Suzumura Consistency

Walter Bossert

1 Introduction

Binary relations are at the heart of much of economic theory, both in the context of individual choice and in multi-agent decision problems. A fundamental coherence requirement imposed on a relation is the well-known transitivity axiom. If a relation is interpreted as a goodness relation, transitivity postulates that whenever one alternative is at least as good as a second and the second alternative is, in turn, at least as good as a third, then the first alternative is at least as good as the third. However, from an empirical as well as a conceptual perspective, transitivity is frequently considered too demanding and weaker notions of coherence have been proposed in the literature. Two alternatives that have received a considerable amount of attention are quasi-transitivity and acyclicity. Quasi-transitivity demands that the asymmetric factor of a relation (the betterness relation) is transitive, whereas acyclicity rules out the presence of betterness cycles. Quasi-transitivity is implied by transitivity and implies acyclicity. The reverse implications are not valid.

Suzumura (1976b) introduced an interesting alternative weakening of transitivity and showed that it can be considered a more intuitive property than quasi-transitivity. This notion of coherence, which Suzumura introduced under the name consistency, rules out the presence of cycles with at least one instance of betterness. Thus, the axiom is stronger than acyclicity and weaker than transitivity. It is equivalent to transitivity in the presence of reflexivity and completeness but independent of quasi-transitivity. Because the term consistency is used in various other contexts in economic theory (see, for instance, Thomson (1990)), I propose to refer to the axiom as *Suzumura consistency*.

Suzumura consistency is exactly what is needed to avoid the phenomenon of a money pump. If Suzumura consistency is violated by an agent's goodness relation,

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there exists a cycle with at least one instance of betterness. In this case, the agent under consideration is willing to trade an alternative for another alternative (where 'willing to trade' is interpreted as being at least as well-off after the trade as before), the second alternative for a third and so on, until an alternative is reached such that getting back the original alternative is better than retaining possession of the last alternative in the chain. Thus, at the end of such a chain of exchanges, the agent is willing to give up the last alternative and, in addition, to pay a positive amount to get back the original alternative.

An important property of Suzumura consistency is that it is necessary and sufficient for the existence of an ordering extension of a relation. Szpilrajn (1930) showed that, for any asymmetric and transitive relation, there exists an asymmetric, transitive and complete relation that contains the original relation. An analogous result applies if asymmetry is replaced with reflexivity. Suzumura (1976b) has shown that the transitivity assumption can be weakened to Suzumura consistency without changing the conclusion regarding the existence of an ordering extension. Moreover, Suzumura consistency is the weakest possible property that guarantees this existence result. Because extension theorems are of considerable importance in many applications of set theory, this is a fundamental result and illustrates the significance of the property.

The purpose of this paper is to review the uses of Suzumura consistency in a variety of applications and to provide some new observations, with the objective of further underlining the importance of this axiom. The first step is a statement of Suzumura's (1976b) extension theorem in the following section, followed by an application in the theory of rational choice due to Bossert, Sprumont, and Suzumura (2005a) in Sect. 3. The last two sections provide new observations. In Sect. 4, a variant of the welfarism theorem that assumes Suzumura consistency instead of transitivity is provided, and Sect. 5 illustrates how an impossibility result in population ethics can be turned into a possibility by weakening transitivity to Suzumura consistency.

2 Relations and Extensions

Suppose X is a non-empty set of alternatives and $R \subseteq X \times X$ is a (binary) relation on X which is interpreted as a *goodness* relation, that is, $(x,y) \in R$ means that x is considered at least as good as y by the agent (or society) under consideration. The *diagonal relation* Δ on X is defined by

$$\Delta = \{(x,x) \mid x \in X\}.$$

The *asymmetric factor* of a relation R is defined by

$$P(R) = \{(x,y) \mid (x,y) \in R \text{ and } (y,x) \notin R\}$$

and the *symmetric factor* of R is

$$I(R) = \{(x,y) \mid (x,y) \in R \text{ and } (y,x) \in R\}.$$

Given the interpretation of R as a goodness relation, $P(R)$ is the *better-than* relation corresponding to R and $I(R)$ is the *equally-good* relation associated with R.

The *transitive closure* $\text{tc}(R)$ of a relation R is defined by

$$\begin{aligned}\text{tc}(R) = \{(x,y) \mid & \text{ there exist } M \in \mathbb{N} \text{ and } x^0,\ldots,x^M \in X \text{ such that} \\ & x = x^0, (x^{m-1},x^m) \in R \text{ for all } m \in \{1,\ldots,M\} \text{ and } x^M = y\}.\end{aligned}$$

As is straightforward to verify,

$$R \subseteq Q \;\Rightarrow\; \text{tc}(R) \subseteq \text{tc}(Q) \tag{1}$$

for any two relations R and Q.

To illustrate the transitive closure, consider the following examples. First, let $X = \{x,y,z\}$ and $R = \{(x,x),(x,y),(y,y),(y,z),(z,x),(z,z)\}$. We obtain $\text{tc}(R) = X \times X$. In addition to the pairs in R, the pair (x,z) must be in the transitive closure of R because we have $(x,y) \in R$ and $(y,z) \in R$. Analogously, (y,x) must be an element of $\text{tc}(R)$ because $(y,z) \in R$ and $(z,x) \in R$, and (z,y) must be in $\text{tc}(R)$ because $(z,x) \in R$ and $(x,y) \in R$. Now let $X = \{x,y,z\}$ and $R = \{(x,y),(y,z)\}$. As it is straightforward to verify, we have $\text{tc}(R) = \{(x,y),(y,z),(x,z)\}$.

A relation R is *reflexive* if, for all $x \in X$,

$$(x,x) \in R$$

and R is *asymmetric* if

$$R = P(R).$$

Furthermore, R is *complete* if, for all $x,y \in X$,

$$x \neq y \;\Rightarrow\; (x,y) \in R \text{ or } (y,x) \in R$$

and R is *transitive* if, for all $x,y,z \in X$,

$$(x,y) \in R \text{ and } (y,z) \in R \;\Rightarrow\; (x,z) \in R.$$

R is *Suzumura consistent* if, for all $x,y \in X$,

$$(x,y) \in \text{tc}(R) \Rightarrow (y,x) \notin P(R).$$

A *quasi-ordering* is a reflexive and transitive relation and an *ordering* is a complete quasi-ordering.

The notion of Suzumura consistency is due to Suzumura (1976b) and it is equivalent to the requirement that any cycle must be such that all relations involved in this cycle are instances of equal goodness – betterness cannot occur. Clearly, this requirement implies (but is not implied by) the well-known *acyclicity* axiom which

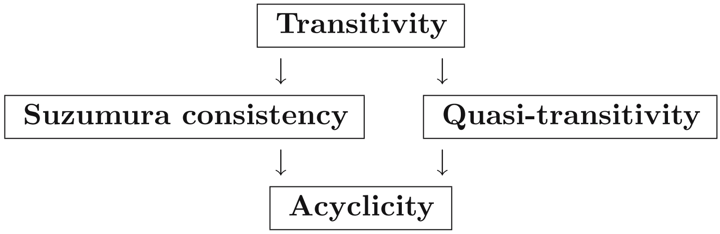

Fig. 1 Logical relationships

rules out the existence of betterness cycles (cycles where *all* relations involve the asymmetric factor of the relation). Suzumura consistency and *quasi-transitivity,* which requires that $P(R)$ is transitive, are independent. Transitivity implies Suzumura consistency but the reverse implication is not true in general. However, if R is reflexive and complete, Suzumura consistency and transitivity are equivalent. Figure 1 illustrates the relationships among transitivity and the above-mentioned weakenings of this property. Each arrow represents a direct implication, and these implications together with those resulting from chains of arrows are the only ones that are valid in the absence of further properties imposed on R.

