Finite Size Stability Analysis for Stochastic Cellular Automata

Yukio Sakisaka[1], Yukio Iwamura[1], Nariyuki Nakagiri[2], Jin Yoshimura[3],
and Kei-ichi Tainaka[3]

[1] Ibaraki Prefectural University of Health Sciences, Ami 300-0394, Japan
[2] University of Hyogo, Himeji 670-0092, Japan
[3] Shizuoka University, Hamamatsu 432-8561, Japan

Abstract. Real simulations are performed on a finite size of lattice. It is therefore very difficult to predict a phase diagram on an infinitely large lattice. Here, we present a Finite Size Stability Analysis (FSSA) to know whether the phase is sustainable or not. Although this analysis is a hypothesis, it enables us to determine the boundary of phase diagram. We apply FSSA to multi-state system. For example we study ten-species system in ecology. From computer simulations on various sizes of lattices, we obtain the waiting time τ to extinction. The system is found to have two phases: the coexistence of all species is either unstable or marginally (neutrally) stable. In the latter case, τ diverges on a power law with the increase of lattice size.

1 Introduction

In recent decades, stochastic cellular automata have been extensively studied in various fields [1,2]. The present paper focuses on the problem "how to determine the phase diagram?" When each site takes one of many states, then there are many phases in stationary state (equilibrium). However, it is not easy to know what kinds of states coexist in each phase. When a lattice size is small, annihilation of almost all states always occurs. To determine the phase diagram is one of important problems. In this paper, we present a "Finite Size Stability Analysis" (FSSA). From the result of real simulations, FSSA predicts whether the coexistence of several states is sustainable or not on an infinitely large lattice.

For example, we consider two-state model: each lattice site takes one of two states A or B. There are three possible phases in the final equilibrium; namely, the survival phase of A (phase A), the survival phase of B (phase B) and the coexistence of both states (phase A & B). In the case of contact process, the phase boundaries can be determined by mathematics, and the dynamics is known for various sizes of lattices [3,4].

The contact process has been introduced by Harris [5] as a simple model for the spread of infection. It has been extensively studied in various fields, such as mathematics [3,6,7,8,9], physics [10,11,12] and ecology [13,14]. Consider a single species on a d-dimensional regular lattice $\mathbf{Z}^d$; each lattice site takes one of two

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states: either empty (E) or occupied (X) by an individual. Birth and death processes are respectively defined by

$$X + E \longrightarrow 2X \qquad (\text{rate:}\, r) \tag{1a}$$

and

$$X \longrightarrow E \qquad (\text{rate:}\, m), \tag{1b}$$

where the parameters r and m represent the reproduction and mortality rates of an individual, respectively ($r > 0$, $m > 0$). So far, many researchers have mathematically studied the contact process, and reported the following properties [6,7,9,10,12,15,16]:

1) When the lattice size is infinite, the system dynamics goes into one of two equilibrium phases; namely the species X survives (phase X & E) or goes extinct (phase E).
2) When the lattice size is finite, the annihilation always occurs. The waiting time τ to annihilation increases with the increase of the lattice size (L) as shown in Fig. 1. In the case of phase X & E, the value τ diverges on an exponential scale with the increase of L. In the case of phase E, the value τ diverges on an log scale with the increase of L.
3) If the system locates on a phase boundary, a power law ($\tau \propto L^{\alpha}$) holds, where the exponent α is a positive constant.

We apply this idea to multiple-state model. Here the waiting time τ is defined by the period until the first annihilation of a state occurs.

2 Model

We study multiple contact process on a finite size (L) of a 2-dimensional lattice [17]. Each lattice site is empty (E) or occupied by species i (X_i), where $i = 1, \ldots, N$. Overall reactions are defined as follows:

$$X_i + E \longrightarrow 2X_i \qquad (\text{rate:}\, r) \tag{2a}$$

and

$$X_i \longrightarrow E \qquad (\text{rate:}\, m). \tag{2b}$$

Here the parameters r and m are assumed to take the same value for any species, respectively. The model of (2a) and (2b) was first presented by Miyazaki et al [18] to explain the paradox of plankton. Many plankton species can coexist in a pond or a lake, although they compete with each other. This contradiction is the orgin of “paradox of plankton” [19]. The simulation is carried out in two different ways: global and local interactions. We first describe the simulation method for the local interaction version of lattice model:

