# Inferring a Possibility Distribution from Very Few Measurements

Gilles Mauris

Laboratoire d'Informatique, Systèmes, Traitement de l'Information et de la Connaissance (LISTIC), Polytech'Savoie, Annecy Le Vieux, France

**Abstract.** This paper considers the problem of the possibility representation of measurement uncertainty in the cases of information shortage: very few measurements, poor knowledge about the underlying probability distribution. After having related possibility distribution to probability confidence intervals, we present a procedure to build a possibility distribution for one measurement issued from an unimodal probability distribution. We consider then the addition of other measurements and more knowledge about the probability distribution. The key role of the uniform distribution as the probability distribution leading to the least specific possibility distribution is highlighted. The approach is compared and discussed versus the conventional one based on the Student distribution.

**Keywords:** Possibility theory, Probability theory, Uncertainty, Scarce measurements.

## 1 Introduction

In many application domains, it is important to take the measurement uncertainties into account [11], [16], especially in order to define around the measurement result an interval which will contain the real value of the considered entity with specified confidence [8], that is, a confidence interval [9]. Such an interval allows to define decision risks later, as for example the risk to exceed an alarm threshold, etc. In practice, two main theories are considered to deal with measurement uncertainty: interval calculus [14] and probability theory [9]. As interval calculus only supplies the confidence interval with 100% confidence, probability theory seems to be required to supply the other confidence intervals. But to handle the whole set of confidence intervals (with all the confidence levels) is quite complex by a probability approach. And choosing a particular confidence level (e.g. 95% which means a .05 probability for the value to be out of the interval) is rather arbitrary. Thus a possibility approach has been proposed in [5, 6, 15] and further developed by a few authors in a measurement context [2, 7, 12, 13, 17].

This paper further explores the connection between possibility distribution and confidence intervals and addresses the possibility expression of measurement uncertainty for situations where only very limited knowledge is available: very few measurements, unknown unimodal probability density. In Section 2, we recall how a possibility distribution can be built from confidence intervals. In the third section, we present the main contribution of the paper, i.e. how to define confidence intervals where only limited knowledge is available about the underlying probability density (unimodal bounded/non-bounded,

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symmetric or not). The results are then applied to expression of uncertainty when only very few measurements are available. The key role of the uniform distribution as the probability distribution leading to the least specific possibility distribution is highlighted. The approach is compared and discussed versus the conventional one based on the Student distribution. Some concluding remarks point out the interest of the approach and some future developments.

## 2 Possibility Distribution Versus Confidence Intervals

### 2.1 Basics of the Possibility Theory

The possibility theory is one of the modern theories available to represent uncertainty when information is scarce and/or imprecise [18]. The basic notion is the possibility distribution, denoted $\pi$. Here, we consider possibility distributions defined on the real line, i.e. $\pi$ is an upper semi-continuous mapping from the real line to the unit interval. Thus $\pi$ is a fuzzy subset but with specific semantics for the membership function. Indeed, a possibility distribution describes the more or less plausible values of some uncertain variable $X$. The possibility theory provides two evaluations of the likelihood of an event, for instance whether the value of a real variable $X$ does lie within a certain interval: the possibility $\Pi$ and the necessity $N$. The normalized measures of possibility $\Pi$ and necessity $N$ are defined from the possibility distribution $\pi : \mathbb{R} \rightarrow [0, 1]$ such that $\sup_{x\in\mathbb{R}} \pi(x) = 1$ as follows:

$$\forall A \subset \mathbb{R},\ \Pi(A) = \sup_{x\in A} \pi(x) \quad \text{and} \quad \forall A \subset \mathbb{R},\ N(A) = 1 - \Pi(\bar{A}) = \inf_{x\in A}(1 - \pi(x)).$$

The possibility measure $\Pi$ satisfies $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$, $\forall A, B \subset \mathbb{R}$.

The necessity measure $N$ satisfies $N(A \cap B) = \min(N(A), N(B))$, $\forall A, B \subset \mathbb{R}$.

