Combined Effect of Topology and Synchronism Perturbation on Cellular Automata: Preliminary Results*

Jean-Baptiste Rouquier[1] and Michel Morvan[1,2]

[1] Université de Lyon, IXXI and ENS Lyon, France
[2] EHESS and Santa Fe Institute

Abstract. The aim of this paper is to experimentally study the combined effect of the introduction of two kinds of structural perturbations to the behavior of cellular automata. We present the results obtained by simultaneously perturbing synchronism and topology of elementary cellular automata. We show that very interesting and different behaviors appear, including phase transitions and non monotonicity (i.e. introduction of both perturbations is less effective than the introduction of only one of them). These results lead us to think that this study is worth to be now developed more accurately.

1 Introduction

The last decade has seen the development of an interest to the effect of structural perturbation applied to discrete dynamical model [1,2,3,4,5]. Almost all such studies were focused on the introduction of one kind of perturbation. However, in [6], it is shown that combining two perturbations, namely the introduction of asynchronism and a topology perturbation, could lead to interesting new behaviors of the Game of Life. In particular, it has been shown that while the Game of Life was very sensitive to asynchronism, this sensitivity was strongly decreased when coupled with the breaking of a small percentage of links in the grid.

The aim of this paper is to develop an exhaustive study of the coupling of the same kind of perturbations for elementary cellular automata. We will show that very different behaviors appear. On the one hand, part of these results were expected: some ECA are robust to both perturbations, even when applied simultaneously, other are sensitive to only one, the other having no effect, and some are sensitive to both of them when they are applied simultaneously. On the other hand, some interesting and more unexpected behavior appear in this last case. For example, the combination of both perturbation can in some cases lead either to the vanishing of a phase transition induced by one perturbation alone or to the appearance of a new phase transition. Moreover, rule 45 exhibits the same behavior as the Game of Life: one perturbation makes the system robust

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to the introduction to the other, and so the effects of the perturbations are not combined in a monotonous way.

The main conclusion of this exploratory paper is that the robustness to the combination of different perturbations is a subject which deserves to be studied in more details, even in very simple systems as ECA.

The paper is organized as follows. In Section 2, we describe the definitions and in particular the definition of the perturbations we will apply to ECA. We also present the experimental protocol. Section 3 contains the result and brief analysis of the experimental results. We classify the different observed behavior and highlight those that are, in our opinion, the most interesting for future studies. We end by a discussion in which we propose promising research directions.

2 Definitions and Experimental Protocol

2.1 Perturbed Cellular Automata

Definition 1. *A Cellular Automata (CA) with periodic boundary condition is a tuple consisting in*

- Q *a set of* states*,*
- $\mathcal{U} := (\mathbb{Z}/n\mathbb{Z})^d$ *the finite set of* cells *(d is the dimension, n is a parameter),*
- $V := \{\vec{v}_1, \ldots, \vec{v}_{|V|}\}$ *is the* neighborhood*, a finite set of vectors of* $\mathbb{Z}^d$*,*
- $\delta : Q^{|V|} \to Q$ *is the* update rule.

A *configuration* specifies the state of each cell, and is thus a function $\mathcal{U} \to Q$. The usual dynamics on a CA is the following: considering a configuration c_t, the next configuration c_{t+1} is defined by $\forall z, c_{t+1}(z) := \delta(c_t(z+\vec{v}_1), \ldots, c_t(z+\vec{v}_{|V|}))$.

This early works focuses on a particular class of CA, namely the Elementary Cellular Automata (ECA), the simplest ones: $Q = \{0, 1\},\ d = 1,\ V = \{-1, 0, 1\}$. We use the classical notation introduced by Wolfram: an ECA is denoted by the code $\delta(0,0,0) \times 2^0 + \delta(0,0,1) \times 2^2 + \delta(0,1,0) \times 2^2 + \cdots + \delta(1,1,1) \times 2^7$.

The fact that it is one dimensional implies that n is also the number of cells.