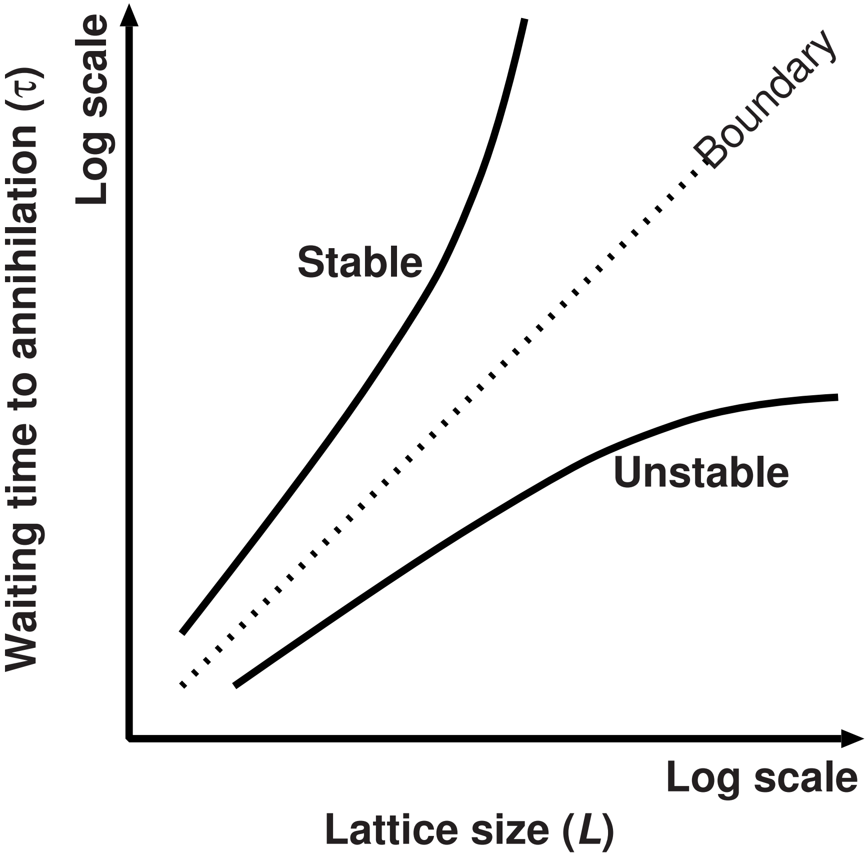

Fig. 1. A schematic diagram of finite size stability analysis (FSSA). We obtain the waiting time τ until a state annihilate for various values of lattice size L. 1) In the unstable phase of coexistence, τ diverges on a log-log scale with the increase of L. 2) In the stable phase, τ exponentially diverges with L. 3) The boundary case between unstable and stable is represented by straight dotted line on a log-log scale (i.e. power law).

1) Initially, we distribute individuals on a lattice.
2) Reactions (2a) and (2b) are performed in the following two steps:
    (i) We perform two-body reaction (2a). Choose one lattice site randomly, and then randomly specify one of $2d$ neighboring sites. If both chosen sites are X_i and E, then E is changed to X_i with probability r.
    (ii) We perform one-body reaction (2b). Choose one lattice point randomly; if the site is occupied by X_i, the site will become E with probability m.
3) Repeat step 2) L^2 times, where L^2 is the total number of a square lattice site. This total number is called the Monte Carlo step (MCS) [20].
4) Repeat step 3) until the first species goes extinct.

Next, we describe the method of global interaction in which the interaction occurs between any pair of lattice sites. The global simulation is very similar to local case, but step (i) in 2) is replaced as follows:

(i′) A pair of lattice sites is chosen not only randomly but also independently.

Under global interaction, the population dynamics of the system of (2a) and (2b) is given by the mean-field theory. Let x_i be the overall density of species i, then we have the following dynamics:

$$\frac{dx_i}{dt} = -mx_i + rx_i e, \tag{3}$$

where e is the density of empty site ($e = 1 - \sum_i x_i$). The first and second terms on the right hand side of (3) comes from death and birth processes, respectively. By summing up (3) with respect to i, we have the following logistic equation:

$$\frac{dS}{dt} = RS(1 - S/K), \tag{4}$$

where S is the total density ($\sum_i x_i$) and $K = 1 - m/r$. Thus, when $m/r < 1$, then S is stable; namely, $S \to K$ for $t \to \infty$. When $m/r \geq 1$, all species go extinct ($S \to 0$ for $t \to \infty$).

