Changing Neighborhoods of CA: Reduced Local Structures and Embeddings for Universality

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Abstract. This paper consists of two parts. In the first we pick up again the question under which circumstances different pairs of a local function and a neighborhood give rise to the same global behavior of CA and disprove a conjecture made in an earlier paper. In the second part we reconsider a construction showing that one can achieve universality by only changing the (positions in the) neighborhood of a CA, while not providing any information about the CA to be simulated in the initial configuration. The construction uses an embedding which in some sense is "non-local". We show that under mild conditions this is necessary.

1 Introduction

Usually investigations of cellular automata without further discussion assume some standard neighborhood because it is "without loss of generality". In general this is correct, except, of course, when one is interested in questions specifically concerning neighborhoods. This is the guiding line of the current paper. It is a continuation e. g. of [3] and [5] (an extended version will appear in [4]).

The rest of the paper is organized as follows: In Section 2 we introduce the notions and notations used throughout the paper. Sections 3 and 4 concerned with a "normal form" of pairs (f, ν) of a local function and a neighborhood and the question under which circumstances different such pairs can give rise to the same global behavior of CA. Finally, in Section 5 we take a second look at a construction showing some kind of universality [5] and prove that the embedding used for the simulation of CA necessarily has to have a certain propoerty.

2 Basics

We assume that readers are familiar with the basic concepts of cellular automata (CA). In the first part of this paper we will consider d-dimensional Euclidean CA for any $d \in \mathbb{N}_+$. We will write R for the set $\mathbb{Z}^d$ of all cells. In the second part for the sake of simplicity we will assume that $d = 1$. The set of states of a

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H. Umeo et al. (Eds): ACRI 2008, LNCS 5191, pp. 252–259, 2008.

single cell will usually be denoted by Q (or $Q_A, \ldots$). A local transition function is of the form $f : Q^n \to Q$. Of course, one is interested in the case $n \geq 1$. But for technical reasons (a simpler proof of Lemma 3) we also allow $n = 0$; in this case $f : Q^0 \to Q$ simply is a constant.

A *neighborhood* is a mapping $\nu : \mathbb{P}_n \to R$, where $\mathbb{P}_n$ denotes the set $\{1, 2, \ldots, n\}$ of positive integers. As a special case we use $\mathbb{P}_0$ to denote the empty set. This can equivalently be seen as a list ν with n components, written as $\nu(1), \ldots, \nu(n)$. Note that for the use with a local function the order may be important. The set of *all* neighborhoods of size n will be denoted as $\mathcal{N}_n$.

A pair (f, ν) of a local function $f : Q^n \to Q$ and a neighborhood $\nu \in \mathcal{N}_n$ is called a *local structure*. We call n the *arity* of the local structure.

Since Q and R are already implicit in f and ν we will simply speak of a CA $\mathcal{A} = (f, \nu)$. As usual, each local structure $\mathcal{A} = (f, \nu)$ induces a global transition function $Q^R \to Q^R$ (which we also denote by $\mathcal{A}$) of a CA by

$$\forall\, x \in R : \mathcal{A}(c)(x) = f(c(x + \nu(1)), c(x + \nu(2)), \ldots, c(x + \nu(n))) \; .$$

The general question we are interested in is:

> What can and what cannot happen when changing the neighborhood of a local function, i. e. when going from a local structure (f, ν) to a local structure (f, ν')?

A restricted version of this question is: What can and what cannot happen when permuting the neighbors of a local function, i. e. when going from a local structure (f, ν) to a local structure (f, ν'), where ν' is a permutation of ν?

3 Reduced Local Structures

Definition 1. *A local structure is called* reduced*, if and only if the following conditions are fulfilled:*

- *f depends on each argument, i. e. for each $i \in \mathbb{P}_n$ there are $q_1, \ldots, q_{i-1}$, q_i, q_i', $q_{i+1}, \ldots, q_n \in Q$ such that*

$$f(q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_n) \neq f(q_1, \ldots, q_{i-1}, q_i', q_{i+1}, \ldots, q_n) \; .$$

- *ν is injective, i. e. no $x \in R$ appears twice in the list. Such neighborhoods are called* non-degenerate *in [3].*

A somewhat special case is a *constant* local function f and the empty neighborhood. In this case f is of the form $f : Q^0 \to Q$, i. e. the only "argument list" for f is the empty list, and $\nu : \mathbb{P}_0 \to R$ is the empty list as well. This local structure is reduced.

Obviously, for $n \geq 1$ the transition function of each reduced local structure is non-constant.

Definition 2. *Two local structures* (f,ν) *and* (f',ν') *are called* equivalent *if and only if they induce the same global function. In that case we sometimes write* $(f,\nu) \approx (f',\nu')$.

