Controlling the Dynamics of the Fuzzy Cellular Automaton Rule 90, I.

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Abstract. Control problems on Cellular Automata (CA) models have been introduced in a rigorous mathematical framework [10]. In this paper, we attempt to apply the control theory concept to the special class of fuzzy CA for which more freedom is gained using a continuum state space. Focusing on the case of fuzzy rule 90, we investigate the possibility of finding a control $u = (u_0, u_1, \cdots, u_{T-1})$ which forces the system at a localized cell, to achieve a given desired state at time T. The problem is studied starting from an initial configuration consisting of a single seed on a zero background.

Keywords: Fuzzy Cellular automata, long-term evolution, Control.

1 Introduction

Cellular automata constitute a very interesting modelling approach which has been explored from mathematical and computational points of view for theoretical as well as practical aspects [17,4,3,13,14,19]. However, from the mainstream literature, CA in their classical form are treated as closed systems, as they do not take into account the interaction between the system and its environment. Considering control problems on systems using CA approaches should be beneficial for this field of research and makes connections with the field of systems theory.

The basic idea of control theory states that systems behavior is caused by a response to an outside stimulus and may be influenced so as to achieve a desired goal [18]. In order to implement this influence, engineers build devices that incorporate various mathematical techniques.

An appropriate way of introducing controls in CA models in order to make them more useful in systems theory has been given in [10,6]. Some concepts related to the control theory (regional controllability, identification, spreadability) has been studied mainly in the case of additive CA [7,8,9]. However, the problem of obtaining analytical results is still posed.

We consider in this paper the fuzzy version of CA which constitute a direct generalization of the classical binary CA models. These elementary cellular

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automata (ECA) which have been studied by Wolfram and others, are good examples of systems with simple rules that may produce unusually complex behavior. We investigate the *Fuzzy CA* (FCA) as a real-valued version of ECA which seem to provide the best results regarding the control problems.

Considering a FCA evolution on a time interval $[0, T]$, we will address the question whether some particular target state is reachable starting from a specific initial condition. The control value to be found is given by the vector $u = (u_0, u_1, \cdots, u_{T-1})$. We enlarge the state space from $[0, 1]$ to $\mathbb{R}$ in order to obtain more flexibility regarding the control values and then a necessary and sufficient condition is found. The same result is obtained when working on $[0, 1]$ but only for specific desired states or small values of T.

2 Basic Definitions

2.1 Cellular Automata

A cellular automaton (CA) may be thought of as a linear collection of cells where all cells share the same local space (i.e., the set of values for the cells) the same neighborhood structure (i.e., the cells on each side of a cell), and the same local function or rule (i.e., the function defining the effect of the neighbors on each cell, also called the transition function or rule function). The global evolution of the CA is defined by the synchronous update of all cell values according to repeated applications of the local function to the neighborhood of each cell. A configuration of the automaton is a state of all lattice cells [22].

Cellular automata were one of the first abstract models for parallel computing. Conceived by John von Neumann [17] in the early 1950's to investigate self-reproduction, CA have been used mainly for studying parallel computing methods and the formal properties of model systems.

Given a bi-infinite lattice of cells on a line, the local space $\{0, 1\}$, the usual neighborhood structure $\langle$ left neighbor, itself, right neighbor $\rangle$, and a rule function $g : \{0, 1\}^3 \longrightarrow \{0, 1\}$, the global dynamics of an *elementary CA* are defined by:

$$f : \{0, 1\}^{\mathbb{Z}} \longrightarrow \{0, 1\}^{\mathbb{Z}}$$
$$\forall i \in \mathbb{Z}, f(x)_i = g(x_{i-1}, x_i, x_{i+1}).$$

The rule function or *local rule* is then defined by the 8 possible local configurations a cell detects in its direct neighborhood:

$$(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (r_0, \cdots, r_7),$$

where each triplet above represents a local configuration of the left neighbor, the cell itself, and the right neighbor. In general, the value of the sum $\sum_{i=0}^{7} 2^i r_i$ is used as the name of the rule. It is well-known that the local rule of any boolean CA can be expressed canonically as a *disjunctive normal form (DNF)*, that is,

$$g(x_1, x_2, x_3) = \vee_{i|r_i=1} \wedge_{j=1:3} x_j^{d_{ij}}$$

where d_{ij} is the j-th digit, read from left to right, of the binary expression of i, and x^0 (resp. x^1) stands for $\neg x$ (resp. x).