3 Simulation Results

We assume that our system contains ten kinds of species ($N = 10$) on a finite size of square lattice. To obtain the waiting time to extinction, we set the simulation condition as follows.

1) The values of reproduction rates for both global and local interactions are such chosen that the stationary densities take the same value.
2) Initially, all species have the identical density which is just the same as the steady-state density.

For various sizes of lattices, we obtain a waiting time (τ) until the first species goes extinct. A typical result is illustrated in Fig. 2, where the vertical axis denotes the waiting time (τ), and the horizontal axis means the lattice size (L). In Fig. 3, we can easily know the approximation lines of both global and local simulations on a log-log scale. All statistical analyses for Figs. 2 and 3 were conducted in R Development Core Team [21]. From Figs. 2 and 3, we find the following power law:

$$\tau \propto L^{\alpha}. \tag{5}$$

The value of exponent takes $\alpha = 2.1$ for global and $\alpha = 2.6$ for local interactions. Hence, simulations for both methods exhibit the similar critical behavior. Equation (5) always holds for any values of r or m, so long as the condition $m/r < c$ is satisfied. Here c is the critical value; $c = 1$ for global and $c \approx 0.625$

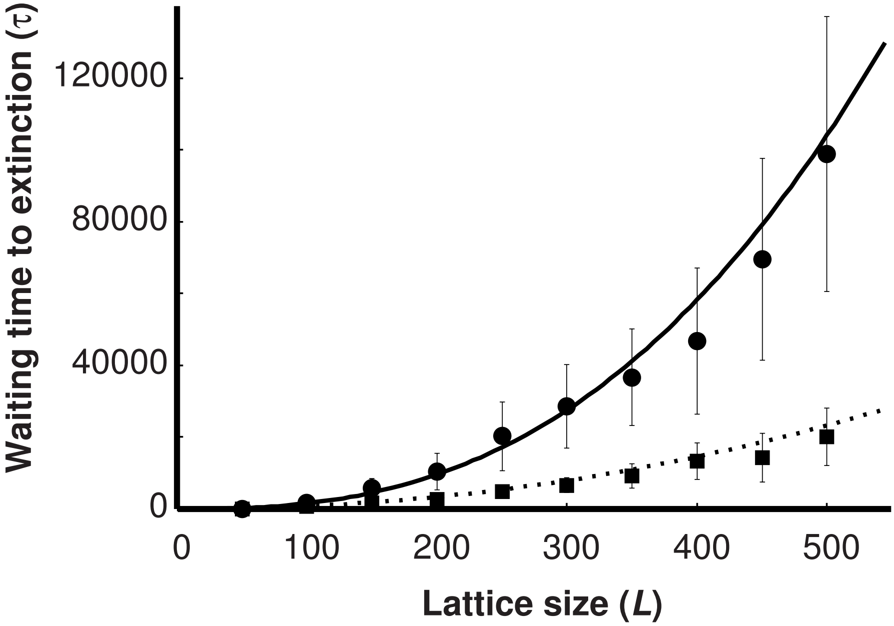

Fig. 2. Results of multiple contact process simulations ($m = 0.3$). All reproduction rates (r) and mortality rates (m) of ten species take the same value, respectively ($N = 10$). The waiting time (τ) until the first species becomes extinct is plotted against the lattice size (L). The plots of filled circles represent the results of local interaction, where r takes values of 0.9. The plots of filled squares represent the results of global interaction, where r takes values of 0.786. Bars indicate standard deviation (SD) of each 20 simulation runs. Almost all plots are well approximated by $\tau = cL^{\alpha}$, where we have $(c, \alpha) = (0.010,\ 2.6)$ for local interaction (solid curve) and (0.050, 2.1) for global interaction (broken curve).

for local interactions [12]. In contrast, when $m/r \geq c$, all species go extinct. Note that all species coexist in neutrally stable phase, but there is no stable phase.

Previous works [4] for multiple contact process reported the following outcomes: when the lattice size is infinitely large ($L \to \infty$), the equilibrium phases were well known. In the case of global interaction, coexistence of all species is possible. However, in local interaction, coexistence of plural number of species is impossible for any condition. Only one species can survive at most. After the extinctions, the system is eventually equivalent to the contact process (single-species or two-state system). However, Fig. 2 and Fig. 3 indicates peculiar results never predicted from the conclusion for equilibrium analyses:

(i) Although extinction is unavoidable for local interaction, the critical behavior is observed.
(ii) The exponent α for local interaction takes a lager value than that for global interaction.