We perturb this dynamics in two ways. First, not all cells are updated at each time step. Instead, at each time step, each cell has a fixed probability p to get updated, independently of the other cells. p is a parameter. The case $p = 1$ is the unperturbed (synchronous) case. Updating only one cell at a time would also be a classical model [7].

Second, a cell might not always "see" all its neighbors. Instead, at each time step, when the cell z requests the states of its neighbors to compute its next state, it gets no answer with probability r. It then assumes the neighbor is in state q. q and r are parameters. The case $r = 0$ (no link cut) is the unperturbed case.

The final dynamics is thus:

$$c_{t+1} := \begin{cases} \delta\big(c'_t(z-1),\ c_t(z),\ c'_t(z+1)\big) & \text{with probability } p \\ c_t(z) & \text{with probability } 1-p \end{cases}$$
$$\text{where } c'_t(y) := \begin{cases} q & \text{with probability } r \\ c_t(y) & \text{with probability } 1-r \end{cases} \tag{1}$$

Motivation of the Model Not updating a cell can be seen as a delay before updating. It can also be seen as a defective cell, where we chose that a defective cell keeps its state. When a cell does not sees one of its neighbor, it corresponds to a defective (oriented) link between the cells.

All defects are temporary. This is necessary for ECA since otherwise a defect would prevent all communication between its left and right sides.

2.2 Experimental Protocol

We call *run* the temporal evolution of a CA when all parameters are chosen. The macroscopic measure we use on a run is the density of cells in state 1. This measure has been shown relevant in [8]. For a configuration c, we note the density

$$\rho(c) := \#\{z \in \mathcal{U} \mid c(z) = 1\} \;/\; |\mathcal{U}|$$

Sampled Parameters. We do one run for each possible combination of the following parameters.

For a given choice of the rule (including the parameter q), we sample many values of p and r, that is, we combine many strengths of synchronism and topology perturbation. For the former, we sample the whole space of synchrony rates, including the synchronous case, by steps of 0.01: $p \in \{0.01, 0.02, \dots 1.00\}$. For the latter, cutting 10% of the links is already a strong perturbation, and we do not perturb the model more. That is, $r \in \{0.00, 0.01, 0.02, \dots 0.10\}$.

We sample both values (0 and 1) for the parameter q. For a given rule, e.g. 110, we note the parameter q as a subscript, e.g. 110_0 and 110_1. For autoconjugate rules (i.e. rules that are unchanged under the exchange of state 0 and 1), both choices of the parameter q leads to the same dynamics (actually to symmetric ones). We therefore study only one, which we note 105_a.

When working on ECA, one usually considers only 88 rules among the 256 possible ones, thanks to symmetry considerations. The left/right symmetric of a rule δ (i.e. the rule $\delta'(a, b, c) := \delta(c, b, a)$) leads to configurations symmetric to the ones obtained with . Therefore, we will only consider one rule among each pair (δ, δ'). However, the other usual symmetry, namely exchanging the states 0 and 1, is no longer possible. Indeed, the parameter q has introduced an asymmetry between both states.

Fixed Parameters. The following parameters are identical for all runs.

- The ring size is n=10 000 cells. [4] estimated that 2 000 was already sufficient in a similar problem.
- The initial configuration is random (each cell is in state 0 with probability 0.5, independently from the other cells) and distinct for each run.
- To speed up computation, we do not measure at each time step, instead we do the same number of measurements (roughly 200) in each decade.
- To detect when the density has settled to a constant value, we do a linear regression on the last 200 measure points and wait for the fitted line to be almost horizontal (we actually do three independent fits on three portions of the history). We then average the density over the history.

– If the density has not stabilized after 10^6 steps, we stop the simulation and ignore this point.

Limits of the Protocol. The density might be seen as a quite rough measure, and so we will certainly miss some phenomenons occuring in this model. However, the variety of results shows that this measure is able to detect one kind of sensibility to the perturbations applied here.

Another limit is that this protocol does not measure the true asymptotic density, but instead a long enough stage where the density is stable. This is the only stage observable experimentally if the asymptotic regime occur only after exponential time. Nevertheless, decision to stop a run might occur early if the density is very slowly decreasing.