A relation R' is an *extension* of a relation R if

$$R \subseteq R' \text{ and } P(R) \subseteq P(R').$$

If an extension R' of R is an ordering, we refer to R' as an *ordering extension* of R. One of the most fundamental results on extensions of binary relations is due to Szpilrajn (1930) who showed that any transitive and asymmetric relation has a transitive, asymmetric and complete extension. The result remains true if asymmetry is replaced with reflexivity, that is, any quasi-ordering has an ordering extension. Arrow (1963, p. 64) stated this generalization of Szpilrajn's theorem without a proof and Hansson (1968) provided a proof on the basis of Szpilrajn's original theorem.

While the property of being a quasi-ordering is sufficient for the existence of an ordering extension of a relation, this is not necessary. As shown by Suzumura (1976b), Suzumura consistency is *necessary and sufficient* for the existence of an ordering extension. This observation is stated formally in the following theorem, see Suzumura (1976b, pp. 389–390) for a proof.

Theorem 1. *A relation R has an ordering extension if and only if R is Suzumura consistent.*

Theorem 1 is an important result. It establishes that Suzumura consistency is the weakest possible property of a relation that still guarantees the existence of an ordering extension. Note that quasi-transitivity (which, as mentioned earlier, is logically independent of Suzumura consistency) has nothing to do with the possibility of extending a binary relation to an ordering.

3 Rational Choice

Suzumura consistency has recently been examined in the context of *rational choice.* Observed (or observable) choices are *rationalizable* if there exists a relation such that, for any feasible set, the set of chosen alternatives coincides with the set of greatest or maximal elements according to this relation.

Following the contributions of Hansson (1968), Richter (1966, 1971), Suzumura (1976a, 1977, Chap. 2 in 1983) and others, the approach to rational choice analyzed in this paper is capable of accommodating a wide variety of choice situations because no restrictions (other than non-emptiness) are imposed on the domain of a choice function. Letting $\mathcal{X}$ denote the power set of X excluding the empty set, a *choice function* is a mapping $C\colon \Sigma \to \mathcal{X}$ such that $C(S) \subseteq S$ for all $S \in \Sigma$, where $\Sigma \subseteq \mathcal{X}$ with $\Sigma \neq \emptyset$ is the domain of C.

The *direct revealed preference relation* $R_C \subseteq X \times X$ of a choice function C with an arbitrary domain Σ is defined as

$$R_C = \{(x,y) \mid \text{there exists } S \in \Sigma \text{ such that } x \in C(S) \text{ and } y \in S\}.$$

The *(indirect) revealed preference relation* of C is the transitive closure $\text{tc}(R_C)$ of the direct revealed preference relation R_C.

A choice function C is *greatest-element rationalizable* if there exists a relation R on X such that

$$C(S) = \{x \in S \mid (x,y) \in R \text{ for all } y \in S\}$$

for all $S \in \Sigma$. If such a relation R exists, it is called a *rationalization* of C. The most common alternative to greatest-element rationalizability is *maximal-element* rationalizability which requires the existence of a relation R such that, for all feasible sets S, $C(S)$ is equal to the set of maximal elements in S according to R, that is, no element in S is better than any element in $C(S)$. Bossert, Sprumont, and Suzumura (2005b) provide a detailed analysis of maximal-element rationalizability. Logical relationships between, and characterizations of, various notions of rationalizability, both on arbitrary domains and under more specific domain assumptions, can be found in Bossert, Sprumont, and Suzumura (2006).

To interpret a rationalization as a goodness relation, it is usually required that it satisfy additional properties such as the *richness* axioms reflexivity and completeness, or one of the *coherence* properties acyclicity, quasi-transitivity, Suzumura consistency and transitivity. The full set of rationalizability notions that can be obtained by combining one or both (or none) of the richness properties with one (or none) of the coherence properties is analyzed in Bossert and Suzumura (2008). They show that, if all these combinations are available, it is sufficient to restrict attention to greatest-element rationalizability: for each notion of maximal-element rationalizability, there exists a notion of the greatest-element rationalizability (possibly involving different richness and coherence properties) that is equivalent. Thus, restricting attention to greatest-element rationalizability does not involve any loss of generality.

Bossert, Sprumont, and Suzumura (2005a) have characterized all notions of rationalizability when the coherence property required is Suzumura consistency. As mentioned earlier, Suzumura consistency and transitivity are equivalent in the presence of reflexivity and completeness. Thus, greatest-element rationalizability by a reflexive, complete and Suzumura-consistent relation is equivalent to greatest-element rationalizability by an ordering and Richter's (1966, 1971) results apply; see Theorem 2. Moreover, greatest-element rationalizability by a complete and Suzumura-consistent relation implies greatest-element rationalizability by a reflexive, complete and Suzumura-consistent relation, and greatest-element rationalizability by a Suzumura-consistent relation implies greatest-element rationalizability by a reflexive and Suzumura-consistent relation. Analogous observations apply in the case of maximal-element rationalizability; see Bossert, Sprumont, and Suzumura (2005a, Theorem 1). As pointed out in Bossert, Sprumont, and Suzumura (2006), as soon as the coherence properties quasi-transitivity or acyclicity are imposed, reflexivity no longer is guaranteed as an additional property of a rationalization. Thus, Suzumura consistency stands out from these alternative weakenings of transitivity in this regard: as is the case for transitive greatest-element (or maximal-element) rationalizability, any notion of Suzumura-consistent greatest-element (or maximal-element) rationalizability is equivalent to the definition that is obtained if reflexivity is added as a property of a rationalization.

Richter (1971) showed that the following axiom is necessary and sufficient for greatest-element rationalizability by a transitive relation and by an ordering. Thus, the existence of a rationalizing relation that is not merely a quasi-ordering but an ordering follows from greatest-element rationalizability by a transitive relation. This observation sets transitive greatest-element rationalizability apart from other notions of greatest-element rationalizability involving weaker coherence requirements.

Transitive-closure coherence. For all $S \in \Sigma$ and for all $x \in S$,

$$(x,y) \in \mathrm{tc}(R_C) \text{ for all } y \in S \;\Rightarrow\; x \in C(S).$$

We now obtain the following result; see Bossert, Sprumont, and Suzumura (2005a).

Theorem 2. *C is greatest-element rationalizable by a (reflexive,) complete and Suzumura-consistent relation if and only if C satisfies transitive-closure coherence.*

Proof. To prove the 'only-if' part, suppose C is greatest-element rationalizable by a complete and Suzumura-consistent relation R. We prove that C is greatest-element rationalizable by a reflexive, complete and Suzumura-consistent relation. Together with the observation that Suzumura consistency and transitivity are equivalent in the presence of reflexivity and completeness and Richter's (1971) result, this establishes that transitive-closure coherence is satisfied.

