A Construction Method of Moore Neighborhood Number-Conserving Cellular Automata

Naonori Tanimoto and Katsunobu Imai

Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, Japan
{naonori,imai}@iec.hiroshima-u.ac.jp

Abstract. A number-conserving cellular automaton (NCCA) is a cellular automaton such that all states of cells are represented by integers and the total of the numbers (states) of all cells of a global configuration is conserved throughout its computing process. It can be thought to be a kind of modelization of the physical conservation law of mass or energy. In this paper, we show a sufficient condition for a Moore neighborhood CA to be number-conserving. According to this condition, the local function of rotation-symmetric NCCA is expressed by a summation of quaternary functions. On this framework, we construct a 6-state logically universal NCCA.

Keywords: cellular automata, number-conservation, logical universality.

1 Introduction

A number-conserving cellular automaton (NCCA) is a cellular automaton such that all states of cells are represented by integers and the total of the numbers (states) of all cells of a global configuration is conserved throughout its computing process. It can be thought to be a kind of modelization of the physical phenomena, for example, fluid dynamics and highway traffic flow[6].

Boccara et al[2]. studied number conservation of one-dimensional CAs on circular configurations. Durand et al[3]. studied the two-dimensional case and the relation between several boundary conditions. Although their theorems are useful for deciding if a given CA is number-conserving or not, it is quite difficult to design NCCAs with complex transition rules.

As for von Neumann neighborhood NCCAs with and without rotation-symmetry, necessary and sufficient conditions to be number-conserved are shown respectively[7]. According to these conditions, the local function of a rotation-symmetric NCCA is expressed by a summation of two binary functions. Designing the two binary functions, we can construct a rotation-symmetric NCCA easily. On this framework, we constructed 14-state logically universal NCCA with rotation-symmetry. (A logically universal NCCA can simulate any boolean circuits. If it has a cyclic configuration, it can simulate a universal Turing machine.)

It seems that there is a trade-off between the size of state-number of a universal NCCA and the size of its neighborhood, and there is no rotation-symmetric von Neumann neighborhood NCCA of which the element number of the state

H. Umeo et al. (Eds): ACRI 2008, LNCS 5191, pp. 244–251, 2008.

set (state set size) is less than five[8]. To find a smaller state universal NCCA, we consider Moore neighborhood which is also used widely. A necessary and sufficient condition for Moore neighborhood CA to be number-conserving is shown by Durand et al[3]. But it is still difficult to construct Moore neighborhood NCCAs by using the condition directly.

Consequently, we show another sufficient condition for a CA to be number-conserving. According to this condition, the local function of rotation-symmetric NCCA is expressed by a summation of quaternary functions ϕ. ϕ indicates the behavior of values within a block composed of four cells. Assigning a function ϕ is enough to design a rotation-symmetric Moore neighborhood NCCA. It can be thought that the transition of an NCCA can be expressed based on local flows of value such as ϕ, whatever the neighborhood is. Although the shape of partition is not unique for a neighborhood, ϕ is the most natural one in case of the Moore neighborhood. Employing this method, we construct a 6-state logically universal NCCA.

2 Definitions

Definition 1. *A deterministic two-dimensional Moore neighborhood cellular automaton is a system defined by $A = (\mathbf{Z}^2, Q, f, q)$, where $\mathbf{Z}$ is the set of all integers, Q is a non-empty finite set of internal states of each cell, $f : Q^9 \to Q$ is a mapping called a local function and $q \in Q$ is a quiescent state that satisfies $f(q, q, q, q, q, q, q, q, q) = q$.*

A *configuration* over Q is a mapping $\alpha : \mathbf{Z}^2 \to Q$ and the set of all configurations over Q is denoted by Conf(Q), i.e., $\mathrm{Conf}(Q) = \{\alpha | \alpha : \mathbf{Z}^2 \to Q\}$. The function $F : \mathrm{Conf}(Q) \to \mathrm{Conf}(Q)$ is defined as follows and called the *global function* of A.