2.2 Fuzzy Cellular Automata

The initial string now consists of a set of *fuzzy* states, that is, a collection of arbitrary but fixed *real numbers* in the closed interval $[0, 1]$ (as opposed to the two-point set $\{0, 1\}$). The process of *fuzzification* described below entails redefining the local rule g above so that it can now act on triples of real numbers (as opposed to triples of boolean numbers) and map the unit box $[0, 1]^3 \in \mathbb{R}^3$ into the unit interval $[0, 1]$.

Inherent in this procedure is the fact that fuzzification will allow one to move from the discrete (boolean CA) to the continuous (fuzzy CA, or FCA) by extending the domain of definition of the rule in such a way that the new "rule" agrees with the original rule when we restrict its domain to the boolean set $\{0, 1\}$. We describe a natural method of fuzzifying a given boolean rule herewith, the source of which is in [2]. We adopt the now standard terminology from Flocchini *et al*, [11].

Definition 1. *A "fuzzy" cellular automaton or fuzzy CA or FCA for brevity, is obtained by* fuzzifying *the local function of a given boolean CA in the following way: For real numbers $a, b \in [0, 1]$ we redefine the quantities $(a \vee b)$ to be $(a + b)$, $(a \wedge b)$ to be (ab), and $(\neg a)$ to be $(1 - a)$ in the DNF. Thus $a \vee b = a + b$, $a \wedge b = a \cdot b$, and $\neg a = 1 - a$, where $+$ and "$\cdot$" are ordinary addition and multiplication of real numbers.*

The case under consideration is rule 90. Since $90 = 2^1 + 2^3 + 2^4 + 2^6$ we see that its rule number, $90 = \sum_{i=0}^{7} r_i \, 2^i$, forces $r_i = 1$ for $i = 1, 3, 4, 6$. Using the DNF above gives us the rule function of FCA 90 in the form

$$g(x, y, z) = x + z - 2xz, \tag{1}$$

for $(x, y, z) \in [0, 1]^3$. We emphasize that in "fuzzifying" the DNF (**??**), we replaced $\neg x$ by $1 - x$, $x \vee y$ by $x + y$, and $x \wedge y$ in (**??**) by their product, $x \cdot y$.

In this case, the local fuzzy rule maps the triples of zeros and ones as follows:

$$000, 001, 010, 011, 100, 101, 110, 111 \rightarrow 0, 0, 1, 1, 1, 0, 1, 0.$$

3 Control of Fuzzy Rule 90

3.1 Problem Statement

CA have been used extensively as a modelling tool to approximate nonlinear discrete and continuous dynamical systems in a variety of applications. However the inverse problem of determining a/the CA that satisfies some specified constraints has received very little attention. A possible formulation for an inverse problem involves the search of an appropriate CA rule capable of carrying a given system from an initial state to a desired final configuration during a time horizon T.

If the rule has the form $s_{t+1} = F_u(s_t, u_t)$, the problem, usually referred to as the *controllability problem*, consists in finding a control $u = (u_0, u_1, \cdots, u_{T-1})$ in an appropriate control space such that, for some $T \geq 0$,

$$s_T = S_d$$

where S_d is the desired state, given in a suitable space of so-called *reachable states* and s_T is the CA configuration at time T.

3.2 The Case of an Excited Cell with a Single Initial Seed in a Background of Zeros

The notion of controllability is geared to the possibility of forcing a system into a particular state by using one or more appropriate control signals. In this work we consider the case where the signals are applied at $t = 0, 1, \ldots, T-1$ so as to influence the space-time diagram of the rule in order to achieve a desired state $A^T_{-T}, \cdots, A^T_{-1}, A^T_0$, at time T (cf., Table 1 below).

