L' \preceq_x L$ means that L is at least as appropriate as a label for x as L'.

For any labels $L_i, L_j \in LA$ if $L_i \preceq_x L_j$ it follows that if $L_j \in \mathscr{D}_x$ then $L_i \in \mathscr{D}_x$ and consequently when $\preceq_x$ is a total ordering then the mass function m_x must be nested. In that case the following theorem shows that the min and max rules for conjunction and disjunction hold for a restricted class of expressions:

Theorem 2 ([6, 11])
Let $LE^{\wedge,\vee} \subseteq LE$ denote those expressions generated recursively from LA using only the connectives $\wedge$ and $\vee$. If the appropriateness of the basic labels as descriptions for x is ranked according to a total ordering $\preceq_x$ on LA then $\forall \theta, \varphi \in LE^{\wedge,\vee}$ it holds that $\mu_{\theta\wedge\varphi}(x) = \min\left(\mu_\theta(x), \mu_\varphi(x)\right)$, $\mu_{\theta\vee\varphi}(x) = \max\left(\mu_\theta(x), \mu_\varphi(x)\right)$.

3 A Prototype Theory Interpretation of Label Semantics

Suppose that a distance metric d is defined on Ω such that $d : \Omega^2 \rightarrow [0, \infty)$ and satisfies $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all elements $x, y \in \Omega$. For each label $L_i \in LA$ let

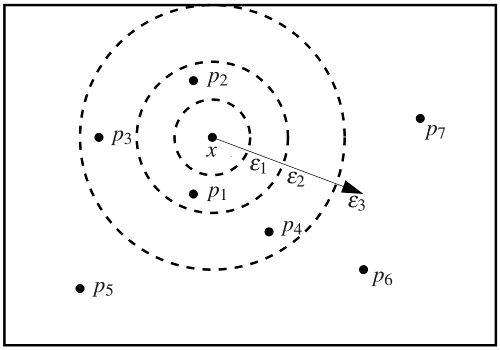

Fig. 1. Identifying $\mathscr{D}_x^\varepsilon$ as ε varies; For ε_1, ε_2 and ε_3 shown in the diagram $\mathscr{D}_x^{\varepsilon_1} = \emptyset$, $\mathscr{D}_x^{\varepsilon_2} = \{L_1, L_2\}$, $\mathscr{D}_x^{\varepsilon_3} = \{L_1, L_2, L_3, L_4\}$

there be a single element $p_i \in \Omega$[1] corresponding to a prototypical case for which L_i is certainly an appropriate description. Within this framework L_i is deemed to be appropriate to describe an element $x \in \Omega$ provided x is sufficiently close or similar to the prototypical element p_i. This is formalized by the requirement that x is within a maximal distance threshold ε of p_i. i.e. L_i is appropriate to describe x if $d(x, p_i) \leq \varepsilon$ where $\varepsilon > 0$. From this perspective an agent's uncertainty regarding the appropriateness of a label to describe a value x is characterised by his or her uncertainty regarding the distance threshold ε. Here we assume that this uncertainty is represented by a probability density function δ for ε defined on $[0, \infty)$. Within this interpretation a natural definition of the description of an element $\mathscr{D}_x$ and the associated mass function m_x can be given as follows:

Definition 6. *For* $\varepsilon \in [0, \infty)$ $\mathscr{D}_x^\varepsilon = \{L_i \in LA : d(x, p_i) \leq \varepsilon\}$ *and* $m_x(F) = \delta(\{\varepsilon : \mathscr{D}_x^\varepsilon = F\})$[2].

Intuitively speaking $\mathscr{D}_x^\varepsilon$ identifies the set of labels with prototypes lying within ε of x. Figure 1 shows $\mathscr{D}_x^\varepsilon$ in a hypothetical conceptual space as ε varies. Notice that the sequence $\mathscr{D}_x^\varepsilon$ as ε varies generates a nested hierarchy of label sets. Furthermore, the distance metric d naturally generates a total ordering on the appropriateness of labels for any element x, according to which label L_j is as least as appropriate to describe x as label L_i if x is closer (or equidistant) to p_j than to p_i i.e. $L_i \preceq_x L_j$ iff $d(x, p_i) \geq d(x, p_j)$. The following theorem shows that this ordering constrains the labels contained in $\mathscr{D}_x^\varepsilon$ as suggested in Section 2:

Theorem 3. *If* $L_i \preceq_x L_j$ *(as defined above) then* $\forall \varepsilon \geq 0$ $L_i \in \mathscr{D}_x^\varepsilon$ *implies that* $L_j \in \mathscr{D}_x^\varepsilon$.