$$\forall(x, y) \in Z^2,$$
$$F(\alpha)(x, y) = f(\alpha(x-1, y-1), \alpha(x, y-1), \alpha(x+1, y-1), \alpha(x-1, y), \alpha(x, y), \alpha(x+1, y), \alpha(x-1, y+1), \alpha(x, y+1), \alpha(x+1, y+1)).$$

In the case of von Neumann neighborhood, the above definition is the same except that its local function f is $f : Q^5 \to Q$ and its global function F is defined as follow.

$$\forall(x, y) \in Z^2, F(\alpha)(x, y) = f(\alpha(x, y), \alpha(x, y+1), \alpha(x+1, y), \alpha(x, y-1), \alpha(x-1, y)).$$

We assume a quiescent state q such that $f(q, q, q, q, q) = q$.

In this paper, we only consider CAs with *finite configurations*, i.e., the number of cells which states are not quiescent is finite. A is said to be *number-conserving* when it satisfies $\sum_{(x,y)\in\mathbf{Z}^2}\{F(\alpha)(x, y) - \alpha(x, y)\} = 0$ for all finite configurations α.

Next we define some symmetry conditions.

Definition 2. *CA A is said to be rotation-symmetric if its local function f satisfies the following condition.*

$$\forall c, u, r, d, l, ul, ur, dl, dr \in Q, f(ul, u, ur, l, c, r, dl, d, dr) = f(dl, l, ul, d, c, u, dr, r, ur).$$

In the case of von Neumann neighborhood, these symmetries is defined as above.

3 Von Neumann Neighborhood Number-Conserving Cellular Automata

We showed a necessary and sufficient condition for a von Neumann neighborhood CA to be number-conserving in [7].

Theorem 1. *[7] A deterministic two-dimensional von Neumann neighborhood CA $A = (\mathbf{Z}^2, Q, f, q)$ is number-conserving iff f satisfies*

$$\begin{aligned}
&\exists g_U, g_R, g_D, g_L, h_{UR}, h_{RD}, h_{DL}, h_{LU} : Q^2 \to \mathbf{Z}, \forall c, u, r, d, l \in Q,\\
&f(c, u, r, d, l) = c + g_U(c, u) + g_R(c, r) + g_D(c, d) + g_L(c, l)\\
&\qquad\qquad +h_{UR}(u, r) + h_{RD}(r, d) + h_{DL}(d, l) + h_{LU}(l, u),\\
&g_U(c, u) = -g_D(u, c), g_R(c, r) = -g_L(r, c),\\
&h_{UR}(u, r) = -h_{DL}(r, u), h_{RD}(r, d) = -h_{LU}(d, r).
\end{aligned}$$

Next, we derived a necessary and sufficient condition for a rotation-symmetric CA to be number-conserving from the condition.

Corollary 1. *[7] A deterministic two-dimensional rotation-symmetric von Neumann neighborhood CA $A = (\mathbf{Z}^2, Q, f, q)$ is number-conserving iff f satisfies*

$$\begin{aligned}
&\exists g, h : Q^2 \to \mathbf{Z}, \forall c, u, r, d, l \in Q,\\
&f(c, u, r, d, l) = c + g(c, u) + g(c, r) + g(c, d) + g(c, l)\\
&\qquad\qquad +h(u, r) + h(r, d) + h(d, l) + h(l, u),\\
&g(c, u) = -g(u, c), h(u, r) = -h(r, u).
\end{aligned}$$

According to this condition, a local function of a rotation-symmetric NCCA is expressed by a summation of two binary function g and h. The binary function g indicates the number flow between two cells in a vertical or horizontal direction, and h does in a diagonal direction. In a vertical or horizontal flow, a value moves on two cells of which states are arguments of g. But in the diagonal flow case, cells on which a value moves don't correspond to arguments of h. This causes divergence of the state set size.

In order to design a rotation-symmetric NCCA, we only have to define g and h. Although it may not be a CA for divergence of the state set size, there exists a procedure to modify these functions for construction of an NCCA after designing g and h.

4 A Sufficient Condition Based on Quaternary Functions

Durand et al[3]. showed a necessary and sufficient condition for an NCCA in the case of $n \times m$ neighborhood. We show the condition which is restricted to Moore neighborhood.