Hence the work presented here is related to the most general problem of control theory using Cellular Automata models. We solve it in the case where the initial string consists of two cells, one of which is a given fuzzy state $x^0_0 = a$, which may or may not be in $[0, 1]$, and the other of which is a control, $x^0_1 = u_0$, under which, in the cells x^i_1, $i = 1, 2, \ldots, T-1$, there is a string of $T-1$ other controls, all immersed in a background of zeros (the so-called *homogeneous background* case). The space-time diagram under consideration is of the form of Table 1

Table 1. The space time diagram showing the evolution of a rule function starting from a single seed **a** with a column of values assigned to the controlled cell

	$-T$	$\cdots$	-3	-2	-1	0	1
0	0	$\cdots$	0	0	0	a	u_0
1	0	$\cdots$	0	0	a	u_0	u_1
2	0	$\cdots$	0	a	u_0	$.$	u_2
$\vdots$			$\cdots$			$\cdots$	$\vdots$
i	$\cdots$	0	a	u_0	$\ldots$	$\cdots$	u_i
$\vdots$							$\vdots$
$T-1$	0	a	$\cdots$	$\cdots$	$\cdots$	$\cdots$	u_{T-1}
T	x^T_{-T}	$\cdots$	x^T_{-3}	x^T_{-2}	x^T_{-1}	x^T_0	

and the problem is to reach a state $(x^T_{-T}, \cdots, x^T_{-1}, x^T_0)$ which coincides with a desired one $A = (A^T_{-T}, \cdots, A^T_{-1}, A^T_0) \in \mathbb{R}^{T+1}$ at time T. In other words, the so-called *input-to-final-state reachability map*

$$\begin{aligned} K: \quad & \mathbb{R}^T \longrightarrow \mathbb{R}^{T+1} \\ & u = (u_0, u_1, \ldots, u_{T-1}) \longrightarrow x_T = (x^T_{-T}, \cdots, x^T_{-1}, x^T_0) \end{aligned}$$

is surjective on some appropriate subset of the range. Prior to formulating our results we need a few lemmas.

Lemma 1. *The rule function $g(x, y, z) \neq 1/2$ if and only if $x \neq 1/2$ and $z \neq 1/2$.*

Proof. Sufficiency: Assume, on the contrary, that $g(x, y, z) = 1/2$. Then $x + z - 2xz = 1/2$; and this implies that $x(1 - 2z) = 1/2 - z = (1 - 2z)/2$ or since $z \neq 1/2$ then $x = 1/2$ which is impossible. On the other hand, if either $x = 1/2$ or $z = 1/2$, the rule function $g(x, y, z) = 1/2$ which contradicts the assumption.

Our first result deals with the case where the domain of rule 90 is enlarged to all of $\mathbb{R}^3$. By doing so, we can obtain a necessary and sufficient condition for the controllability of the system.

Theorem 1. *Let $T > 0$ be a given (final) time, $A = (A^T_{-T}, \cdots, A^T_{-1}, A^T_0) \in \mathbb{R}^{T+1}$ be a given desired state. Consider the controllability problem associated with Table 1 with $a \in \mathbb{R}$, $g(x, y, z) = x + z - 2xz$ defined on all of $\mathbb{R}^3$.*

1. *If $A^T_{-T} \neq a$ then the control problem has no solution.*
2. *If $A^T_{-T} = a$, then the control problem has a (unique) solution in the admissible set $\mathcal{U}_{ad} = \mathbb{R}^T \setminus \{(1/2, 1/2, \ldots, 1/2)\}$ if and only if all the coordinates of A are different from $1/2$,* i.e., *for every N, $0 \leq N \leq T$ we have $A^T_{-T+N} \neq 1/2$.*

Proof. From Table 1 it is clear that $A^T_{-T} = a$, this proves the first claim. Hence this last equality is in force throughout. We proceed on a case-by-case basis for $i = 1, 2, 3$ and then appeal to an induction argument for the general case. The case $i = 0$ being trivial we proceed immediately to the next case.

In the case where $i = 1$ we observe that $x^i_{-i+1} = x^1_0 = g(0, a, u_0) = u_0$, for every i, $0 \leq i \leq T$.