Lemma 3. *For each local structure* (f,ν) *there is an equivalent reduced local structure* (f',ν').

Proof. Let n denote the arity of (f,ν). Assume that (f,ν) is not reduced. We will see that $n \geq 1$ and show how to construct an equivalent local structure (f',ν') with arity $n-1$.

Case 1: ν is not injective. Then clearly $n \geq 2$. Let i and j be indices such that $i < j$ and $\nu_i = \nu_j$. Define $\nu' \in \mathcal{N}_{n-1}$ as

- $\nu'_k = \begin{cases} \nu_k & \text{iff } k < j \\ \nu_{k+1} & \text{iff } k \geq j \end{cases}$, i. e. drop the j-th component of ν, and
- $f' : Q^{n-1} \to Q$ by $f'(q_1,\ldots,q_{n-1}) = f(q_1,\ldots,q_{j-1},q_i,q_j,\ldots,q_{n-1})$

For any configuration $c \in Q^R$ holds:

$$\begin{aligned} F'(c)(0) &= f'(c(\nu'_1),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu'_1),\ldots,c(\nu'_{j-1}),c(\nu'_i),c(\nu'_j),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu_1),\ldots,c(\nu_{j-1}),c(\nu_i),c(\nu_{j+1}),\ldots,c(\nu_n)) \\ &= f(c(\nu_1),\ldots,c(\nu_{j-1}),c(\nu_j),c(\nu_{j+1}),\ldots,c(\nu_n)) \\ &= F(c)(0) \end{aligned}$$

Since application of local functions commutes with shifts, it follows $F'(c)(x) = F(c)(x)$ for all $x \in R$.

Case 2: f does not depend on all arguments. Then clearly $n \geq 1$. Assume that it does not depend on argument i, $1 \leq i \leq n$. Define $\nu' \in \mathcal{N}_{n-1}$ as

- $\nu'_k = \begin{cases} \nu_k & \text{iff } k < i \\ \nu_{k+1} & \text{iff } k \geq i \end{cases}$, and
- $f'(q_1,\ldots,q_{n-1}) = f(q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_{n-1})$ for any $q \in Q$. Since f does not depend on the i-th argument, f' is well defined.

For any configuration $c \in Q^R$ holds:

$$\begin{aligned} F'(c)(0) &= f'(c(\nu'_1),\ldots,c(\nu'_{i-1}),c(\nu'_i),c(\nu'_{i+1}),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu'_1),\ldots,c(\nu'_{i-1}),q,c(\nu'_i),c(\nu'_{i+1}),\ldots,c(\nu'_{n-1})) \\ &= f(c(\nu_1),\ldots,c(\nu_{i-1}),q,c(\nu_{i+1}),c(\nu_{i+2}),\ldots,c(\nu_n)) \\ &= f(c(\nu_1),\ldots,c(\nu_{i-1}),c(\nu_i),c(\nu_{i+1}),c(\nu_{i+2}),\ldots,c(\nu_n)) \\ &= F(c)(0) \end{aligned}$$

As in case 1 it follows that $F'(c)(x) = F(c)(x)$ for all $x \in R$. □

The construction above does *not* imply that the equivalent reduced local structure itself is unique. In fact in general it is not: As a simple example consider the local function $f : \{0,1\}^2 \to \{0,1\} : (x_1, x_2) \mapsto x_1 \wedge x_2$. Since the order of the arguments x_i does not matter for the value $f(x_1, x_2)$ the local structures $(f, (0,1))$ and $(f, (1,0))$ are equivalent. At the same time both are obviously reduced.

Open problem 4. Given any non-reduced local structure (f, ν) and an equivalent reduced local structure (f', ν'), is there always a sequence of operations as in the proof of Lemma 3 that transforms (f, ν) into (f', ν')?

4 Equivalence of Local Structures

Definition 5. *For $n \geq 1$ let $\pi \in S_n$ denote a permutation of the numbers in $\mathbb{P}_n$.*

- *For a neighborhood ν denote by ν^π the neighborhood defined by $\nu^\pi_{\pi(i)} = \nu_i$.*
- *For an n-tuple $\ell \in Q^n$ denote by ℓ^π the permutation of ℓ such that $\ell^\pi(i) = \ell(\pi(i))$ for $1 \leq i \leq n$.*
- *For a local function $f : Q^n \to Q$ denote by f^π the local function $f^\pi : Q^n \to Q$ such that for all ℓ: $f^\pi(\ell) = f(\ell^\pi)$.*

In the first part of the definition we have preferred the given specification over the equally possible $\nu^\pi_i = \nu_{\pi(i)}$, because the former leads (in our opinion) to a slightly nicer formulation of the following lemma.