Theorem 2. *[3] A deterministic two-dimensional Moore neighborhood CA* $A = (\mathbf{Z}^2, Q, f, q)$ *is number-conserving iff* f *satisfies*

$$\begin{aligned}
&\forall c, u, r, d, l, ul, ur, dl, dr, \in Q, \\
&f(ul, u, ur, l, c, r, dl, d, dr) = ul + f(q, u, ur, q, c, r, q, d, dr) + f(q, q, u, q, q, c, q, q, d) \\
&\quad + f(q, q, q, q, u, ur, q, c, r) + f(q, q, q, q, q, u, q, q, c) + f(q, q, q, q, q, q, q, u, ur) \\
&\quad + f(q, q, q, q, q, q, q, q, u) + f(q, q, q, l, c, r, dl, d, dr) + f(q, q, q, q, l, c, q, dl, d) \\
&\quad + f(q, q, q, q, q, l, q, q, dl) + f(q, q, q, q, q, q, l, c, r) + f(q, q, q, q, q, q, q, l, c) \\
&\quad + f(q, q, q, q, q, q, q, q, l) \\
&\quad - f(q, ul, u, q, l, c, q, dl, d) - f(q, q, ul, q, q, l, q, q, dl) - f(q, q, q, ul, u, ur, l, c, r) \\
&\quad - f(q, q, q, q, ul, u, q, l, c) - f(q, q, q, q, q, ul, q, q, l) - f(q, q, q, q, q, q, ul, u, ur) \\
&\quad - f(q, q, q, q, q, q, q, ul, u) - f(q, q, q, q, q, q, q, q, ul) - f(q, q, q, q, c, r, q, d, dr) \\
&\quad - f(q, q, q, q, q, c, q, q, d) - f(q, q, q, q, q, q, q, c, r) - f(q, q, q, q, q, q, q, q, c). \quad (1)
\end{aligned}$$

It is difficult to design f which satisfies the condition. Because when we allocate one value of f, we have to consider other 24 values at most. In addition, each 24 values must satisfies the condition. Therefore we introduce another representation of sufficient condition to be number-conserving and show the condition is usable to design Moore neighborhood NCCAs.

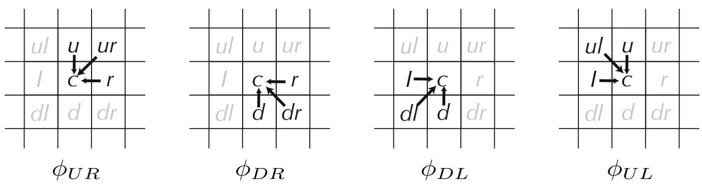

Fig. 1. An interpretation of ϕ_{XX}

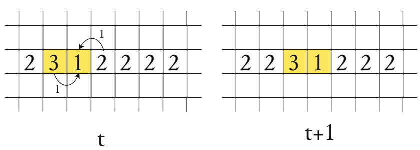

Fig. 2. A configuration of a wire and a signal in A_{ex}

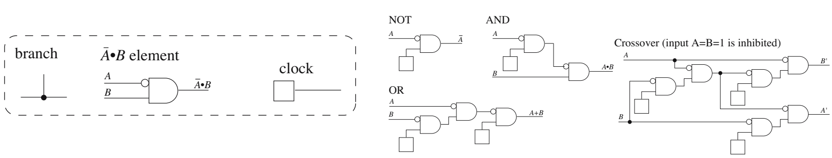

Fig. 3. A construction of basic logic elements[4]

Corollary 2. *A deterministic two-dimensional Moore neighborhood CA* $A = (\mathbf{Z}^2, Q, f, q)$ *is number-conserving if* f *satisfies*

$$\exists \phi_{UL}, \phi_{UR}, \phi_{DL}, \phi_{DR} : Q^4 \to \mathbf{Z}, \forall c, u, r, d, l, ul, ur, dl, dr \in Q,$$

$$f(ul, u, ur, l, c, r, dl, d, dr) = c + \phi_{UR}(c, u, ur, r) + \phi_{DR}(c, r, dr, d) + \phi_{DL}(c, d, dl, l) + \phi_{UL}(c, l, ul, u), \quad (2)$$

$$\phi_{UR}(c, u, ur, r) + \phi_{DR}(u, ur, r, c) + \phi_{DL}(ur, r, c, u) + \phi_{UL}(r, c, u, ru) = 0. \quad (3)$$

Proof. We assume a CA A of which f satisfies the equations (2) and (3). Then this f suffuses (1). Therefore the CA A is number-conserving.