When $i = 2$ note that $x^1_1 = u_1$, and since $x^2_0 = g(x^1_{-1}, x^1_0, x^1_1)$ by definition, we get

$$x^2_0 = x^1_{-1} + (1 - 2x^1_{-1})u_1. \tag{2}$$

Similarly,

$$\begin{aligned} x^3_{-1} &= g(x^2_{-2}, x^2_{-1}, x^2_0), \\ &= x^2_{-2} + (1 - 2x^2_{-2})x^2_0, \\ &= x^2_{-2} + (1 - 2x^2_{-2})(x^1_{-1} + (1 - 2x^1_{-1})u_1), \quad \text{(by (2))} \\ &= x^2_{-2} + (1 - 2x^2_{-2})x^1_{-1} + (1 - 2x^2_{-2})(1 - 2x^1_{-1})u_1. \end{aligned} \tag{3}$$

The form of the next term, x^4_{-2} necessary for our purposes is found similarly. Thus, substituting (3)) for x^3_{-1} we find,

$$\begin{aligned} x^4_{-2} &= g(x^3_{-3}, x^3_{-2}, x^3_{-1}), \\ &= x^3_{-3} + (1 - 2x^3_{-3})x^3_{-1}, \\ &= x^3_{-3} + (1 - 2x^3_{-3})(x^2_{-2} + (1 - 2x^2_{-2})x^1_{-1} + (1 - 2x^2_{-2})(1 - 2x^1_{-1})u_1), \\ &= x^3_{-3} + (1 - 2x^3_{-3})x^2_{-2} + (1 - 2x^3_{-3})(1 - 2x^2_{-2})x^1_{-1} + \\ &\qquad +(1 - 2x^3_{-3})(1 - 2x^2_{-2})(1 - 2x^1_{-1})u_1. \end{aligned}$$

Continuing in this way we define the terms x^n_{-m} when $n - m = 2$ recursively and find that the cell values along this diagonal are given by

$$x^n_{-n+2} = x^{n-1}_{-n+1} + \sum_{k=1}^{n-2} \prod_{j=k}^{n-2} (1 - 2x^{j+1}_{-j-1})\, x^k_{-k} + u_1 \prod_{j=1}^{n-1} (1 - 2x^j_{-j}). \tag{4}$$

We calculate one more case, $i = 3$, prior to stating the general form of the x^T_{-T+i} for any i and for any diagonal. Note that $x^2_1 = u_2$. Next,

$$x^3_0 = x^2_{-1} + (1 - 2x^2_{-1})u_2, \tag{5}$$

and

$$\begin{aligned} x^4_{-1} &= g(x^3_{-2}, x^3_{-1}, x^3_0), \\ &= x^3_{-2} + (1 - 2x^3_{-2})x^3_0, \\ &= x^3_{-2} + (1 - 2x^3_{-2})(x^2_{-1} + (1 - 2x^2_{-1})u_2), \quad \text{(by (5))} \\ &= x^3_{-2} + (1 - 2x^3_{-2})x^2_{-1} + (1 - 2x^3_{-2})(1 - 2x^2_{-1})u_2. \end{aligned}$$

The term x^5_{-2} is found as above. This gives,

$$\begin{aligned} x^5_{-2} &= g(x^4_{-3}, x^4_{-2}, x^4_{-1}), \\ &= x^4_{-3} + (1 - 2x^4_{-3})x^4_{-1} \\ &= x^4_{-3} + (1 - 2x^4_{-3})(x^3_{-2} + (1 - 2x^3_{-2})x^2_{-1} + (1 - 2x^3_{-2})(1 - 2x^2_{-1})u_2) \\ &= x^4_{-3} + (1 - 2x^4_{-3})x^3_{-2} + (1 - 2x^4_{-3})(1 - 2x^3_{-2})x^2_{-1} + \\ &\quad +(1 - 2x^4_{-3})(1 - 2x^3_{-2})(1 - 2x^2_{-1})u_2. \end{aligned}$$