Last, for each value of the parameters, we compute only one data point. The smoothness of most surfaces shows that variance among runs is low, and that we do not need to average over several data points.

3 Exhaustive Study of the ECA Space

We now apply this protocol to each ECA and plot ρ against p and r (Fig. 1). Reviewing the sampling surfaces suggests the classification in four classes of Table 1, based on the sensitivity to each perturbation. The subclasses a/, b/ and c/ are explained in Section 3.4. We now review each class.

Table 1. Classification of the ECA from their robustness against each perturbation

		Topology perturbation	
		insensitive	sensitive
Synchronism perturbation	insensitive	0_0 0_1 2_0 8_0 8_1 10_0 23_a 24_0 28_0 32_0 34_0 40_0 42_0 51_a 56_0 60_0 60_1 74_0 105_a 128_0 128_1 130_0 136_0 136_1 138_0 152_0 154_1 160_0 162_0 168_0 200_1 204_a	4_1 5_0 12_1 13_0 15_a 29_0 29_1 32_1 36_1 40_1 44_1 72_0 72_1 76_1 78_1 104_0 104_1 132_1 140_1 160_1 164_1 168_1 170_a 172_1 184_0 184_1 200_0 232_a
	sensitive	1_0 4_0 5_1 6_0 9_0 11_0 11_1 12_0 18_0 19_0 19_1 22_0 26_0 33_0 35_0 35_1 36_0 38_0 41_0 41_1 44_0 46_0 50_0 54_0 58_0 76_0 90_1 108_0 110_0 110_1 132_0 134_0 146_0 164_0	a/ 2_1 6_1 10_1 24_1 34_1 38_1 42_1 45_0 45_1 46_1 56_1 73_0 73_1 74_1 77_a 94_0 94_1 130_1 134_1 138_1 152_1 154_0 162_1 b/ 18_1 26_1 50_1 58_1 106_0 106_1 146_1 178_a c/ 1_1 3_0 3_1 7_0 7_1 9_1 13_1 14_0 14_1 22_1 25_0 25_1 27_0 27_1 28_1 30_0 30_1 33_1 37_0 37_1 43_a 54_1 57_0 57_1 62_0 62_1 78_0 90_0 108_1 122_0 122_1 126_0 126_1 140_0 142_a 150_a 156_0 156_1 172_0

3.1 Rules Insensitive to Both Perturbations

Most of the rules in this class have a surface that is just 0 everywhere (Fig 1.a), sometimes with an exception in the unperturbed case ($p = 1$, $r = 0$). A few