The local function f is expressed by a summation of four quaternary functions $\phi_{UR}, \phi_{DR}, \phi_{DL}$ and ϕ_{UL}, see Equation (2). Each function represents the received value by the central cell from the other three cells, see Fig. 2.

When we design the function ϕs, we have only to take into account the interactions of four cells appeared in each ϕ at a time. The interaction should satisfy the equation (3). Especially if $\phi \equiv \phi_{UR} = \phi_{DR} = \phi_{DL} = \phi_{UL}$, the NCCA is rotation-symmetric. To obtain the actual state-number of the NCCA by this construction method, we need to follow the procedure as below.

1. Choose a *partial* state set $\tilde{Q}$ of size $k(> 0)$ and design $\phi(i, j, k, l)$ for $(i, j, k, l) \in \tilde{Q}^4$.
2. Determine the state set Q by extending $\tilde{Q}$ to Q. This is performed by calculating the set Q' which contains all next values of local function $f(ul, u, ur, l, c, r, dl, d, dr)$ for all $(ul, u, ur, l, c, r, dl, d, dr) \in \tilde{Q}^9$ and $Q := Q' \cup \tilde{Q}$. The domain of ϕ is extended from $\tilde{Q}$ to Q. (Note that $\phi(x_0, x_1, x_2, x_3) \equiv 0$ for any element of $\{(x_0, x_1, x_2, x_3) \in Q | \exists i, x_i \in (Q - \tilde{Q})\}$)

Example 1. Let's consider an NCCA A_{ex} depicted in Fig. 2. It realizes structures which can be regarded as wires and signals. A wire is a connected set of cells of width 1 of which the states are 2. A signal is encoded by two cells whose states are 1 and 3, and it flows along a wire at speed 1, and at the end of the wire(i.e., when it faces to three quiescent cells), the signal is terminated. The state set $\tilde{Q}_{ex} = \{0, 1, 2, 3\}$ is 4-state and the function $\phi_{ex}(i, j, k, l), (i, j, k, l) \in \tilde{Q}_{ex}$ has non-zero values only at $(i, j, k, l) = (3, 1, 0, 0), (1, 0, 0, 3), (1, 2, 0, 0), (2, 0, 0, 1)$, thus $\phi_{ex}(3, 1, 0, 0) = -1, \phi_{ex}(1, 0, 0, 3) = 1$, $\phi_{ex}(1, 2, 0, 0) = 1$, and $\phi_{ex}(2, 0, 0, 1) = -1$ realize the movement of signals. By calculating its local function f_{ex} for all $(ul, u, ur, l, c, r, dl, d, dr) \in \tilde{Q}_{ex}$, its state set $Q_{ex} = \tilde{Q}_{ex}$. Thus the obtained NCCA is 4-state Moore neighborhood NCCA $A_{ex} = (\mathbf{Z}^2, Q_{ex}, f_{ex}, 0)$.

Table 1. Defined non-zero values of g_{14} and h_{14}

$g_{14}(1,2) = 1$,	$g_{14}(1,3) = 1$,	$g_{14}(1,4) = 1$,	$g_{14}(1,6) = 1$,	$g_{14}(7,5) = 1$,	
$g_{14}(4,3) = 1$,	$g_{14}(5,4) = 1$,	$g_{14}(4,8) = 1$,	$g_{14}(1,5) = 1$,	$g_{14}(5,11) = 1$,	
$g_{14}(-2,8) = 1$,	$h_{14}(8,5) = 1$				