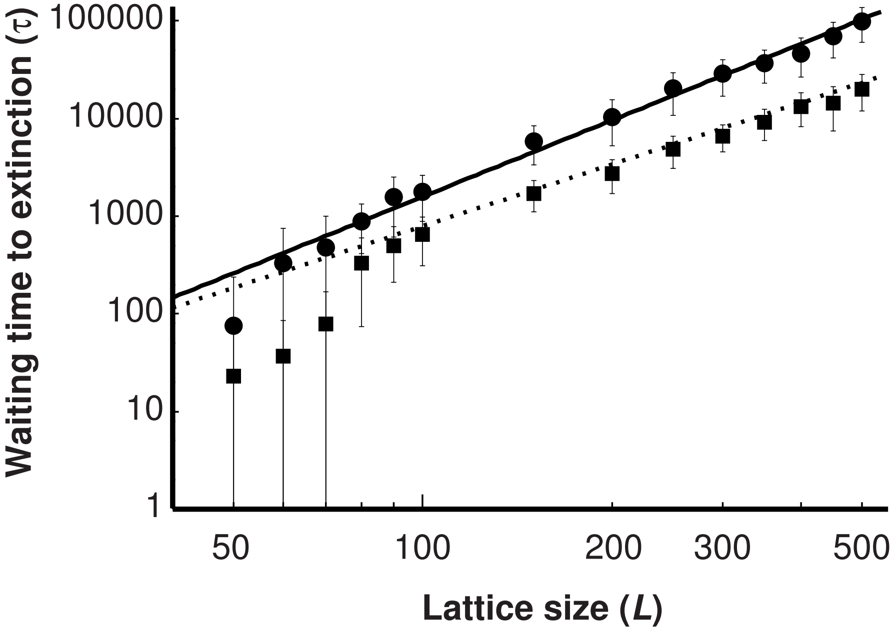

Fig. 3. The same data of Fig. 2 plotted on a log-log sacale. Both vertical and horizontal axises are correspond to Fig. 1's. The function $\tau = cL^{\alpha}$ forms a line on this scale.

It is therefore found that species survive for a very long time, so long as the local interaction is applied.

4 Conclusions and Discussions

We present a new method called finite size stability analysis (FSSA). By the use of FSSA, we can determine whether multiple states is stable or not in each phase. We obtain the waiting time (τ) until the first state annihilates against the lattice size (L) as shown in Fig. 1. If the power law (5) holds, then the coexistence is neutrally stable (boundary between different phases).

An example is an ecological model of (2a) and (2b) to explain the coexistence of plankton species. In this system, there are eleven states: each lattice site takes empty (E) or individual (X_i) of species i. This system evolves into one of two phases: extinct or survival phases. When $m/r \geq c$ (extinct phase), all lattice sites become empty (E). Here $c = 1$ for global and $c \approx 0.625$ for local interactions. In contrast, when $m/r < c$, the coexistence of all species becomes neutrally stable: (5) always holds for any values of r or m. From the mean-field theory, readers may have the following question: why the coexistence is neutrally stable in spite of logistic equation (4). The answer is simple. Equation (4) predicts the stability of the total density S, but the density of each species is neutrally stable [17].

We here suggest the origin of the power law (5) for local interaction. This law can be observed for other values of parameters, as long as all species have the same reproduction rates and same mortality rates, respectively. Moreover, our results never change for different number of species. We carry out simulations for 9-, 10- and 11-species systems. When species number increases, the waiting time (τ) decreases. However, the value of exponent never changes.

In the two-species contact process, Neuhauser [4] indicated that the problem of coexistence or competitive exclusion can be replaced by that of collision of random walking particles. Random walking and extinction processes of many particles on a lattice space are studied extensively in the field of diffusion-limited reactions. According to these studies, the number of particles decreases proportionally to $t^{-\beta}$ in most cases (β is a positive constant) [22,23,24]. By assuming that extinction occurs at about $1/L$ particles, the extinction time τ satisfies the relation $1/L \sim \tau^{-\beta}$. Therefore, we can expect the power law $\tau \sim L^{\alpha}$. Note that it holds for large values of L.

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