# Characterization of the Maximin Choice Function in a Simple Dynamic Economy

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## 1 Introduction

In the literature of intergenerational equity, Rawlsian maximin principle is one of the most well-known criteria for distributive justice among generations.[1] Since this principle has an intuitive appeal to egalitarian writers, several attempts to characterize the principle have been made in welfare economics. Arrow (1973), Dasgupta (1974a, b), and Riley (1976) scrutinized the performance thereof in the context of optimal growth. Arrow shows that the utility path as well as the consumption path generated by the maximin principle has a saw-tooth shape. Dasgupta shows that it gives rise to a logical deficit such as time-inconsistency. The other line of researches has been stimulated by the axiomatic approaches of Hammond (1976, 1979) and Sen (1970, 1977). In this line, researchers extended axiomatizations of the maximin principle and applied them to intergenerational equity. The maximin path is characterized by a constant path, which emphasizes its egalitarian perspective.[2]

In a previous discussion on this topic, Suga and Udagawa (2004) addressed the question of how to characterize the maximin principle axiomatically in a simple

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[1] See Rawls (1971) for the description of the maximin principle. Rawls himself denies a direct application of the principle, and emphasizes that each generation has a paternalistic concern to the payoff of his immediate descendant. Since the number of generations are infinite so that the minimum utility may not exist, we usually evaluate consumption plans not by the minimum but by the infimum.

[2] See, for example, Asheim, Buchholz, and Tungodden (2001), Epstein (1986a,b) and Lauwers (1997).

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dynamic economy, called Arrow–Dasgupta economy, where each generation has a paternalistic concern to the descendants. In Suga and Udagawa (2004), the axioms are imposed on intergenerational preference relations over the set of consumption paths. They supposed that there exists a hypothetical social planner who judges the consumption paths, and characterized the maximin principle by some axioms on the planner's intergenerational preference relation.

In this chapter, on the other hand, we examine the same question of axiomatic characterization of the maximin principle by applying axioms in a choice function framework. That is, we consider a choice-theoretic model of infinite horizon economy in which a choice function selects a consumption path from the set of feasible paths from the viewpoint of a social planner. We suppose that the social planner adopts the maximin principle as a criterion to construct an intertemporal choice function. We focus our attention to a simple dynamic economy with linear technology à la Arrow (1973) and Dasgupta (1974a,b) to characterize the maximin choice function on the set of consumption paths. We employ this choice-theoretic approach to give another look at characteristics of consumption paths derived by the maximin principle under the feasibility conditions.

With a similar motivation, Asheim, Bossert, Sprumont, and Suzumura (2006) propose a choice-theoretic model for intergenerational equity. They provide characterizations of all infinite-horizon choice functions satisfying either efficiency or time-consistency, and identify all choice functions with both properties. Their results show that the choice-theoretic approach to intergenerational resource allocation provides an interesting and viable alternative to the models based on establishing intergenerational preference relations of utility paths.

Our purpose in this chapter is to characterize the maximin principle in an infinite horizon economy. Axioms are imposed not on intergenerational preference relations but on choice functions themselves. Some of the axioms are similar to those in characterization of the maximin principle in intra-generational equity, that is, Pareto principle and extended Hammond equity. Others are conditions $\alpha$ and $\beta$, which are often used in choice theory to describe consistent choices.

The chapter is organized as follows. Section 2 is the description of the economy, which provides a canvas for our analysis. Axioms are stated in Sect. 3. The main theorem, the lemmas, and their proofs are contained in Sect. 4. Section 5 provides related examples. We conclude the chapter with some final remarks in Sect. 6.

## 2 Simple Dynamic Economy

Let $\mathcal{Z}_+$ be the set of all nonnegative integers, each element of which is used to represent a generation or time period. For simplicity, we assume that each time period consists of one generation, and each generation consists of one representative individual. There is a private good, which can either be consumed or invested to be capital that bears a return. $k_t$ denotes the accumulated capital at the beginning of time period $t \in \mathcal{Z}_+$. In that period a fraction $x_t$ is consumed and the remainder $k_t - x_t$ is used in production. The production technology is assumed to be linear. Then each unit used in production brings $\gamma$ units of the good at the end of the period, and are transferred to the next period $t+1$. Hence

$$k_{t+1} = \gamma(k_t - x_t). \tag{1}$$

We assume that the economy is productive, so that

$$\gamma > 1. \tag{2}$$

The following feasibility condition for production is assumed. For all $t \in \mathcal{Z}_+$

$$k_t \geq 0. \tag{3}$$

A feasibility condition for consumption is also assumed. That is, any individual cannot survive without consumption. Hence, for all $t \in \mathcal{Z}_+$