Table 2. Defined non-zero values of ϕ

$\phi(3,1,0,0) = -1,$	$\phi(1,0,0,3) = 1,$	$\phi(1,2,0,0) = 1,$	$\phi(2,0,0,1) = -1,$
$\phi(4,1,0,0) = -1,$	$\phi(1,0,0,4) = 1,$	$\phi(1,2,0,2) = 1,$	$\phi(2,0,2,1) = -1,$
$\phi(1,2,2,0) = 1,$	$\phi(2,2,0,1) = -1,$	$\phi(1,2,3,3) = 1,$	$\phi(3,3,1,2) = -1,$
$\phi(2,2,1,3) = -1,$	$\phi(2,1,3,2) = 1,$	$\phi(3,1,2,0) = -1,$	$\phi(1,2,0,3) = 1,$
$\phi(4,1,0,1) = -1,$	$\phi(1,0,1,4) = 1$		

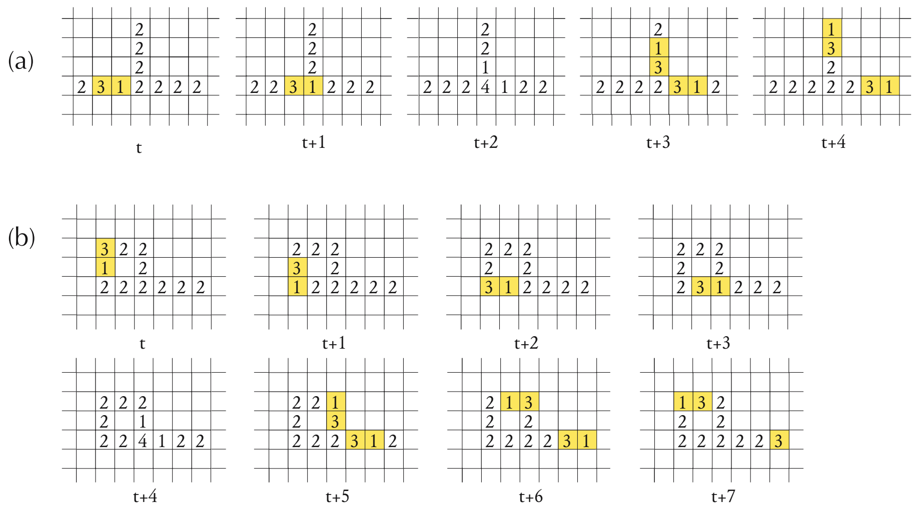

Fig. 4. Configurations of a junction(a) and a clock(b)[5]

5 A Logically Universal NCCA

Banks[1] shows a two-state logically universal von Neumann neighborhood CA, i.e., it is possible to embed any boolean circuit into its cellular space as a configuration and its transition process can simulate the circuit. His model has configurations of a signal and a wire. Wires are allowed to have branching points and there are two types of branching points, a junction and a $\bar{A} \cdot B$ logic element. His model also has a clock (periodic signal generator). He shows that above elements can be embedded into his two-state von Neumann neighborhood CA and combining these elements, he also shows that it is possible to realize AND, OR, NOT, and signal crossover elements depicted in Fig. 3.

In NCCA case, we showed a 14-state logically universal CA A_{14} with von Neumann neighborhood and rotation-symmetry. $A_{14} = (\mathbf{Z}^2, N_{[-2,11]}, f_{14}, 0)$[7]. The flow functions g_{14}, h_{14} of f_{14} have the values in Table 1 and satisfy $g_{14}(x,y) = -g_{14}(y,x)$ and $h_{14}(x,y) = -h_{14}(y,x)$ for all $x, y \in N_{[-2,11]}$. Values not defined by Table 1 are 0. Fig. 4 shows configurations of a junction and a period 9 clock.