Let

$$\begin{aligned} R' = {} & [R \cup \Delta \cup \{(y,x) \mid x \notin C(\Sigma) \text{ and } y \in C(\Sigma)\}] \\ & \setminus \; \{(x,y) \mid x \notin C(\Sigma) \text{ and } y \in C(\Sigma)\}. \end{aligned}$$

Clearly, R' is reflexive by definition.

To show that R' is complete, let $x,y \in X$ be such that $x \neq y$ and $(x,y) \notin R'$. By definition of R', this implies

$$(x,y) \notin R \text{ and } [x \notin C(\Sigma) \text{ or } y \in C(\Sigma)]$$

or

$$x \notin C(\Sigma) \text{ and } y \in C(\Sigma).$$

If the former applies, the completeness of R implies $(y,x) \in R$ and, by definition of R', we obtain $(y,x) \in R'$. If the latter is true, $(y,x) \in R'$ follows immediately from the definition of R'.

Next, we show that R' is Suzumura consistent. Let $(x,y) \in \text{tc}(R')$. By definition, there exist $M \in \mathbb{N}$ and $x^0,\ldots,x^M \in X$ be such that $x = x^0$, $(x^{m-1},x^m) \in R'$ for all $m \in \{1,\ldots,M\}$ and $x^M = y$. Clearly, we can, without loss of generality, assume that $x^{m-1} \neq x^m$ for all $m \in \{1,\ldots,M\}$. We distinguish two cases.

(i) $x^0 \notin C(\Sigma)$. In this case, it follows that $x^1 \notin C(\Sigma)$; otherwise we would have $(x^1,x^0) \in P(R')$ by definition of R', contradicting our hypothesis. Successively applying this argument to all $m \in \{1,\ldots,M\}$, we obtain $x^m \notin C(\Sigma)$ for all $m \in \{1,\ldots,M\}$. By definition of R', this implies $(x^{m-1},x^m) \in R$ for all $m \in \{1,\ldots,M\}$. By the Suzumura consistency of R, we must have $(x^M,x^0) \notin P(R)$. Because $x^M \notin C(\Sigma)$, this implies, according to the definition of R', $(x^M,x^0) \notin P(R')$.

(ii) $x^0 \in C(\Sigma)$. If $x^M \notin C(\Sigma)$, $(x^M,x^0) \notin P(R')$ follows immediately from the definition of R'. If $x^M \in C(\Sigma)$, it follows that $x^{M-1} \in C(\Sigma)$; otherwise we would have $(x^{M-1},x^M) \notin R'$ by definition of R', contradicting our hypothesis. Successively applying this argument to all $m \in \{1,\ldots,M\}$, we obtain $x^m \in C(\Sigma)$ for all $m \in \{1,\ldots,M\}$. By definition of R', this implies $(x^{m-1},x^m) \in R$ for all $m \in \{1,\ldots,M\}$. By the Suzumura consistency of R, we must have $(x^M,x^0) \notin P(R)$. Because $x^0 \in C(\Sigma)$, this implies, according to the definition of R', $(x^M,x^0) \notin P(R')$.

Finally, we show that R' is a rationalization of C. Let $S \in \Sigma$ and $x \in S$.

Suppose first that $(x,y) \in R'$ for all $y \in S$. If $|S| = 1$, $x \in C(S)$ follows immediately because $C(S)$ is non-empty. If $|S| \geq 2$, we obtain $x \in C(\Sigma)$ by definition of R'. Because R is a rationalization of C, this implies $(x,x) \in R$. By definition of R', $(x,z) \in R$ for all $z \in C(S)$. Therefore, $(x,z) \in R$ for all $z \in C(S) \cup \{x\}$. Suppose, by way of contradiction, that $x \notin C(S)$. Because R is a rationalization of C, it follows that there exists $y \in S \setminus (C(S) \cup \{x\})$ such that $(x,y) \notin R$. The completeness of R implies $(y,x) \in P(R)$. Let $z \in C(S)$. It follows that $(z,y) \in R$ because R is a rationalization of C and, as established earlier, $(x,z) \in R$. This contradicts the Suzumura consistency of R.

To prove the converse implication, suppose $x \in C(S)$. Because R is a rationalization of C, we have $(x,y) \in R$ for all $y \in S$. In particular, this implies $(x,x) \in R$ and, according to the definition of R', we obtain $(x,y) \in R'$ for all $y \in S$.

The 'if' part of the theorem follows immediately from the equivalence of transitive-closure coherence and greatest-element rationalizability by a reflexive, complete and transitive rationalization established by Richter (1971) and the observation that Suzumura consistency and transitivity coincide in the presence of reflexivity and completeness. □

If completeness is dropped as a requirement imposed on a rationalization, a weaker notion of greatest-element rationalizability is obtained. In contrast to greatest-element rationalizability by a quasi-transitive or an acyclical relation which leads to much more complex necessary and sufficient conditions (see Bossert and Suzumura (2008)), requiring a rationalization to be Suzumura consistent preserves the intuitive and transparent nature of the characterization stated in Theorem 2. There is a unique minimal Suzumura-consistent relation that has to be respected by any Suzumura-consistent rationalization, namely, the *Suzumura-consistent closure* of R_C. The Suzumura-consistent closure $\mathrm{sc}(R)$ of a relation R is defined by

$$\mathrm{sc}(R) = R \cup \{(x,y) \mid (x,y) \in \mathrm{tc}(R) \text{ and } (y,x) \in R\}.$$

Clearly, $R \subseteq \mathrm{sc}(R) \subseteq \mathrm{tc}(R)$. Just as $\mathrm{tc}(R)$ is the unique smallest transitive relation containing R, $\mathrm{sc}(R)$ is the unique smallest Suzumura-consistent relation containing R; see Bossert, Sprumont, and Suzumura (2005a).

To see that this is the case, we first establish that $\mathrm{sc}(R)$ is Suzumura consistent. Suppose $M \in \mathbb{N}$ and $x^0, \ldots, x^M \in X$ are such that $(x^{m-1}, x^m) \in \mathrm{sc}(R)$ for all $m \in \{1, \ldots, M\}$. We show that $(x^M, x^0) \notin P(\mathrm{sc}(R))$. Because $\mathrm{sc}(R) \subseteq \mathrm{tc}(R)$, $(x^{m-1}, x^m) \in \mathrm{tc}(R)$ for all $m \in \{1, \ldots, M\}$, and the transitivity of $\mathrm{tc}(R)$ implies

$$(x^0, x^M) \in \mathrm{tc}(R). \tag{2}$$

If $(x^M, x^0) \notin \mathrm{sc}(R)$, we immediately obtain $(x^M, x^0) \notin P(\mathrm{sc}(R))$ and we are done. Now suppose that $(x^M, x^0) \in \mathrm{sc}(R)$. By definition of $\mathrm{sc}(R)$, we must have

$$(x^M, x^0) \in R \text{ or } [(x^M, x^0) \in \mathrm{tc}(R) \text{ and } (x^0, x^M) \in R].$$

If $(x^M, x^0) \in R$, (2) and the definition of $\mathrm{sc}(R)$ together imply $(x^0, x^M) \in \mathrm{sc}(R)$ and, thus, $(x^M, x^0) \notin P(\mathrm{sc}(R))$. If $(x^M, x^0) \in \mathrm{tc}(R)$ and $(x^0, x^M) \in R$, $(x^0, x^M) \in \mathrm{sc}(R)$ follows because $R \subseteq \mathrm{sc}(R)$. Again, this implies $(x^M, x^0) \notin P(\mathrm{sc}(R))$ and the proof that $\mathrm{sc}(R)$ is Suzumura consistent is complete.