A possibility distribution $\pi_1$ is more specific than $\pi_2$ as soon as $\pi_1 \leq \pi_2$ (in the usual definition of inclusion of fuzzy sets), i.e. $\pi_1$ is more informative than $\pi_2$. In fact, possibility measures are set functions similar to probability measures, but they rely on axioms which involve the operations “maximum” and ”minimum” instead of the operations “addition” and “product” (if the measures are decomposable [3]).

### 2.2 Possibility Representation of Confidence Intervals

Let us assume that the random variable associated to the measurement results is denoted $X$ (a realization of $X$ is denoted $x$), is continuous on the set of reals and is described by a probability density function $p$, $F$ being its corresponding probability distribution function with $F^{-1}$ its inverse function if it exists (otherwise the pseudo-inverse function can be considered [9]). For every possible confidence level $\beta \in [0,1]$, the corresponding confidence interval is defined as an interval that contains the measurand (i.e. the physical entity to be determined denoted $\mu$) with probability $\geq \beta$. In other words, a confidence interval of confidence level $\beta$ (denoted $I_\beta$) is defined as an interval for which the probability $P_{\text{out}}$ to be outside this interval $I_\beta$ does not exceed $\alpha \stackrel{\text{def}}{=} 1 - \beta$, i.e. $P(\mu \notin I_\beta) = \alpha$.

It is possible to link confidence intervals and possibility distribution in the following way. A unimodal numerical possibility distribution may be viewed as a nested set of confidence intervals, which are the $\alpha$ cuts $[\underline{x}_\alpha, \overline{x}_\alpha] = \{x, \pi(x) \geq \alpha\}$ of $\pi$. The degree of certainty that $[\underline{x}_\alpha, \overline{x}_\alpha]$ contains $\mu$ is $N([\underline{x}_\alpha, \overline{x}_\alpha])$ (if continuous). Obviously, the confidence intervals built around the same point $x^*$ are nested. It has been proven in [12] that stacking confidence intervals of a probability distribution on top of one another leads to a possibility distribution (denoted $\pi^*$ having $x^*$ as modal value). In fact, in this way, the $\alpha$-cuts of $\pi^*$, i.e. $A_\alpha = \{x, | \pi^*(x) \geq \alpha\}$ are identified with the confidence interval $I^*_\beta$ of confidence level $\beta = 1 - \alpha$ around the nominal value $x^*$. Thus, the possibility distribution $\pi^*$ encodes the whole set of confidence intervals in its membership function. Moreover, this possibility distribution satisfies $\Pi^*(A) \geq P(A), \forall A \subset \mathbb{R}$, with $\Pi^*$ and $P$ the possibility and probability measures associated respectively to $\pi^*$ and $p$ (the underlying probability density function of the measurement results).

A closed form expression of the possibility distribution $\pi^M(x)$ induced by confidence intervals around the mode $x^* = M$ is obtained for unimodal continuous probability densities $p(x)$ strictly increasing on the left and decreasing on the right of $M$ [4]:

$$\pi^M(x) = \int_{-\infty}^{x} p(y)dy + \int_{\phi(x)}^{+\infty} p(y)dy = F(x) + 1 - F(\phi(x)) = \pi^M(\phi(x)) \tag{1}$$

for all $x \in [-\infty, M]$, where $\phi$ is a decreasing mapping $\phi : [-\infty, M] \mapsto [M, \infty] \,|\, \phi(M) = M$. $\pi^M(x)$ is the probability that the measurand $\mu$ is outside the interval $[x, \phi(x)]$, i.e. $1 - \pi^M(x)$ is the confidence level of this interval.

## 3 Inferring a Possibility Distribution from a Small Sample

We will consider confidence intervals associated with an underlying probability density being unimodal (i.e. having only one maximum, both local and global) with different assumptions: bounded and non bounded, symmetric or not. Most of the following results are based on trivial properties of unimodal distribution described below.