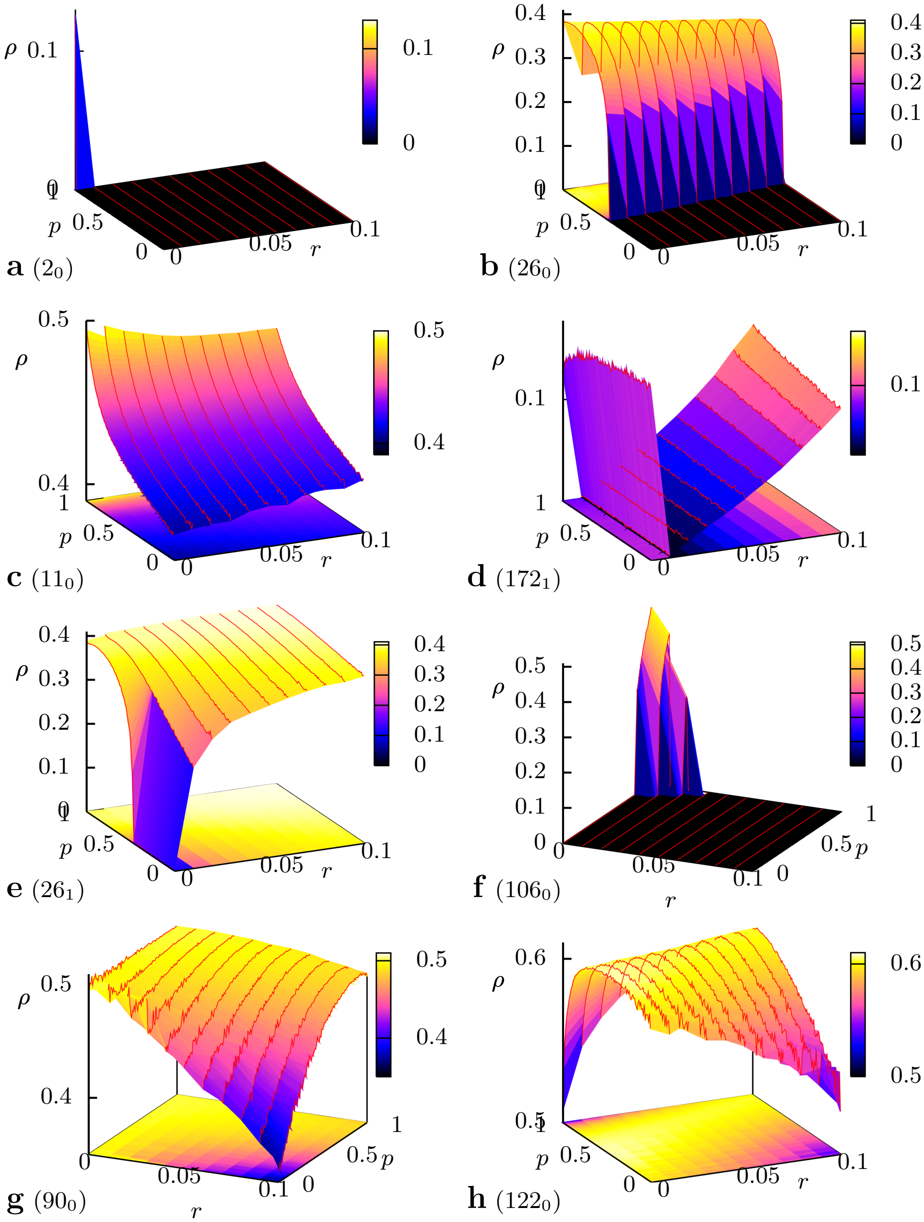

Fig. 1. Some sampling surfaces (color online). Axis x (width) represents topology perturbation, y (depth) synchronism perturbation, z (height) asymptotic density. The difference between Figures 1.a and 1.f is that in the former only the unperturbed case ($p = 1, r = 0$) is non zero, whereas in the latter there is a continuous transition from non-zero to zero, with respect to both p and r.

rules (9%) outside this class also have a noticeably different behavior in the unperturbed case, but this is left for further study.

Since they are by definition insensitive to the perturbations, the other rules of this class also have a horizontal surface, (the altitude is often around $= 0.5$).

3.2 Rules Sensitive Only to Synchronism Perturbation

Some rules of this class undergo a phase transition when p changes (Fig 1.b). The conditions of a conjecture by [9] are fulfilled, so this transition should belong to the universality class of directed percolation. (For an introduction to directed percolation in CA see [3], and for a detailed review of phase transitions see [10].)

It is interesting to note that the topology perturbation slightly shifts the threshold (the position of the transition) of most of those rules.

The remaining surfaces of this class are smooth, for instance like Fig 1.c. Since these rules are almost insensitive to topology perturbation, all slices along a red line (grey for a B&W version) are nearly identical.

3.3 Rules Sensitive Only to Topology Perturbation

Some rules of this class (4_1, 5_0, 12_1, 13_0, 36_1, 44_1, 72_0, 72_1, 76_1, 78_1, 104_0, 104_1, 132_1, 140_1, 164_1, 170_a, 172_1, 184_0, 184_1, 200_0, 232_a) are sensitive to the introduction of topology perturbation, i.e., they have a brutal change of behavior between $r = 0$ and $r > 0$ (Fig 1.d). A few rules actually have a density that depends on p, but only when $r = 0$.

The other surfaces of this class are smooth. The only phase transitions of this class are thus first order phase transitions (i.e. discontinuity of ρ),and they are all located at the introduction of topology perturbation.