To show that $\mathrm{sc}(R)$ is the smallest Suzumura-consistent relation containing R, suppose that Q is an arbitrary Suzumura-consistent relation containing R. To complete the proof, we establish that $\mathrm{sc}(R) \subseteq Q$. Suppose that $(x,y) \in \mathrm{sc}(R)$. By definition of $\mathrm{sc}(R)$,

$$(x,y) \in R \text{ or } [(x,y) \in \mathrm{tc}(R) \text{ and } (y,x) \in R].$$

If $(x,y) \in R$, $(x,y) \in Q$ follows because R is contained in Q by assumption. If $(x,y) \in \mathrm{tc}(R)$ and $(y,x) \in R$, (1) and the assumption $R \subseteq Q$ together imply that $(x,y) \in \mathrm{tc}(Q)$ and $(y,x) \in Q$. If $(x,y) \notin Q$, we obtain $(y,x) \in P(Q)$ in view of $(y,x) \in Q$. Since $(x,y) \in \mathrm{tc}(Q)$, this contradicts the Suzumura consistency of Q. Therefore, we must have $(x,y) \in Q$.

The property of $\mathrm{sc}(R)$ just established is crucial in obtaining a clear-cut and intuitive rationalizability result even without imposing completeness (in which case Suzumura-consistent greatest-element rationalizability is not equivalent to transitive greatest-element rationalizability). In contrast, there is no such thing as a quasi-transitive closure or an acyclical closure of a relation, which explains why rationalizability results involving these coherence properties are much more complex.

The following examples illustrate the Suzumura-consistent closure and its relation to the transitive closure. First, let $X = \{x,y,z\}$ and $R = \{(x,x),(x,y),(y,y),(y,z)$ $(z,x),(z,z)\}$. We obtain $\mathrm{sc}(R) = \mathrm{tc}(R) = X \times X$. Now let $X = \{x,y,z\}$ and $R = \{(x,y),(y,z)\}$. We have $\mathrm{sc}(R) = R$ and $\mathrm{tc}(R) = \{(x,y),(y,z),(x,z)\}$. In the first example, the Suzumura-consistent closure coincides with the transitive closure, whereas in the second, the Suzumura-consistent closure is a strict subset of the transitive closure.

Greatest-element rationalizability by means of a Suzumura-consistent (and reflexive but not necessarily complete) relation can now be characterized by employing a natural weakening of transitive-closure coherence: all that needs to be done is replacing the transitive closure of the direct revealed preference relation by its Suzumura-consistent closure.

Suzumura-consistent-closure coherence. For all $S \in \Sigma$ and for all $x \in S$,

$$(x,y) \in \mathrm{sc}(R_C) \text{ for all } y \in S \;\Rightarrow\; x \in C(S).$$

The following characterization is also due to Bossert, Sprumont, and Suzumura (2005a).

Theorem 3. *C is greatest-element rationalizable by a (reflexive and) Suzumura-consistent relation if and only if C satisfies Suzumura-consistent-closure coherence.*

Proof. To prove the 'only-if' part of the theorem, suppose R is a Suzumura-consistent rationalization of C and let $S \in \Sigma$ and $x \in S$ be such that $(x,y) \in \mathrm{sc}(R_C)$ for all $y \in S$. Consider any $y \in S$. By definition,

$$(x,y) \in R_C \text{ or } [(x,y) \in \mathrm{tc}(R_C) \text{ and } (y,x) \in R_C].$$

If $(x,y) \in R_C$, there exists $T \in \Sigma$ such that $x \in C(T)$ and $y \in T$. Because R greatest-element rationalizes C, this implies $(x,y) \in R$. If $(x,y) \in \mathrm{tc}(R_C)$ and $(y,x) \in R_C$, there exist $M \in \mathbb{N}$ and $x^0,\ldots,x^M \in X$ such that $x = x^0$, $(x^{m-1},x^m) \in R_C$ for all $m \in \{1,\ldots,M\}$ and $x^M = y$. As in the argument just used, the assumption that R greatest-element rationalizes C implies $(x^{m-1},x^m) \in R$ for all $m \in \{1,\ldots,M\}$ and, thus, $(x,y) \in \mathrm{tc}(R)$. Furthermore, $(y,x) \in R_C$ implies $(y,x) \in R$ because R is a rationalization of R. If $(x,y) \notin R$, it follows that $(y,x) \in P(R)$ in view of $(y,x) \in R$. Because $(x,y) \in \mathrm{tc}(R)$, this contradicts the Suzumura consistency of R. Therefore, $(x,y) \in R$. Because $y \in S$ has been chosen arbitrarily, this is true for all $y \in S$ and, as a consequence of the assumption that R greatest-element rationalizes C, we obtain $x \in C(S)$.

To prove the 'if' part, suppose C satisfies Suzumura-consistent-closure coherence. We first show that $\mathrm{sc}(R_C)$ is a Suzumura-consistent rationalization of C. That $\mathrm{sc}(R_C)$ is Suzumura consistent has already been established. To prove that $\mathrm{sc}(R_C)$ is a rationalization of C, suppose first that $S \in \Sigma$ and $x \in S$. Suppose $(x,y) \in \mathrm{sc}(R_C)$

for all $y \in S$. Suzumura-consistent-closure coherence implies $x \in C(S)$. Conversely, suppose $x \in C(S)$. By definition, this implies $(x,y) \in R_C$ for all $y \in S$ and, because $R_C \subseteq \text{sc}(R_C)$, we obtain $(x,y) \in \text{sc}(R_C)$ for all $y \in S$. The proof is completed by showing that

$$R' = (\text{sc}(R_C) \cup \Delta) \setminus \{(x,y) \mid x \notin C(\Sigma) \text{ and } x \neq y\}$$

is a reflexive and Suzumura-consistent rationalization of C.

That R' is reflexive is obvious. To prove that R' is Suzumura consistent, suppose $(x,y) \in \text{tc}(R')$. Thus, there exist $M \in \mathbb{N}$ and $x^0,\ldots,x^M \in X$ such that $x = x^0$, $(x^{m-1},x^m) \in R'$ for all $m \in \{1,\ldots,M\}$ and $x^M = y$. Clearly, we can without loss of generality assume that $x^{m-1} \neq x^m$ for all $m \in \{1,\ldots,M\}$. By definition of R', $x^0 \in C(\Sigma)$. If $x^M \notin C(\Sigma)$, $(x^M,x^0) \notin P(R')$ follows immediately from the definition of R'. If $x^M \in C(\Sigma)$, it follows that $x^{M-1} \in C(\Sigma)$; otherwise we would have $(x^{M-1},x^M) \notin R'$ by definition of R', contradicting our hypothesis. Successively applying this argument to all $m \in \{0,\ldots,M-1\}$, we obtain $x^m \in C(\Sigma)$ for all $m \in \{0,\ldots,M-1\}$. By definition of R', this implies $(x^{m-1},x^m) \in \text{sc}(R_C)$ for all $m \in \{1,\ldots,M\}$. By the Suzumura consistency of $\text{sc}(R_C)$, we must have $(x^M,x^0) \notin P(\text{sc}(R_C))$. Because $x^0 \in C(\Sigma)$, this implies, according to the definition of R', $(x^M,x^0) \notin P(R')$.