Let us consider a unimodal probability density $p$ with the mode $M$ that will be identified to the measurand. Thus, $p$ is non increasing for its argument values greater than $M$, and non decreasing for its argument values less than $M$. Therefore, for any values superior to $M$ such that $x_3 \geq x_2 \geq x_1$, the average of $p$ over $[x_2, x_3]$ must be less than or equal to its average over $[x_1, x_3]$:

$$\frac{\int_{x_2}^{x_3} p(x)dx}{x_3 - x_2} \leq \frac{\int_{x_1}^{x_3} p(x)dx}{x_3 - x_1} \tag{2}$$

Similarly, for any values less than $M$ such that $x_1 \leq x_2 \leq x_3$:

$$\frac{\int_{x_1}^{x_2} p(x)dx}{x_2 - x_1} \leq \frac{\int_{x_1}^{x_3} p(x)dx}{x_3 - x_1} \tag{3}$$

Note that the equality in (2) and (3) holds if $p$ is constant on the considered domain.

### 3.1 Bounded Probability Density

Let us consider that $X$ is defined by a probability density, its mode is denoted $M$ and its support $[M - a, M + b]$. Then the mode and the support of $X - M$ are respectively 0 and the interval $[-a, b]$. We have the following result:

**Proposition 1.** $\forall t \in [0,1]$, $\Pr[X - ta \leq M \leq X + tb] \geq t$.

*Proof.* $\Pr[X - ta \le M \le X + tb] = \Pr[-ta \le X - M \le tb]$ and $\Pr[-ta \le X - M \le tb] = 1 - \int_{-a}^{-ta} p(x)dx - \int_{tb}^{b} p(x)dx$.

Then by applying (3) to $x_1 = -a, x_2 = -ta, x_3 = 0$ and (2) to $x_1 = 0, x_2 = tb, x_3 = b$, we obtain:

$$\int_{-a}^{-ta} p(x)dx \le (1-t)\int_{-a}^{0} p(x)dx \quad \text{and} \quad \int_{tb}^{b} p(x)dx \le (1-t)\int_{0}^{b} p(x)dx$$

Therefore:

$$\int_{-a}^{-ta} p(x)dx + \int_{tb}^{b} p(x)dx \le (1-t)\int_{-a}^{b} p(x)dx = 1-t$$

Then: $\forall t \in [0,1]$, $\Pr[X - ta \le M \le X + tb] \ge 1 - (1-t) = t$. □

Therefore the corresponding possibility distribution is defined by:

$$\forall x \in [M-a, M],\ \pi^M(x) \le \frac{x+a-M}{a} \quad \text{and} \quad \forall x \in [M, M+b],\ \pi^M(x) \le \frac{-x+M+b}{b}$$

Therefore, the possibility distribution defined by the triangular possibility distribution having for support $[M-a, M+b]$ is consistent with all the unimodal probability distributions (symmetric or not) having $M$ as modal value and $[M-a, M+b]$ as support.

Note that the triangular possibility distribution is also the possibility distribution associated to the uniform probability density. Moreover, the triangular symmetric possibility distribution with support $[M-a, M+b]$ and mode $M$, is the least upper bound of all the possibility transforms of symmetric probability distributions having $M$ for modal value and $[M-a, M+b]$ for support. This result has been previously stated in [4] but in another way.

### 3.2 Non Bounded Probability Density

As the support is known as infinite, the intervals have to be built from other information from the random variable. Thus, we will consider intervals of the form $X \pm t|X|$. In fact, instead of starting from the support as for bounded distributions, we propose to start from the mode.

The following result holds for any unimodal distribution [1, 10]:

**Proposition 2**

$$\Pr[X - t|X| \le M \le X + t|X|] \ge 1 - \frac{2}{1+t} \quad \textit{for } t \ge 1 \tag{4}$$

*Proof*

$$\Pr[|X-M| \le t|X|] = \Pr\left[\left|1 - \frac{M}{X}\right| \ge t\right] = \Pr\left[\frac{M}{X} \in 1 \pm t\right]$$
$$= \Pr\left[X \in M\frac{1}{1 \pm t}\right] = \Pr\left[X - M \in M\left(\frac{1}{1 \pm t} - 1\right)\right]$$