[1] For simplicity of notation we assume that each label has a single prototype. However, the case where there is a set of prototypes P_i for L_i can easily be accommodated by extending the distance metric d such that $d(x, P_i) = \inf\{d(x, p_i) : p_i \in P_i\}$.

[2] For Lesbegue measurable set I, we denote $\delta(I) = \int_I \delta(\varepsilon) d\varepsilon$.

Proof. Suppose $\exists x \in \Omega$ for which $L_i \preceq_x L_j$ and $\exists \varepsilon \geq 0$ such that $L_i \in \mathscr{D}_x^\varepsilon$ and $L_j \notin \mathscr{D}_x^\varepsilon$. From this it follows that $d(x, p_i) \leq \varepsilon$ and $d(x, p_j) > \varepsilon$ and hence $L_i \npreceq_x L_j$ which is a contradiction as required. □

Also notice from Definition 6, that for $L_i \in LA$ the appropriateness measure $\mu_{L_i}(x)$ is given by $\delta(\{\varepsilon : L_i \in \mathscr{D}_x^\varepsilon\})$. Consequently, if we view $\mathscr{D}_x^\varepsilon$ as a random set from $[0,\infty)$ into 2^{LA} then $\mu_{L_i}(x)$ corresponds to the single point coverage function of $\mathscr{D}_x^\varepsilon$. This provides us with a link to the random set interpretation of fuzzy sets (See [1], [2] or [4] for an exposition) except that in this case the random set maps to sets of labels rather than sets of elements. Hence, the interpretation of label semantics as proposed above provides a link between random set theory and prototype theory.

The following results show how the appropriateness of an expression $\theta \in LE$ to describe an element x is equivalent to a constraint $\varepsilon \in I(\theta, x)$, for some measurable subset $I(\theta, x)$ of $[0, \infty)$.

Definition 7. *$\forall x \in \Omega$ and $\theta \in LE$, $I(\theta,x) \subseteq [0,\infty)$ is defined recursively as follows: $\forall L_i \in LA$, $\forall \theta, \varphi \in LE$; $I(L_i,x) = [d(x,p_i),\infty)$; $I(\theta \wedge \varphi, x) = I(\theta,x) \cap I(\varphi,x)$; $I(\theta \vee \varphi,x) = I(\theta,x) \cup I(\varphi,x)$; $I(\neg\theta,x) = I(\theta,x)^c$.*

Theorem 4. $\forall \theta \in LE, \forall x \in \Omega$ $I(\theta,x) = \{\varepsilon : \mathscr{D}_x^\varepsilon \in \lambda(\theta)\}$.

Corollary 1. $\forall \theta \in LE$, $\forall x \in \Omega$ $\mu_\theta(x) = \delta(I(\theta,x))$.

Definition 8. *We define $k : LE^{\wedge,\vee} \times \Omega \to [0,\infty)$ recursively as follows: $\forall x \in \Omega$, $\forall L_i \in LA$, $\forall \theta, \varphi \in LE^{\wedge,\vee}$; $k(L_i,x) = d(x,p_i)$; $k(\theta \wedge \varphi, x) = \max(k(\theta,x), k(\varphi,x))$ and $k(\theta \vee \varphi, x) = \min(k(\theta,x), k(\varphi,x))$.*

Theorem 5. $\forall x \in \Omega$, $\forall x \in LE^{\wedge,\vee}$, *then* $I(\theta,x) = [k(\theta,x),\infty)$.

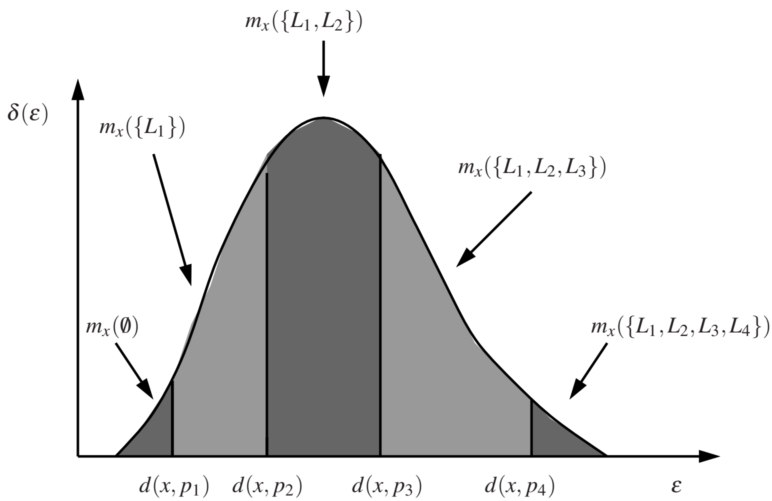

Fig. 2. Let $LA = \{L_1, L_2, L_3, L_4\}$ and $L_4 \preceq_x L_3 \preceq_x L_2 \preceq_x L_1$. This figure shows the values of m_x as areas under δ.