$$x_t \geq 0.^3 \tag{4}$$

Now we describe our problem to find a consumption path that is selected by Rawlsian maximin principle for intergenerational justice. For the convenience of description, we adopt the following notation: let $\mathcal{L}_+^\infty = \{(x_0, x_1, \ldots, x_t, \ldots) | \; \forall t \in \mathcal{Z}_+ : x_t \geq 0\}$. Denote a consumption path by the capital letter, for example, $X = (x_0, x_1, \ldots)$. $_{\text{rep}}(x_1, \ldots, x_n)$ represents the path $(x_1, \ldots, x_n, \; x_1, \ldots, x_n, \ldots)$, which consists of $(x_1, \ldots, x_n)$ repeated infinitely many times. By the feasibility condition, consumption paths ought to be chosen from the set $\mathcal{X} = \{X \in \mathcal{L}_+^\infty | \; \forall t \in \mathcal{Z}_+ : \; 0 \leq k_{t+1} = \gamma(k_t - x_t)\}$ given $k_0 > 0$.[4] It is convenient, however, to use the following equivalent form: for any given $k_0$ and $\gamma$, the set of feasible consumption paths are given by

$$\mathcal{X} = \left\{ X = (x_0, x_1, \ldots) \in \mathcal{L}_+^\infty \;\middle|\; \sum_{t=0}^{\infty} \gamma^{-t} x_t \leq k_0 \right\}.$$

We denote the utility function of generation $t \in \mathcal{Z}_+$, or often called individual $t$, by $W_t(X)$ when the consumption path $X$ is attained. We assume that generation $t$ derives utility from her own consumption $x_t$ and also from her immediate $n-1$ descendants' satisfaction, where $n \geq 2$, so that her utility function depends on the consumption stream of $n$ generations beginning with her own. We also assume that the utility function $W_t$ is the same for all generations $t \in \mathcal{Z}_+$, that is, $W_t = W$ for all $t$. Following Arrow (1973) and Dasgupta (1974a, b), we assume that $W$ is additively separable as to generations for simplicity, that the felicity ascribed by individual $t$ to individual $t+i$ is the same as that ascribed by individual $t+i$ to herself, that the

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[3] Any path on which at least one generation survives is meaningful for the discussion of intergenerational equity. It is possible for a generation to exhaust the whole amount of the good inherited from the past. Here we impose a mild requirement on the feasibility of the consumption.

[4] $\mathcal{X}$ depends on the initial capital stock $k_0$. But in the following discussion, $k_0$ is given from the outside at the outset so that we employ the notation $\mathcal{X}$ instead of $\mathcal{X}(k_0)$.

felicity function is the same for all $t$, and that the felicity of the future generations is discounted in the utility of the present generation. That is,

$$W_t(X) = W(x_t, x_{t+1}, \ldots, x_{t+n-1}) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}), \tag{5}$$

where $\rho_0 = 1$ and $\rho_i$ $(1 \leq i \leq n-1)$ are a parameters reflecting the weight each generation attach to the future generations. We assume that the weight of a farther future generation is smaller, that is, $\rho_i \geq \rho_{i+1}$ $(0 \leq i \leq n-1)$. The felicity function $U$ is assumed to satisfy the following conditions: (a) $U : \mathfrak{R}_+ \to \mathfrak{R}$ is twice continuously differentiable; (b) $U'(x) > 0$ and $U''(x) < 0$.

We focus our concern on the case in which the optimal consumption path for the maximin principle has a saw-tooth shape.[5] Therefore, we assume

$$\gamma^i \rho_i < \gamma^j \rho_j \quad (0 \leq i < j \leq n). \tag{6}$$

This assumption requires that each generation obtains more utility if she bequeaths capital to the next generation than that if she consumes it by herself. Although the utility of the next generation is discounted by $\rho$, the total utility will go up if the increase in production is included.

Then, the maximin principle of justice gives a solution to the problem

$$\begin{aligned} \max_{X \in S} \min_t W_t(X) &\Longleftrightarrow \max_{X \in S} \min_t W(x_t, x_{t+1}, \cdots, x_{t+n-1}), \\ &\Longleftrightarrow \max_{X \in S} \min_t \sum_{i=0}^{n-1} \rho_i U(x_{t+i}), \end{aligned} \tag{7}$$

where $S$ is any subset of $\mathscr{X}$.

Now we present Arrow's theorem on the maximin path. Let $\hat{x}$ be the consumption level that allows to bequeath the same amount of capital as the initial level to the next generation, that is,

$$\hat{x} = \frac{\gamma - 1}{\gamma} k_0. \tag{8}$$

Clearly, the consumption path $_{\text{rep}}(\hat{x})$ satisfies the feasibility condition. In other words, the constant consumption $\hat{x}$ will cause $k_t$ to remain constant at the initial level $k_0$.