As before, we proceed by induction to find that, for this third left-diagonal (here $n - m = 3$)

$$x^n_{-m} = x^{n-1}_{-m-1} + \sum_{k=1}^{m} x^{k+1}_{-k} \prod_{j=k}^{m} (1 - 2x^{j+2}_{-j-1}) + u_2 \prod_{j=1}^{m+1} (1 - 2x^{j+1}_{-j}),$$

or, upon setting $m - 3$, we obtain

$$x^n_{-n+3} = x^{n-1}_{-n+2} + \sum_{k=1}^{n-3} x^{k+1}_{-k} \prod_{j=k}^{n-3} (1 - 2x^{j+2}_{-j-1}) + u_2 \prod_{j=1}^{n-2} (1 - 2x^{j+1}_{-j}). \tag{6}$$

This argument extends to the general case of the i-th left diagonal, where $i \leq T - 1$. In this case every cell value of this diagonal is given by terms of the form

$$\begin{aligned} x^n_{-n+i} = x^{n-1}_{-n+i-1} &+ \sum_{k=1}^{n-i} x^{i+k-2}_{-k} \prod_{j=k}^{n-i} (1 - 2x^{i+j-1}_{-j-1}) + \\ &+ u_{i-1} \prod_{j=1}^{n-i+1} (1 - 2x^{i+j-2}_{-j}). \end{aligned} \tag{7}$$

where $n = 0, 1, 2, \ldots, T$. So, for the desired final row (see Table 1), $n = T$ and $i = N$ therefore

$$x^T_{-T+N} = x^{T-1}_{-T+N-1} + \sum_{k=1}^{T-N} x^{N+k-2}_{-k} \prod_{j=k}^{T-N} (1 - 2x^{N+j-1}_{-j-1}) + \\ +u_{N-1} \prod_{j=1}^{T-N+1} (1 - 2x^{N+j-2}_{-j}), \tag{8}$$

where $N = 0, 1, \ldots, T$. Let $A^T_{-T+N} = x^T_{-T+N}$ be the given final state at time T, where $0 \le N \le T$.

For $N = 1$ the coefficient of u_0 in (8) is 1 by Table 1. Hence a control u_0 exists with the property that the space-time diagram of Table 1 will reach A^T_{-T+1} at time T. This unique control u_0 is now fixed.

The case $N = 2$ gives that the coefficient of u_1 in (8) is given by

$$\prod_{j=1}^{T-1} (1 - 2x^j_{-j}) = \prod_{j=1}^{T-1} (1 - 2a) = (1 - 2a)^{T-1}.$$

Since $A^T_{-T} = a \neq 1/2$ by hypothesis, it follows that u_1 exists and is unique. This now fixes the control u_1.

When $N = 3$, since $x^{j+1}_{-j} = u_0$ for all j, the coefficient of u_2 in (8) is

$$\prod_{j=1}^{T-2} (1 - 2x^{j+1}_{-j}) = \prod_{j=1}^{T-2} (1 - 2u_0) = (1 - 2u_0)^{T-2}.$$

It follows that the control u_2 exists and is unique. The three controls u_0, u_1, u_2 now found will bring the system in Table 1 to the given values $A^T_{-T}, A^T_{-T+1}, A^T_{-T+2}$ at time T.

For general N the coefficient of u_{N-1} is

$$\prod_{j=1}^{T-N+1} (1 - 2x^{N+j-2}_{-j})$$

where we require that the $x^{N+j-2}_{-j} \neq 1/2$ for all j, $j = 1, 2, \ldots, T - N + 1$. We show next that this is always the case.

If possible, let j be a subscript with $x^{N+j-2}_{-j} = 1/2$. Then

$$\begin{aligned} x^{N+j-1}_{-j-1} &= g(x^{N+j-2}_{-j-2}, x^{N+j-2}_{-j-1}, x^{N+j-2}_{-j}) \\ &= x^{N+j-2}_{-j-2} + 1/2 - 2x^{N+j-2}_{-j-2}/2 \\ &= 1/2. \end{aligned}$$

Similarly, we get that $x^{N+j}_{-j-2} = 1/2$, and so on for all the cells down this diagonal (cells whose terms are necessarily of the form x^{N+j-k}_{-j+k-2}, $k = 0, 1, 2, \ldots$). Since this diagonal must intersect the row of A's in Table 1, we see that there must be some M such that $A^T_{-T+M} = 1/2$ and this contradicts the hypothesis. It now follows that a unique control u_{N-1} exists satisfying the requirement of controllability stated at the outset. Incidentally, this same argument shows that the $u_{N-1} \neq 1/2$. Since N is arbitrary, we get that all $u_N \neq 1/2$, for $N = 0, 1, \ldots, T-1$.