Thus

$$\Pr\left[X - M \in M\left(\frac{1}{1 \pm t} - 1\right)\right] = \begin{cases} F\left(M\frac{-t}{t+1}\right) - F\left(M\frac{-t}{t-1}\right) & \text{for } M \ge 0 \\ F\left(M\frac{-t}{t-1}\right) - F\left(M\frac{-t}{t+1}\right) & \text{for } M \le 0 \end{cases}$$

Then by applying (2) to $x_1 = \frac{-tM}{t-1}$, $x_2 = \frac{-tM}{t+1}$, $x_3 = M$ and (3) to $x_1 = \frac{-tM}{t+1}$, $x_2 = \frac{-tM}{t-1}$, $x_3 = M$, we obtain respectively:

$$F\left(M\frac{-t}{t+1}\right) - F\left(M\frac{-t}{t-1}\right) \leq \frac{2}{t+1} \text{ for } M \geq 0$$

$$F\left(M\frac{-t}{t-1}\right) - F\left(M\frac{-t}{t+1}\right) \leq \frac{2}{t+1} \text{ for } M \leq 0$$

Therefore: $\Pr[|X-M| \leq t|X|] \leq \frac{2}{t+1}$ and finally we obtain: $\Pr[X - t|X| \leq M \leq X + t|X|] \geq 1 - \frac{2}{1+t}$. □

By the same reasoning, we obtain for a symmetric unimodal probability density:

$$\Pr[X - t|X| \leq M \leq X + t|X|] \geq 1 - \frac{1}{1+t} \quad \text{for} t \geq 1 \tag{5}$$

Note that the equality (5) holds for $p$ uniform, and thus this probability distribution is the least favourable in the sense that it gives the least specific possibility distribution (for $t \geq 1$). If the shape of the probability distribution is known, the inequality can be reduced for high values of $t$. For example, if it is Gaussian, the bound in (5) can be improved [1]:

$$\Pr[X - t|X| \leq M \leq X + t|X|] \geq 1 - \frac{0.484}{t-1} \quad \text{for } t \geq 1 \tag{6}$$

### 3.3 Case of One Measurement

Let us consider the case where only one single measurement is available. In this case it is natural to consider that the observed value corresponds to the mode of the underlying probability density. If the density is assumed to be non symmetric, we have from (4) $\pi(x_0 - tx_0) = \pi(x_0 + tx_0) = \frac{2}{1+t}$ for $t \geq 1$. If it is symmetric, we have from (5) $\pi(x_0 - tx_0) = \pi(x_0 + tx_0) = \frac{1}{1+t}$ for $t \geq 1$. If it is Gaussian, we have from (6) $\pi(x_0 - tx_0) = \pi(x_0 + tx_0) = \frac{0.484}{t-1}$ for $t \geq 0.484$.

Let us consider for example the case where a sensor provides a single value of 30°C the associated probability distribution is supposed to be unimodal. Figure 1a) highlights the reduction of confidence interval lengths according to the amount of available knowledge: when the distribution is non symmetric, when the distribution is symmetric. When it is Gaussian, the use of the equation (6) leads to a reduction of confidence interval lengths only for high values of $t$. For low values of $t$, the exact expression will also give reduced intervals but it has not yet been computed; the uniform distribution being the least favorable for $t \geq 1$.

### 3.4 Case of one Measurement and a Guess

By making the variable change of $X$ into $X - A$, in (4), is replaced by $X - A$ and $M$ by $M - A$, then the following result is deduced for any unimodal symmetric distribution:

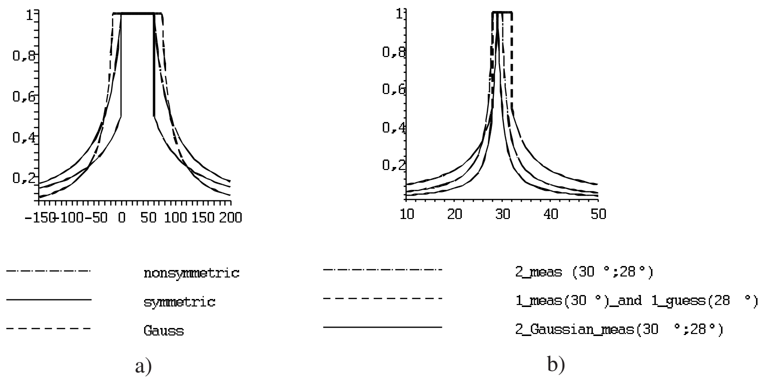


**Fig. 1.** Possibility distributions a) for one measurement b) for two measurements

$$\Pr[X - t|X - A| \leq M \leq X + t|X - A|] \geq 1 - \frac{1}{1+t} \text{ for } t \geq 1 \tag{7}$$