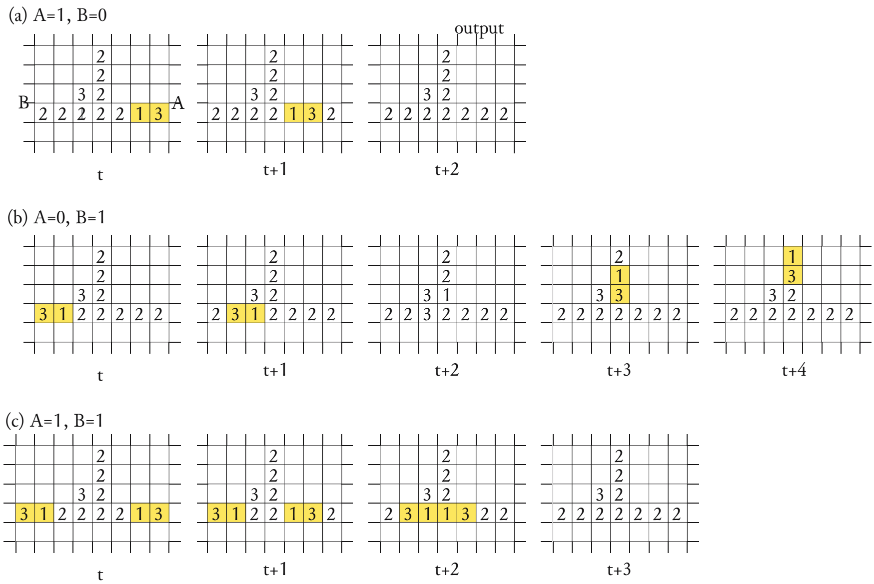

Fig. 5. Configurations of $\bar{A} \cdot B$ logic element for each inputs

In Moore neighborhood NCCA, we construct logically universal CA A_6 with rotation-symmetry. $A_6 = (\mathbf{Z}^2, \{0, 1, 2, 3, 4, 5\}, f_6, 0)$. ϕ which defines f_6 has the values in Table 2. Values not defined by Table 2 are 0. A junction and a clock are same as A_{14} (Fig. 4). Fig. 5 shows configurations of $\bar{A} \cdot B$ logic elements for each inputs.

6 Conclusion

In this paper, we gave a sufficient condition for a Moore neighborhood CA to be number-conserving and showed that the condition is usable to construct a Moore neighborhood NCCA. According to this condition, the local function of rotation-symmetric NCCA is expressed by a summation of quaternary functions ϕ. ϕ indicates the behavior of values within a block composed of four cells. Assigning a function ϕ is enough to design a rotation-symmetric Moore neighborhood NCCA. Employing this method, we constructed a 6-state logically universal NCCA.

An NCCA which is constructed by ϕ has a stronger symmetry than the rotation-symmetry. But the symmetry is not clear so far. In case of a larger radius neighborhood, a construction method of an NCCA may be established by the same approach which separates a neighborhood into several blocks.

Acknowledgments. The authors thank Kenichi Morita and Chuzo Iwamoto for their daily advices. The authors also thank anonymous reviewers for their attentive comments.

References

1. Banks, E.R.: Universality in cellular automata. In: Proc. Eleventh Annual Symposium on Switching and Automata Theory, pp. 194–215 (1970)
2. Boccara, N., Fukś, H.: Number-conserving cellular automaton rules. Fundamenta Informaticae 52, 1–13 (2003)
3. Durand, B., Formenti, E., Grange, A., Róka, Z.: Number conserving cellular automata: new results on decidability and dynamics. In: Morvan, M., Rémila, É. (eds.) Discrete Models for Complex Systems, DMCS 2003. Discrete Mathematics and Theoretical Computer Science Proceedings, vol. AB, pp. 129–140 (2003)
4. Imai, K., Ikazaki, A., Iwamoto, C., Morita, K.: A logically universal number-conserving reversible cellular automaton with a unary table-lookup function. Trans. IEICE E87-D(3), 694–699 (2004)
5. Imai, K., Furuta, K., Iwamoto, C., Morita, K.: A universal number conserving cellular automaton whose number of inter-cell moving particle is restricted. In: 12th International Workshop on Cellular Automata (2006)
6. Nagel, K., Schreckenberg, M.: A cellular automaton model for freeway traffic. Journal of Physics I France 2, 2221–2229 (1992)
7. Tanimoto, N., Imai, K.: A characterization of von Neumann neighbor number-conserving cellular automata. In: Proceedings of 2nd International Workshop on Natural Computing, 10 pages (2007)
8. Tanimoto, N., Imai, K., Iwamoto, C., Morita, K.: On constraints of rotation-symmetry of number-conserving cellular automata on their state numbers. In: LA symposium 2007 winter (in Japanese), pp. 4.1–4.5 (2008)