From Theorem 5 we have that

$$\mu_{\theta\vee\varphi}(x) = \delta([k(\theta\vee\varphi,x),\infty)) = \delta([\min(k(\theta,x),k(\varphi,x)),\infty))$$
$$= \max(\delta([k(\theta,x),\infty)),\delta([k(\varphi,x),\infty))) = \max(\mu_\theta(x),\mu_\varphi(x)).$$

Similarly, $\mu_{\theta\wedge\varphi}(x) = \min(\mu_\theta(x),\mu_\varphi(x))$ as is consistent with Theorem 2.

Example 1. $I(L_i,x) = [d(x,p_i),\infty)$, $I(\neg L_i,x) = [0,d(x,p_i))$, $I(L_i \wedge L_j,x) = [\max(d(x,p_i),d(x,p_j)),\infty)$, $I(L_i \vee L_j,x) = [\min(d(x,p_i),d(x,p_j)),\infty)$. Also $I(L_i \wedge \neg L_j,x) = [d(x,p_i),d(x,p_j))$ provided $d(x,p_i) < d(x,p_j)$ and $= \emptyset$ otherwise.

From Lawry [7] we have that for $F \subseteq LA$ $m_x(F) = \mu_{\alpha_F}(x)$ where $\alpha_F = (\bigwedge_{L\in F} L) \wedge (\bigwedge_{L\notin F} \neg L)$. Hence, $m_x(F) = \delta(I(\alpha_F,x))$ where $I(\alpha_F,x) = [\max\{d(x,p_i) : L_i \in F\}, \min\{d(x,p_i) : L_i \notin F\})$ provided that $\max\{d(x,p_i) : L_i \in F\} < \min\{d(x,p_i) : L_i \notin F\}$ and $= \emptyset$ otherwise.

Figure 2 shows the areas under δ corresponding to the values of the mass function m_x.

4 Conclusions

Label semantics is an epistemic theory of uncertainty for vague concepts based on appropriateness measures and mass functions. The underlying calculus is not truth-functional but can be functional in a weaker sense, with the min and max rules for conjunction and disjunction being preserved for a restricted class of expressions.

Appropriateness measures and mass functions can be interpreted, within prototype theory, as the probability that a distance threshold ε lies in a measurable subset of $[0,\infty)$ as determined by the relevant label or expression. Here ε represents an upper-bound on the distance that an element x can be from the prototype p_i for a label L_i, in order that L_i is still deemed an appropriate description of x.

Acknowledgement. Yongchuan Tang is funded by the National Natural Science Foundation of China (NSFC) under Grant 60604034, the joint funding of NSFC and MSRA under Grant 60776798, and the Science and Technology Program of Zhejiang Province under Grant 2007C23061.

References

1. Dubois, D., Prade, H.: The three semantics of fuzzy sets. Fuzzy Sets Syst. 90(2), 141–150 (1997)
2. Dubois, D., Godo, L., Prade, H., Esteva, F.: An information-based discussion of vagueness. In: Cohen, H., Lefebvre, C. (eds.) Handbook of Categorization in Cognitive Science, pp. 892–913 (2005)
3. Gärdenfors, P.: Conceptual Spaces: The Geometry of Thought. MIT Press, Cambridge (2000)
4. Goodman, I.R., Nguyen, H.T.: Uncertainty Models for Knowledge-Based Systems; A Unified Approach to the Measurement of Uncertainty. Elsevier Science Inc., New York (1985)
5. Kripke, S.: A completeness theorem for modal logic. J. Symbolic. Logic 24, 1–24 (1959)
6. Lawry, J.: A framework for linguistic modelling. Artificial Intelligence 155, 1–39 (2004)

7. Lawry, J.: Modelling and Reasoning with Vague Concepts, vol. 12. Springer, New York (2006)
8. Lawry, J.: Appropriateness measures: An uncertainty model for vague concepts. Synthese 161(2), 255–269 (2008)
9. Rosch, E.: Cognitive representation of semantic categories. J. Exp. Psychol. Gen. 104, 192–233 (1975)
10. Ruspini, E.H.: On the semantics of fuzzy logic. Internat. J. Approx. Reason 5, 45–88 (1991)
11. Tang, Y., Zheng, J.: Linguistic modelling based on semantic similarity relation among linguistic labels. Fuzzy Sets Syst. 157, 1662–1673 (2006)
12. Williamson, T.: Vagueness. Routledge, London (1994)