The necessity is straightforward. For if such controls u_N, $N = 0, 1, \ldots, T-1$ exist satisfying the controllability hypothesis then we claim that $A^T_{-T+N} \neq 1/2$ for all N. Otherwise using Lemma 1 we can "work our way up and to the right" of this cell and deduce that some $u_N = 1/2$. Since this is not an admissible control value by hypothesis this contradiction then proves the necessity.

We give the general idea on how to proceed: if for some N we have $A^T_{-T+N} = 1/2$, then by Lemma 1 either the cell directly above it and to the left or the cell directly above it and to the right must have value equal to 1/2. The worst case scenario is if we keep going up and left, away from the line of controls on the extreme right using repeated applications of Lemma 1 (cf., Table 1). Then at some point in this procedure we must cross the first non-zero diagonal $\mathcal{L}_0^-$ at the extreme left which consists of the quantity "a" only. Since $a \neq 1/2$ by hypothesis, this cannot occur. Hence as we work our way up the space-time diagram using said Lemma, eventually the process must bring us to some control u_N with value necessarily equal to 1/2; and this is a contradiction.

Example 1. Let $a = 1/3$, and $T = 5$ in Table 1. In addition let $A^5_{-5} = 1/3$, $A^5_{-4} = 1/4$, $A^5_{-3} = 1/5$, $A^5_{-2} = 1/6$, $A^5_{-1} = 1/7$, $A^5_0 = 1/8$ be the set of reachable states. Recall that $u_0 = 1/4$. Then a straightforward calculation using the rule function $g(x, y, z) = x + z - 2xz$ gives the following table of values;

Table 2.

t	-5	-4	-3	-2	-1	0	1
0	0	0	0	0	0	1/3	$1/4 = u_0$
1	0	0	0	0	1/3	1/4	$-119/5 = u_1$
2	0	0	0	1/3	1/4	$-38/5$	$-13/6 = u_2$
3	0	0	1/3	1/4	$-11/5$	$-5/6$	$788/1702 = u_3$
4	0	1/3	1/4	$-2/5$	$-1/6$	19/63	$7/32 = u_4$
5	1/3	1/4	1/5	1/6	1/7	1/8	.

When we speak of *controllability on J* where J is either $[0, 1]$ or $\mathbb{R}$ we mean that the admissible controls set $\mathcal{U}_{ad}$ defined in Theorem 1 above is of the form $\mathcal{U}_{ad} = J^T \setminus \{(1/2, 1/2, \ldots, 1/2)\}$ and the $A^T_{-T+N} \in J$ for all $N = 0, 1, \ldots, T-1$. The next theorem gives some results for the more difficult problem of controllability on $[0, 1]$.

Remark 1. Note that in the preceding example $x^5_{-3} = 1/5$ by hypothesis. In this case the resulting equation admits the solution $u_1 = -119/5 < 0$; indeed many such values are negative. Thus the controllability problem on $[0, 1]$ has no solution, although the same problem on $\mathbb{R}$ has a unique solution (as per

Theorem 1) and it is exhibited in Table 2. This example shows us that for controllability we must expect a delicate generally nonlinear interplay between the values of the A's, a and the controls u_i.

Remark 2. We have presented in this paper only the case of $J = \mathbb{R}$. The general controllability problem on $[0, 1]$ for fuzzy rule 90 have also been solved but only for $T \leq 5$. We gather that, in general, we can expect a solution to the controllability problem for $[0, 1]$ provided the A^T_{-T+N}, $N \geq 3$, are very close to but not equal to $1/2$ and the time T is not too large. This is because the exponentially fast asymptotes derived in [11] guarantee that $A^T_{-T+N} \to 1/2$ as $T \to \infty$ for every N. As one can gather from the obtained results the rule of thumb is, the larger the time, the closer the initial conditions are to be to $1/2$, in which case there is some hope for solvability.

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