It remains to be shown that R' is a rationalization of C. Let $S \in \Sigma$ and $x \in S$.

First, suppose $(x,y) \in R'$ for all $y \in S$. By definition of R',

$$(x,y) \in \text{sc}(R_C) \tag{3}$$

for all $y \in S \setminus \{x\}$ and $x \in C(\Sigma)$. Because $\text{sc}(R_C)$ is a rationalization of C, this implies $(x,x) \in \text{sc}(R_C)$. Suppose, by way of contradiction, that $x \notin C(S)$. Because $\text{sc}(R)$ is a rationalization of C, it follows that there exists $y \in S \setminus \{x\}$ such that $(x,y) \notin \text{sc}(R_C)$, contradicting (3).

Finally, suppose $x \in C(S)$. This implies $(x,y) \in \text{sc}(R_C)$ for all $y \in S$ because $\text{sc}(R_C)$ is a rationalization of C. Furthermore, because $C(S) \subseteq C(\Sigma)$, we have $x \in C(\Sigma)$. By definition of R', this implies $(x,y) \in R'$ for all $y \in S$. □

4 Welfarism

Following Arrow's (1951, 2nd ed. 1963) impossibility theorem, one route of escape from its negative consequences that has been chosen in the subsequent literature is to assume that a social ranking is established on the basis of a richer informational framework. In Arrow's setup, the individual goodness relations form the informational basis of collective choice. This approach rules out, in particular, interpersonal comparisons of well-being. An informationally richer environment is obtained if a social ranking is allowed to depend on *utility* profiles instead of profiles of goodness relations, and these utilities can be assumed to carry more than just ordinally measurable and interpersonally non-comparable information regarding the well-being of the agents. Under an implicit regularity assumption that guarantees the existence of representations of the individual goodness relations, the Arrow framework is included as a special case: it corresponds to the informational assumption of ordinal measurability and interpersonal non-comparability.

The universal set of alternatives X is assumed to contain at least three elements. There are a finite number $n \geq 2$ of agents indexed by the first n positive integers, so that the set of agents is $N = \{1, \ldots, n\}$. The set of all utility functions $U \colon X \to \mathbb{R}$ is denoted by $\mathcal{U}$ and its n-fold Cartesian product is $\mathcal{U}^n$. A *utility profile* is an n-tuple $\mathbf{U} = (U_1, \ldots, U_n) \in \mathcal{U}^n$.

A *collective choice functional* is a mapping $F \colon \mathcal{D} \to \mathcal{B}$ where $\mathcal{D} \subseteq \mathcal{U}^n$ is the domain of this functional, assumed to be non-empty, and $\mathcal{B}$ is the set of all binary relations on X. For each utility profile $\mathbf{U} \in \mathcal{D}$, $F(\mathbf{U})$ is the social preference corresponding to $\mathbf{U}$. A *reflexive and Suzumura-consistent collective choice functional* is a collective choice functional F such that $F(\mathbf{U})$ is reflexive and Suzumura consistent for all $\mathbf{U} \in \mathcal{D}$, and a *social-evaluation functional* is a collective choice functional F such that $F(\mathbf{U})$ is an ordering for all $\mathbf{U} \in \mathcal{D}$. Informational assumptions regarding the measurability and interpersonal comparability of individual utilities can be expressed by requiring the collective choice functional to be constant on sets of utility profiles that contain the same information. For example, if utilities are *cardinally measurable and fully comparable,* any utility profile $\mathbf{U}'$ that is obtained from a profile $\mathbf{U}$ by applying the same increasing affine transformation to all individual utility functions carries the same information as $\mathbf{U}$ itself. Thus, the collective choice functional must assign the same social ranking to both profiles. See Blackorby, Donaldson, and Weymark (1984) or Bossert and Weymark (2004) for discussions of information assumptions in social-choice theory.

A fundamental result in this setting is the *welfarism theorem;* see, for instance, d'Aspremont and Gevers (1977) and Hammond (1979). A social-evaluation functional F is *welfarist* if, for any utility profile $\mathbf{U}$ and for any two alternatives x and y, the social ranking of x and y according to the social ordering assigned to the profile $\mathbf{U}$ by F depends on the two utility vectors $\mathbf{U}(x) = (U_1(x), \ldots, U_n(x))$ and $\mathbf{U}(y) = (U_1(y), \ldots, U_n(y))$ only. Thus, a *single* ordering of utility vectors is sufficient to rank the alternatives for *any* profile. The welfarism theorem states that, provided that the domain of the social-evaluation functional consists of all possible utility profiles, welfarism is equivalent to the conjunction of Pareto indifference and independence of irrelevant alternatives.

In this section, it is illustrated that the welfarism theorem has an analogous formulation for reflexive and Suzumura-consistent collective choice functionals: even if every social ranking is merely required to be reflexive and Suzumura consistent rather than an ordering, the conjunction of the two axioms is (under the unlimited-domain assumption) equivalent to the existence of a single reflexive and Suzumura-consistent relation R defined on utility vectors that is sufficient to obtain the social ranking for any utility profile. This relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is referred to as a *social-evaluation relation.* The requisite axioms are the following.

Unlimited domain. $\mathcal{D} = \mathcal{U}^n$.

Pareto indifference. For all $x, y \in X$ and for all $\mathbf{U} \in \mathcal{D}$,

$$U_i(x) = U_i(y) \text{ for all } i \in N \Rightarrow (x,y) \in I(F(\mathbf{U})).$$

Independence of irrelevant alternatives. For all $x,y \in X$ and for all $\mathbf{U}, \mathbf{U}' \in \mathcal{D}$ such that $U_i(x) = U_i'(x)$ and $U_i(y) = U_i'(y)$ for all $i \in N$,

$$\left[(x,y) \in F(\mathbf{U}) \Leftrightarrow (x,y) \in F(\mathbf{U}')\right] \text{ and } \left[(y,x) \in F(\mathbf{U}) \Leftrightarrow (y,x) \in F(\mathbf{U}')\right].$$

The following theorem generalizes the standard welfarism theorem by allowing social relations to be intransitive and incomplete but imposing the Suzumura-consistency requirement.