This result can be used to introduce via $A$ some form of prior information (called a guess, coming for example from an expert) concerning the dispersion. In fact, $|x_1 - A|$ can be viewed as the equivalent of the sample standard deviation used classically (see Section 3.6). The introduction of $A$ allows to reduce the lengths of confidence intervals obtained by one single measurement as it is illustrated in Figure 1b).

### 3.5 Case of Two Measurements

Let us now consider the case where a second measurement $x_2$, coming from the same probability distribution as $x_1$ and considered as being independent from it. We propose (in an equivalent way with classical propositions when two measurements are available) to consider the confidence intervals of the form:

$$\frac{X_1 + X_2}{2} - t\frac{|X_1 - X_2|}{2} \leq M \leq \frac{X_1 + X_2}{2} + t\frac{|X_1 - X_2|}{2}$$

In the case of symmetric unimodal distribution, we obtain by the same reasoning as the one used in Section 3.2:

$$\Pr\left[\frac{X_1 + X_2}{2} - \frac{t}{2}|X_1 - X_2| \leq M \leq \frac{X_1 + X_2}{2} + \frac{t}{2}|X_1 - X_2|\right] \geq 1 - \frac{1}{1+t} \tag{8}$$

The Fig. 1b) illustrates the case where the sensor provides the two measurements $x_1 =$ 30°C and $x_2 = 28$°C .

### 3.6 Discussion Versus the Conventional Probability Approach

The above mathematical derivations formalize the idea that without any appeal to other information (except unimodality), we can compute the actual length of the finite confidence interval. It is remarkable that the confidence intervals thus created have finite lengths, except for the 100% confidence level (see Fig. 1a). Indeed, this result seems to contradict the standard statistical intuition that at least two measurements are required in order to have some idea about the dispersion (i.e. to have an estimation of the standard deviation $\sigma$). Indeed, the conventional probability recommendation to deal with a small number $n$ (but $n > 1$) of measurement consists in using confidence intervals of the form [8]:

$$\bar{X} - tS/\sqrt{n} \leq M \leq \bar{X} + tS/\sqrt{n} \tag{9}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the sample mean, and $S = \left[\sum_{i=1}^{n} (X_i - \bar{X})/(n-1)\right]^{1/2}$ the sample standard deviation.

If the underlying probability distribution is Gaussian, the $t$ value is the one given by the Student distribution for a given confidence level. An interesting remark is that for $n = 2$, (9) has the same form as (8). Indeed, in this case of two measurement, (8) is equivalent to (9) for a Gaussian distribution. The Fig. 1a) gives an example of the effect on the possibility distribution specificity (for high values of $t$) of making the Gaussian assumption.

## 4 Conclusion

A possibility distribution can encode a family of probability distributions. This fact has been used as a basis for a transformation of a probability distribution into a possibility distribution by using the notion of confidence intervals. Thus the possibility distribution has been related to probability inequalities, especially for unimodal bounded (or not) symmetric (or not) probability distributions. The obtained results have been used for a possibility expression of measurement uncertainty in situations where only a very limited knowledge is available: one or two measurements, unknown unimodal probability density. In fact, the proposed approach extends the conventional probability approach of Student to the case of one single measurement and to the case of non Gaussian distribution for two measurements. The results highlight the key role of the uniform probability distribution that leads to the least specific possibility distribution at least for high confidence levels. Further developments will consider how having more measurements allows to shorten the confidence intervals and thus to increase the specificity of the corresponding possibility distribution.

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