Lemma 6. *(f, ν) and (f^π, ν^π) are equivalent for any permutation π.*

Proof. For any configuration c:

$$\begin{aligned} F^\pi(c)(0) &= f^\pi(c(\nu^\pi_1), \ldots, c(\nu^\pi_n)) \\ &= f(c(\nu^\pi_{\pi(1)}), \ldots, c(\nu^\pi_{\pi(n)})) \\ &= f(c(\nu_1), \ldots, c(\nu_n)) \\ &= F(c)(0) \end{aligned}$$

□

We are now going to show that for *reduced (!)* local structures the relationship via a permutation π is the *only* possibility to get equivalence.

Lemma 7. *If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $\nu^\pi = \nu'$.*

It should be noted that e. g. for local structures with a local transition function f which does not depend on an input the claim of the lemma is wrong. Changing the position of the neighbor which "provides" the irrelevant input for f of course does not change the global behavior.

Proof. Assume that there is an x which does not appear in ν' but does appear in ν, say at position i. Since (f, ν) is reduced, f does depend on its i-th argument

and there are two configurations c and $\bar{c}$, which do *only* differ at cell x, such that $F(c)(0) \neq F(\bar{c})(0)$.

Since ν' does not contain x, it is clear that $F'(c)(0) = F'(\bar{c})(0)$. It is therefore impossible that $F(c)(0) = F'(c)(0)$ and simultaneously $F(\bar{c})(0) = F'(\bar{c})(0)$. Hence $F(c) \neq F'(c)$ and $F \neq F'$. □

By choosing different neighborhoods which are not permutations of each other one immediately gets the following; compare Theorem 1 of [3]:

Corollary 8. *For each reduced non-constant local function there are infinitely many compatible neighborhoods inducing pairwise different global CA functions.*

Lemma 9. *If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^\pi, \nu^\pi) = (f', \nu')$.*

Proof. By Lemma 7 we already know that ν and ν' are permutations of each other: $\nu' = \nu^\pi$ for some π; and $(f, \nu) \approx (f', \nu^\pi)$. But it is clear that different local functions induce different global functions, if they use the same neighborhood. Hence if one assumes $f' \neq f^\pi$, then $(f', \nu^\pi) \not\approx (f^\pi, \nu^\pi)$ which together with $(f^\pi, \nu^\pi) \approx (f, \nu)$ (Lemma 6) contradicts $(f, \nu) \approx (f', \nu^\pi)$. □

5 Universality Revisited

In order to keep notation simple, we will only consider one-dimensional CA in this section.

5.1 A Short Review of the Construction

In a previous paper [5] we have sketched a construction which allows to do the following:

Fix an arbitrary set of states Q_A of cardinality $a \geq 2$ and consider the set of *all* CA $\mathcal{A}_i$ with local structures (f_i, ν_i) of arbitrary local functions $f_i : Q_A^{n_i} \to Q_A$ of any arity n_i and compatible arbitrary neighborhoods $\nu_i \in \mathcal{N}_{n_i}$.

Then there is

- one set of states Q_B of cardinality b,
- one embedding $E : Q_A^R \to Q_B^R$ of configurations and
- *one* local function $g : Q_B^k \to Q_B$ ($k = 5$ in [5])

such that

- for each (f_i, ν_i) one can effectively construct a neighborhood $\nu_i' \in \mathcal{N}_k$ such that
- for each configuration $c : R \to Q_A$
- using (g, ν_i') on initial configuration $E(c)$ simulates each step of each cell of the CA given by (f_i, ν_i).

In the construction we use $Q_B = Q_A \times Q'$ where Q' contains the symbols $\circ$ and $\llcorner$ (among others) and the embedding is defined as

$$E(c)(x) = \begin{cases} (c(x), \circ) & \text{if } x = 0 \\ (c(x), \llcorner) & \text{otherwise} \end{cases}$$

5.2 Discussion of the Embedding

In this subsection we will discuss the embedding E in more detail.

The idea of setting a special marker in one cell has appeared in the literature before. Durand-Lose [1] has used it in his construction to simulate irreversible one-dimensional CA on reversible one-dimensional CA for infinite configurations. This result has to be contrasted with a theorem by Hertling [2] stating that such a simulation is impossible. Both authors are right, simply because they use different notions of simulation. In particular, Durand-Lose uses an embedding of configurations which does set a special marker in cell 0, while Hertling only consider embeddings of a certain type called weak morphisms. E is a weak morphims if there is an integer m such that

$$E(\sigma(c)) = \sigma^m(E(c))$$

holds for all configurations c; here σ is the shift $(\forall\, x :\ \sigma(c)(x) = c(x{+}1))$. Clearly, an embedding setting a special marker in cell 0 violates this requirement.