Theorem 4. *Suppose that a reflexive and Suzumura-consistent collective choice functional F satisfies unlimited domain. F satisfies Pareto indifference and independence of irrelevant alternatives if and only if there exists a reflexive and Suzumura-consistent social-evaluation relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that, for all $x,y \in X$ and for all $\mathbf{U} \in \mathcal{U}^n$,*

$$(x,y) \in F(\mathbf{U}) \Leftrightarrow (\mathbf{U}(x), \mathbf{U}(y)) \in R. \tag{4}$$

Proof. The 'if' part of the theorem is straightforward to verify. To prove the converse implication, suppose that F is a reflexive and Suzumura-consistent collective choice functional satisfying unlimited domain, Pareto indifference and independence of irrelevant alternatives. Define the relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$ as follows. For all $u,v \in \mathbb{R}^n$, $(u,v) \in R$ if and only if there exist $x,y \in X$ and $\mathbf{U} \in \mathcal{U}^n$ such that $\mathbf{U}(x) = u$, $\mathbf{U}(y) = v$ and $(x,y) \in F(\mathbf{U})$. That R is well-defined follows as in the standard welfarism theorem; see, for instance, Blackorby, Donaldson, and Weymark (1984) or Bossert and Weymark (2004). Once R is well defined, (4) is immediate and, furthermore, R is reflexive because $F(\mathbf{U})$ is reflexive for all $\mathbf{U} \in \mathcal{U}^n$. The proof is completed by showing that R is Suzumura consistent. Let $u,v \in \mathbb{R}^n$ be such that $(u,v) \in \mathrm{tc}(R)$. By definition of the transitive closure of a relation, there exist $M \in \mathbb{N}$ and $u^0,\ldots,u^M \in \mathbb{R}^n$ such that $u = u^0$, $(u^{m-1},u^m) \in R$ for all $m \in \{1,\ldots,M\}$ and $u^M = v$. By definition of R, there exist $x^0,\ldots,x^M \in X$ and $\mathbf{U}^1,\ldots,\mathbf{U}^M \in \mathcal{U}^n$ such that $\mathbf{U}^{m-1}(x^{m-1}) = u^{m-1}$, $\mathbf{U}^{m-1}(x^m) = u^m$ and $(x^{m-1},x^m) \in F(\mathbf{U}^{m-1})$ for all $m \in \{1,\ldots,M\}$. By unlimited domain, there exists $\mathbf{V} \in \mathcal{U}^n$ such that $\mathbf{V}(x^m) = u^m$ for all $m \in \{0,\ldots,M\}$. Using (4), it follows that $(x^{m-1},x^m) \in F(\mathbf{V})$ for all $m \in \{1,\ldots,M\}$. Because $F(\mathbf{V})$ is Suzumura consistent, it follows that $(x^M,x^0) \notin P(F(\mathbf{V}))$. Thus, by (4), $(v,u) = \left(\mathbf{V}(x^M), \mathbf{V}(x^0)\right) \notin P(R)$ and R is Suzumura consistent. □

5 Population Ethics

The traditional social-choice framework with a fixed population is unable to capture important aspects of many public-policy choices. For example, decisions on funds devoted to prenatal care, the intergenerational allocation of resources and the design of aid packages to developing countries involve endogenous populations: depending on the selected alternative, some individuals may or may not be brought into existence. To address this issue, a social ranking must be capable of comparing alternatives with different population sizes.

The possibility of extending the welfarist approach to a variable-population environment has been examined in a variety of contributions, most notably in applied ethics; see, for instance, Parfit (1976, 1982, 1984). Impossibility results arise frequently in this area, and it is therefore of interest to examine the possibilities of escaping these negative conclusions. The purpose of this section is to illustrate that weakening transitivity to Suzumura consistency turns some of these impossibilities into possibilities. Of course, to ensure that Suzumura consistency is indeed weaker than transitivity, we cannot impose reflexivity, completeness and Suzumura consistency – as mentioned earlier, Suzumura consistency and transitivity coincide in the presence of the two richness conditions. Therefore, the question arises whether reflexivity and completeness rather than transitivity are, to a large extent, responsible for the impossibilities. This is not the case: although most of the impossibility results in this area have been established for orderings, they remain true if reflexivity and completeness are dropped.

A variable-population version of a social-evaluation relation is defined on the set of utility vectors of *any* dimension, that is, it is a relation $R \subseteq \Omega \times \Omega$, where $\Omega = \cup_{n \in \mathbb{N}} \mathbb{R}^n$. The components of a utility vector $u \in \Omega$ are interpreted as the *lifetime* utilities of those alive in the requisite alternative. For an individual who is alive, a *neutral* life is one which is as good as one without experiences. A life above neutrality is worth living, a life below neutrality is not. Following standard practice in population ethics, a lifetime-utility level of zero is assigned to neutrality.

In Blackorby, Bossert, and Donaldson (2006), it is shown that there exists no variable-population social-evaluation ordering satisfying four axioms that are common in the literature. This result can be generalized by noting that it does not make use of reflexivity or completeness – all that is needed is the transitivity of R.

The first axiom is *minimal increasingness.* It requires that, for any fixed population size, if all individuals have the same utility in two utility vectors, then the vector where everyone's utility is higher is better according to R. We use $\mathbf{1}_n$ to denote the vector of $n \in \mathbb{N}$ ones.

Minimal increasingness. For all $n \in \mathbb{N}$ and for all $\beta, \gamma \in \mathbb{R}$,

$$\beta > \gamma \Rightarrow (\beta \mathbf{1}_n, \gamma \mathbf{1}_n) \in P(R).$$

Minimal increasingness is a weak unanimity property: it only applies if everyone has the same utility in both alternatives to be compared.

Another fixed-population axiom is *weak inequality aversion.* This axiom demands that, for any fixed population size, perfect equality is at least as good as any distribution of the same total utility.

Weak inequality aversion. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$,

$$\left(\left(\frac{1}{n} \sum_{i=1}^{n} u_i \right) \mathbf{1}_n, u \right) \in R.$$

Sikora (1978) suggests a variable-population version of the Pareto principle. The axiom usually is defined as the conjunction of the strong Pareto principle and the requirement that the addition of an individual above neutrality to a utility-unaffected population is a social improvement. Because strong Pareto will be introduced as a separate axiom later on and is not needed for the impossibility result, we use the second part of the property only.

Pareto plus. For all $n \in \mathbb{N}$, for all $u \in \mathbb{R}^n$ and for all $a \in \mathbb{R}_{++}$,

$$((u,a),u) \in P(R).$$

In the axiom statement, the population common to u and (u,a) is unaffected and, thus, to defend the axiom on individual-goodness grounds, it must be argued that a level of well-being above neutrality is better than non-existence. Thus, the axiom applies the Pareto condition to situations where a person is not alive in all alternatives to be compared. While it is possible to compare alternatives with different populations from a social point of view (which is the issue addressed in population ethics), it is not clear that such a comparison can be made from the viewpoint of an individual if the person is not alive in one of the alternatives. It is therefore difficult to interpret this axiom as a Pareto condition because it appears to be based on the idea that people who do not exist have interests that should be respected. There is, therefore, an important asymmetry in the assessment of alternatives with different populations. It is perfectly reasonable to say that an individual considers life worth living if the person is alive with a positive level of lifetime well-being, but that does not justify the claim that a person who does not exist gains from being brought into existence with a lifetime utility above neutrality.

As is the case for Pareto plus, the final axiom used in our impossibility result applies to comparisons across population sizes. A variable-population social-evaluation relation leads to the *repugnant conclusion* if population size can always be substituted for well-being, no matter how close to neutrality the utilities of a large population are. That is, mass poverty may be considered superior to some alternatives in which fewer people lead very good lives. This property has been used by Parfit (1976, 1982, 1984) to argue against *classical utilitarianism,* the variable-population social-evaluation ordering that ranks utility vectors on the basis of their total utilities. If Parfit's view is accepted, R should be required to avoid the repugnant conclusion.