3.4 Rules Sensitive to Both Perturbations

This is the most interesting class, as we will see many different behaviors. We examine specifically how both perturbations interact.

a/ Sensitive to Introduction of Perturbation. As in the class 3.3, the surfaces of this subclass exhibit a gap between $r = 0$ and $r > 0$. This is a first order phase transition, and it is not the only phase transition of class 3.4.

b/ Created or Destroyed Phase Transition. We have seen in section 3.2 that one of the two ways for the density to vary when p changes is a phase transition. For the ECA of Section 3.2, this phase transition was robust against topology perturbation. Here it is not the case since the phase transition is destroyed when $r > 0$ (Fig 1.e). We looked at the configurations (not reproduced here due to lack of space) and it seems that cutting a link in a region of cells in state 0 (i.e. in the absorbing state) corresponds to reseeding an active site at this point. The configuration is thus constantly reseeded with active sites and 0^n is no more an absorbing state. This is the case for 18_1, 26_1, 50_1, 58_1, 106_1, 146_1.

The case of 178_a is the opposite: when the topology is not perturbed, this rule is autoconjugate and can't have a preferred state. In [11], without topology perturbation, it is suggested that this rule could have a phase transition of the $\mathbb{Z}_2$ symmetric percolation universality class when p varies. The slightest topology perturbation introduces a bias towards 0^n, which becomes the only absorbing state. The rule then satisfies the conditions of the conjecture by [9], and should thus belong to the universality class of directed percolation. Therefore, we conclude that topology perturbation has created a new phase transition.

c/ Smooth. The last subclass contains the remaining surfaces, which are smooth.

Some Remarkable Rules. We now would like to point out some remarkable rules of the present class (Section 3.4).

Rule 106_0 (Fig 1.f) exhibits two phase transitions, one when p varies and the other when r varies.

Rule 90_0 (Fig 1.g) is almost insensitive to a single perturbation. However, both perturbations combined induce a noticeable decrease of the density. (This study focuses on small r, but it is not hard to see that for high enough r, rule 90_0 has a vanishing density.)

Rule 122_0 (Fig 1.h) has a density close to 0.5 in the unperturbed case. When applying any one of the perturbations, the density tends to 0.6. But when applying both, the density tends to back 0.5. (However, the configurations are quite different from the unperturbed case, only the density has been restored.) One perturbation can be seen as a way to make the CA robust against the other perturbation. This is one of the rare cases where perturbations are not combining in a monotonous way.

Rule 73 (which has the same qualitative behavior for both values of q) has a constant asymptotic density when topology is not perturbed, for all values of p (except $p = 1$). With a small perturbation of the topology, spans the range 0.3 to 0.5. The interesting effect is that introducing topology perturbation has made the rule sensitive to synchronism perturbation. Same remark applies to rules 14_0, 14_1, 57_0, 90_0, 150_a 156_0 156_1 and to almost all rules of subclass a/.

The opposite behavior also exists: 45, with no topology perturbation, has a density that depends on p. With $r > 0$, the density becomes constant for all p: perturbing the topology makes the rule robust against synchronism perturbation.

4 Discussion and Perspectives

By means of *dynamic* perturbation, we studied topology perturbation on ECA and its interplay with synchronism perturbation. Topology perturbation is a perturbation on the neighboring cells, while synchronism perturbation is a perturbation on the central cell. The macroscopic parameter chosen, namely the density, already allows one to see a rich set of behaviors, even if there may be more than what this measure reveals. This is both a confirmation of the relevancy of the density and a confirmation that this model could be studied further, possibly with analytical results.

There does not seem to be any relation between this classification and classical classifications about synchronous deterministic CA.

The topology perturbation introduced in this paper should be studied for itself, possibly with the following generalization of the parameter q. When a cell does not know the state of its neighbor, it currently assumes it is in state q. It could instead decide its state according to a Bernoulli law. Future work will study the combined effect of r and this new parameter q.

A few surfaces contains the characteristic curve of second order phase transition. Future work should check if they belong to the universality class of directed percolation, according to the aforementioned conjecture.

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