Open problem 10. We note in passing that in addition Hertling's notion of simulation is not compatible with that used by Durand-Lose, and that it is an open problem whether all of the differences are really necessary in order to be able to realize a simulation of irreversible CA on reversible CA for infinite configurations.

We now turn to the universality construction mentioned above and have a closer look at spatially period configurations.

Definition 11. *A configuration c is spatially periodic or simply periodic*[1] *if there is a positive integer $p \geq 1$ such that $\sigma^p(c) = c$. If ℓ is the smallest such integer we also say that c is ℓ-periodic.*

It should be clear that the application of a global CA function F maps configurations with period p to configurations with period p again. The smallest positive period may decrease from ℓ to a divisor of ℓ.

Observation 12. If $\sigma^p(c) = c$ and $E(\sigma(c)) = \sigma^m(E(c))$, then $E(c) = E(\sigma^p(c)) = \sigma^{pm}(E(c))$. I. e., under weak morphisms periodic configurations have to be embedded into periodic ones and *the period can increase by at most a constant factor* (m).

For the moment fix some arity n and consider all $\mathcal{A}_i$ with local structures $(f_i, (1, 2, \ldots, n))$ of this arity. There are a^{a^n} different such CA.

We will use a periodic configuration c_n which as its building block uses a de Bruijn sequence of length a^n. That is, the a^n subwords of length n starting at positions 1, 2, ..., a^n in c_n are pairwise different (and their union is Q_A^n). Configuration c_n has smallest positive period a^n. Hence, according to Observation 12 $c'_n = E(c_n)$ has a smallest positive period of at most ma^n.

We are interested in running some CA $\mathcal{B}_i$ with (g, ν'_i) on the periodic configuration c'_n. In general the following holds:

[1] We will not consider periodicity in time in this paper.

Lemma 13. *Let c denote an ℓ-periodic configuration for some CA $\mathcal{B} = (g, \mu)$ and let $\mathcal{B}' = (g, \mu')$ where μ' denotes the neighborhood with $\mu'(i) = \mu(i) \bmod \ell$ for all i. Then for all $t \geq 0$ holds:*

$$\mathcal{B}^t(c) = \mathcal{B}'^t(c) \ .$$

This is so since c and all of its successor configurations have period ℓ and hence it does not matter if the position of a neighbor is shifted by a multiple of ℓ.

Theorem 14. *If the embedding is a weak morphism, it is impossible to achieve universality as described in Subsection 5.1.*

Proof. Configuration c_n has been chosen such that all a^{a^n} different CA produce a^{a^n} different successor configurations. Assume that c'_n is ℓ'-periodic. According to Lemma 13 there are at most ℓ'^k different global behaviors possible for the $\mathcal{B}_i$ when starting from c'_n. Thus one needs $\ell'^k \geq a^{a^n}$. Writing a' for $a^{1/k}$ and denoting the minimum positive period of c_n by ℓ this means $\ell' \geq (a')^\ell$. In other words the periods of configurations have to grow exponentially in ℓ. Since we want to be able to simulate all CA $\mathcal{A}_i$, one can choose n sufficiently large and increase the gap between ℓ and ℓ' arbitrarily. Comparing this with Observation 12 immediately gives the claim. □

Open problem 15. The embedding described in Subsection 5.1 does more than "significantly" increasing the period length of periodic configurations. It completely breaks spatial periodicity. It is an open problem whether that necessarily has to be the case.

For the sake of a simple proof of Theorem 14 we have assumed that one wants to be able to simulate *all* CA. Looking at the proof one can see that we only used the fact for each of the neighborhoods $(1, 2, \ldots, n)$ we used all local functions of arity n. But in fact somewhat weaker conditions suffice: For example the number m_n of local functions using such a neighborhoods does not have to be a^{a^n}. One only needs that the condition $\ell'^k \geq m_n$ implies that ℓ' grows more than linearly in ℓ. To this end $\sqrt[k]{m_n} \in \omega(a^n)$ is sufficient, e. g. $m_n \geq (a^n)^{2k}$.

6 Summary and Outlook

We have shown that "factoring out" relations by permutation different reduced local structures are exactly those which give rise to different global CA functions.

The universality contruction shows what can be achieved by changing the neighborhood if the local function is chosen carefully. We have proven that in order to achieve universality, the embedding cannot ever be a weak morphism.

It remains the main open problem for future research to find ways of characterizing at least partially how much can possibly happen for local functions if they are not designed for a specific task, but coming from a non-trivial set of local functions.

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