Avoidance of the repugnant conclusion. There exist $n \in \mathbb{N}$, $\xi \in \mathbb{R}_{++}$ and $\varepsilon \in (0,\xi)$ such that, for all $m > n$,

$$(\varepsilon \mathbf{1}_m, \xi \mathbf{1}_n) \notin P(R).$$

Blackorby, Bossert, and Donaldson (2006, Theorem 2) show that there exists no variable-population social-evaluation ordering satisfying the above four axioms; see Blackorby, Bossert, and Donaldson (2005), Blackorby, Bossert, Donaldson, and Fleurbaey (1998), Blackorby and Donaldson (1991), Carlson (1998), McMahan (1981), Parfit (1976, 1982, 1984) and Shinotsuka (2008) for similar observations.

The following theorem shows that reflexivity and completeness are not required – transitivity of R is sufficient to generate the impossibility.

Theorem 5. *There exists no transitive variable-population social-evaluation relation satisfying minimal increasingness, weak inequality aversion, Pareto plus and avoidance of the repugnant conclusion.*

Proof. Suppose R satisfies minimal increasingness, weak inequality aversion and Pareto plus. The proof is completed by showing that R leads to the repugnant conclusion. For any population size $n \in \mathbb{N}$, let $\xi, \varepsilon, \delta \in \mathbb{R}_{++}$ be such that $0 < \delta < \varepsilon < \xi$. Choose any integer r such that

$$r > n\frac{(\xi - \varepsilon)}{(\varepsilon - \delta)}. \tag{5}$$

Because both the numerator and denominator on the right-hand side of the inequality are positive, r is positive. By Pareto plus,

$$((\xi \mathbf{1}_n, \delta \mathbf{1}_r), \xi \mathbf{1}_n) \in P(R). \tag{6}$$

Average utility in $(\xi \mathbf{1}_n, \delta \mathbf{1}_r)$ is $(n\xi + r\delta)/(n+r)$ so, by minimal inequality aversion,

$$\left(\left(\frac{n\xi + r\delta}{n+r}\right)\mathbf{1}_{n+r}, (\xi \mathbf{1}_n, \delta \mathbf{1}_r)\right) \in R. \tag{7}$$

By (5),

$$\varepsilon > \frac{n\xi + r\delta}{n+r}$$

and, by minimal increasingness,

$$\left(\varepsilon \mathbf{1}_{n+r}, \left(\frac{n\xi + r\delta}{n+r}\right)\mathbf{1}_{n+r}\right) \in P(R). \tag{8}$$

Combining (6)–(8) and using transitivity, it follows that $(\varepsilon \mathbf{1}_{n+r}, \xi \mathbf{1}_n) \in P(R)$ and avoidance of the repugnant conclusion is violated. □

If transitivity is weakened to Suzumura consistency, the axioms in the theorem statement are compatible. Moreover, three of them can be strengthened and other properties that are commonly imposed in population ethics can be added without obtaining an impossibility.

Expressed in the current setting, the strong Pareto principle is another fixed-population axiom. If everyone alive in two fixed-population alternatives with utility vectors u and v has a utility in u that is at least as high as the utility of this person in v with at least one strict inequality, u is better than v. Clearly, this axiom is a strengthening of minimal increasingness.

Strong Pareto. For all $n \in \mathbb{N}$ and for all $u, v \in \mathbb{R}^n$,

$$u_i \geq v_i \text{ for all } i \in \{1, \ldots, n\} \text{ and } u \neq y \;\Rightarrow\; (u, v) \in P(R).$$

Continuity is a condition that prevents the social-evaluation relation R from exhibiting 'large' changes in response to 'small' changes in a utility vector. Again, the axiom imposes restrictions on fixed-population comparisons only.

Continuity. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$, the sets $\{v \in \mathbb{R}^n \mid (v,u) \in R\}$ and $\{v \in \mathbb{R}^n \mid (u,v) \in R\}$ are closed in $\mathbb{R}^n$.

Weak inequality aversion can be strengthened by requiring the restriction of R to $\mathbb{R}^n \times \mathbb{R}^n$ to be *strictly S-concave* for any population size $n \in \mathbb{N}$; see, for instance, Marshall and Olkin (1979). Strict S-concavity is equivalent to the conjunction of the *strict transfer principle* familiar from the theory of inequality measurement and *anonymity.* The strict transfer principle requires that a progressive transfer increases goodness, provided the relative rank of the individuals involved in the transfer is unchanged; see Dalton (1920) and Pigou (1912). A social-evaluation relation is anonymous if the individuals in a fixed population are treated impartially, without paying attention to their identities; see Sen (1973) for a detailed discussion. A *bistochastic $n \times n$ matrix* is a matrix whose entries are in the closed interval $[0,1]$ and all row sums and column sums are equal to one.

Strict S-concavity. For all $n \in \mathbb{N}$, for all $u \in \mathbb{R}^n$ and for all bistochastic $n \times n$ matrices B,

(i) $(Bu,u) \in R$.
(ii) Bu is not a permutation of $u \Rightarrow (Bu,u) \in P(R)$.

Independence of the utilities of unconcerned individuals is a fixed-population separability property introduced by d'Aspremont and Gevers (1977). It requires that only the utilities of those who can possibly be affected by a choice between two fixed-population alternatives should determine their ranking.

Independence of the utilities of unconcerned individuals. For all $n,m \in \mathbb{N}$, for all $u,v \in \mathbb{R}^n$ and for all $w,s \in \mathbb{R}^m$,

$$((u,w),(v,w)) \in R \Leftrightarrow ((u,s),(v,s)) \in R.$$

We now turn to further variable-population axioms. The *negative expansion principle* is dual to Pareto plus. It requires any utility distribution to be ranked as better than one with the ceteris-paribus addition of an individual whose life is not worth living – that is, with a lifetime utility below neutrality.

Negative expansion principle. For all $n \in \mathbb{N}$, for all $u \in \mathbb{R}^n$ and for all $a \in \mathbb{R}_{--}$,

$$(u,(u,a)) \in P(R).$$

Expansion continuity applies the notion of continuity to pairs of utility vectors of different dimension, particularly pairs of vectors whose dimensions differ by one.

Expansion continuity. For all $n \in \mathbb{N}$ and for all $u \in \mathbb{R}^n$, the sets $\{t \in \mathbb{R} \mid ((u,t),u) \in R\}$ and $\{t \in \mathbb{R} \mid (u,(u,t)) \in R\}$ are closed in $\mathbb{R}$.

Note that, in the presence of Pareto plus and the negative expansion principle, expansion continuity implies *existence of critical levels,* requiring that non-trivial trade-offs between population size and well-being are possible in the sense that, for any utility vector $u \in \Omega$, there exists a utility level $c \in \mathbb{R}$ (which may depend on u) such that the ceteris-paribus addition of an individual with utility level c to an existing population with utilities u is a matter of indifference according to R.

Finally, a strengthening of avoidance of the repugnant conclusion is defined. It is obtained by replacing the existential quantifiers in the original axiom with universal quantifiers and replacing the negation of betterness in the conclusion with the negation of the at-least-as-good-as relation. This is a strong property and one might not want to endorse it; the reason why it is used to replace the weaker condition is that it makes the possibility result logically stronger.

Strong avoidance of the repugnant conclusion. For all $n \in \mathbb{N}$, for all $\xi \in \mathbb{R}_{++}$, for all $\varepsilon \in (0,\xi)$ and for all $m > n$,

$$(\varepsilon \mathbf{1}_m, \xi \mathbf{1}_n) \notin R.$$

We do not impose *avoidance of the sadistic conclusion* or any of its variants (see Arrhenius (2000)) because it is implied by some of the properties already defined.

Theorem 6. *There exists a reflexive and Suzumura-consistent variable-population social-evaluation relation satisfying strong Pareto, continuity, strict S-concavity, independence of the utilities of unconcerned individuals, Pareto plus, the negative expansion principle, expansion continuity and strong avoidance of the repugnant conclusion.*

Proof. An example is sufficient to establish the theorem. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous, increasing and strictly concave function such that $g(0) = 0$ and define the relation R^* by letting, for all $n,m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$,

$$\begin{aligned}(u,v) \in R^* \Leftrightarrow & \left[n = m \text{ and } \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)\right] \\ & \text{or } \left[m = n+1 \text{ and } \exists \alpha \in \mathbb{R}_- \text{ such that } v = (u,\alpha)\right] \\ & \text{or } \left[n = m+1 \text{ and } \exists \beta \in \mathbb{R}_+ \text{ such that } u = (v,\beta)\right].\end{aligned}$$

Strong Pareto is satisfied because g is increasing, continuity is satisfied because g is continuous, strict S-concavity follows from the strict concavity of g and independence of the utilities of unconcerned individuals is satisfied because of the additively separable structure of the criterion for fixed-population comparisons. Pareto plus and the negative expansion principle follow immediately from the definition of R^*. Expansion continuity is satisfied because the comparisons involving vectors of dimensions n and $n+1$ for any $n \in \mathbb{N}$ clearly are performed in accordance with this requirement. Strong avoidance of the repugnant conclusion is satisfied because $(\varepsilon \mathbf{1}_m, \xi \mathbf{1}_n) \notin R^*$ for all $n \in \mathbb{N}$, for all $\xi \in \mathbb{R}_{++}$, for all $\varepsilon \in (0, \xi)$ and for all $m > n$. That R^* is reflexive is immediate.

It remains to show that R^* is Suzumura consistent. The first step is to prove that, for all $n, m \in \mathbb{N}$, for all $u \in \mathbb{R}^n$ and for all $v \in \mathbb{R}^m$,

$$(u,v) \in R^* \Rightarrow \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i) \tag{9}$$

and

$$(u,v) \in P(R^*) \Rightarrow \sum_{i=1}^{n} g(u_i) > \sum_{i=1}^{m} g(v_i). \tag{10}$$

To prove (9), suppose that $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$ and $(u,v) \in R^*$. According to the definition of R^*, there are three possible cases.

Case 1. $n = m$ and $\sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)$. The conclusion is immediate in this case.

Case 2. $m = n+1$ and $\exists \alpha \in \mathbb{R}_-$ such that $v = (u, \alpha)$. Thus,

$$\sum_{i=1}^{m} g(v_i) = \sum_{i=1}^{n} g(u_i) + g(\alpha) \leq \sum_{i=1}^{n} g(u_i),$$

where the inequality follows because $\alpha \leq 0$ and, by the increasingness of g and the property $g(0) = 0$, $g(\alpha) \leq 0$.

Case 3. $n = m+1$ and $\exists \beta \in \mathbb{R}_+$ such that $u = (v, \beta)$. This implies

$$\sum_{i=1}^{n} g(u_i) = \sum_{i=1}^{m} g(v_i) + g(\beta) \geq \sum_{i=1}^{m} g(v_i),$$

where the inequality follows because $\beta \geq 0$ and thus $g(\beta) \geq 0$.

To prove (10), suppose $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ are such that $(u,v) \in P(R^*)$. Again, there are three cases.

Case 1. $n = m$ and $\sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)$. If $\sum_{i=1}^{m} g(v_i) \geq \sum_{i=1}^{n} g(u_i)$, we obtain $(v,u) \in R^*$ and thus a contradiction to our hypothesis $(u,v) \in P(R^*)$. Therefore, $\sum_{i=1}^{n} g(u_i) > \sum_{i=1}^{m} g(v_i)$.

Case 2. $m = n+1$ and $\exists \alpha \in \mathbb{R}_-$ such that $v = (u, \alpha)$. Thus,

$$\sum_{i=1}^{m} g(v_i) = \sum_{i=1}^{n} g(u_i) + g(\alpha) \leq \sum_{i=1}^{n} g(u_i) \tag{11}$$

as established in the proof of (9). If $\alpha = 0$, it follows that $v = (u, 0)$ which leads to $(v,u) \in R^*$, contradicting our hypothesis $(u,v) \in P(R^*)$. Thus, $\alpha < 0$ and $g(\alpha) < 0$ because $g(0) = 0$ and g is increasing. Therefore, the inequality in (11) is strict.

Case 3. $n = m + 1$ and $\exists \beta \in \mathbb{R}_+$ such that $u = (v, \beta)$. This implies

$$\sum_{i=1}^{n} g(u_i) = \sum_{i=1}^{m} g(v_i) + g(\beta) \geq \sum_{i=1}^{m} g(v_i) \tag{12}$$

as established in the proof of (9). If $\beta = 0$, it follows that $u = (v, 0)$ which leads to $(v, u) \in R^*$, again contradicting the hypothesis $(u, v) \in P(R^*)$. Thus, $\beta > 0$ and the inequality in (12) is strict.

To complete the proof, suppose $n, m \in \mathbb{N}$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ are such that $(u, v) \in \mathrm{tc}(R^*)$. By repeated application of (9) and the transitivity of $\geq$, it follows that $\sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i)$. If $(v, u) \in P(R^*)$, (10) implies $\sum_{i=1}^{m} g(v_i) > \sum_{i=1}^{n} g(u_i)$, a contradiction. Thus, $(v, u) \notin P(R^*)$ and R^* is Suzumura consistent. □

Another impossibility result in population ethics is due to Broome (2004, Chap. 10). Broome suggests that existence is in itself neutral and, thus, the ceteris-paribus addition of an individual to a utility-unaffected population should lead to an equally-good alternative, at least as long as the utility of the added person (if brought into being) is within a non-degenerate interval. This intuition, which Broome calls the *principle of equal existence,* is incompatible with strong Pareto, provided that the social-evaluation relation R is transitive. The impossibility persists if transitivity is weakened to Suzumura consistency. The following axiom is a weak form of the principle of equal existence.

Principle of equal existence. There exist $u \in \Omega$ and distinct $\alpha, \beta \in \mathbb{R}$ such that

$$((u, \alpha), u) \in I(R) \text{ and } ((u, \beta), u) \in I(R). \tag{13}$$

We obtain the following impossibility result.

Theorem 7. *There exists no Suzumura-consistent variable-population social-evaluation relation satisfying strong Pareto and the principle of equal existence.*

Proof. Suppose R satisfies strong Pareto and the principle of equal existence. The proof is completed by showing that R cannot be Suzumura consistent. By the principle of equal existence, there exist $u \in \Omega$ and distinct utility levels α and β such that (13) is satisfied. Without loss of generality, suppose $\alpha > \beta$. By strong Pareto, $((u, \alpha), (u, \beta)) \in P(R)$ which, together with (13), leads to a violation of Suzumura consistency. □

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