Let $(x_0^{\mathrm{R}}, x_1^{\mathrm{R}}, \ldots, x_n^{\mathrm{R}})$ be the solution to the problem

$$\max_{x_0, x_1, \ldots, x_{n-1}} \sum_{i=0}^{n-1} \rho_i U(x_i), \tag{9}$$

$$\text{s.t.} \quad \sum_{i=0}^{n-1} \gamma^{-i} x_i = \hat{x} \sum_{i=0}^{n-1} \gamma^{-i}. \tag{10}$$

[5] The inequality (6) is a necessary condition for the case. See Arrow (1973).

Equation (10) is equivalent to the condition $k_n = k_0$. Therefore, this problem can be interpreted as the maximization problem of generation 0's utility, subject to the restriction that generation 0 must bequeath $k_0$ to generation $n$.

Then we have the following theorem by Arrow (1973), which is the most fundamental proposition in this field.

**Theorem 1.** [Arrow (1973); Theorem 3] *Suppose that the utility of any generation $t$ is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

$\gamma^i \rho_i$ *increases with $i$ for $i \leq n-1$, and $\rho_i$ is nonincreasing in $i$. Then the feasible consumption path that maximizes* $\min_t W_t$ *can be characterized as follows. Choose* $x_i^*$ $(i = 0, \dots, n-1)$ *to maximize* $W(x_0, \dots, x_{n-1})$ *subject to the constraint*

$$\sum_{i=0}^{n-1} \gamma^{-i} x_i = \hat{x} \sum_{i=0}^{n-1} \gamma^{-i},$$

*where $\hat{x}$ is given in* (8). *Then at the optimum (i)* $x_{nl+i} = x_i^*$ $(0 \leq i \leq n-1)$ *for any* $l \in \mathcal{Z}_+$. *For this path the following properties hold: (ii)* $x_i^* < x_{i+1}^*$; *(iii)* $W_t = \min_t W_t$, *if $t$ is divisible by $n$; (iv) for all other $t$,* $W_t \geq \min_t W_t$, *and (v) the inequality is strict if* $\rho_i > \rho_{i+1}$ *for some* $i < n-1$.

We define a choice function $C$ that maps any nonempty set $S \subseteq \mathcal{X}$ of feasible consumption paths to its subset, given a utility function $W$. Because $W$ is given and fixed throughout this chapter, a choice function is denoted by $C(S)$. We define the Rawlsian choice function $C^{\mathrm{R}}$, which maps any feasible set $S$ of consumption paths to the set of all maximin consumption paths $X^{\mathrm{R}}$ in $S$, given a utility function $W$. It is not generally true that $C^{\mathrm{R}}(S) \neq \varnothing$ for all $S \subseteq \mathcal{X}$ and $W$, but Arrow (1973) showed $C^{\mathrm{R}}(\mathcal{X}) \neq \varnothing$ under the above utility function $W$.

## 3 Axioms

In this section, we define several axioms for a characterization of the maximin principle in this simple dynamic economy.[6] First, we define two binary relations on $\mathcal{X}$. One is the strict Paretian relation, $\succ^{\mathrm{P}}$, which is given by: for any $X^1, X^2 \in \mathcal{X}$,

$$X^1 \succ^{\mathrm{P}} X^2 \iff \forall t : W_t(X^1) > W_t(X^2).$$

Another is the Hammond equity relation, $\succsim^{\mathrm{H}}$, which is defined as, for any $X^1, X^2 \in \mathcal{X}$

[6] See Hammond (1976, 1979), Sen (1970), and Suzumura (1983) to understand the meanings of the axioms in the classical environment. In the following definitions, we employ those in Suzumura (1983).

$$X^1 \succsim^{\mathrm{H}} X^2 \iff \exists t^1, t^2 \in \mathcal{Z}_+ : \begin{aligned}[t] &\text{(i) } W_{t^1}(X^1) \leq W_{t^2}(X^1), \\ &\text{(ii) } W_{t^1}(X^1) \geq W_{t^1}(X^2), \\ &\text{(iii) } W_{t^2}(X^1) \leq W_{t^2}(X^2), \text{and} \\ &\text{(iv) } W_t(X^1) = W_t(X^2)\ \forall t \neq t^1, t^2. \end{aligned}$$

By extending the Hammond equity principle, we introduce a new concept of equity among groups of generations. It is called the extended Hammond equity principle, which implies a fairness requirement that we should treat two groups of generations equally if they are regarded equal in utility profiles. As an auxiliary step, we follow Suzumura (1983) to introduce the lexicographic ordering $R^{\mathrm{L}}$ on the Euclidean $n$-space $E^n$. For every $v \in E^n$, let $i(v)$ denote the $i$th smallest element, ties being broken arbitrarily, so that we have

$$v_{1(v)} \leq v_{2(v)} \leq \cdots \leq v_{n(v)}.$$

We may then define three binary relations $P^{\mathrm{L}}$, $I^{\mathrm{L}}$, and $R^{\mathrm{L}}$ on $E^n$ by

$$v^1 P^{\mathrm{L}} v^2 \iff \exists r \leq n : \begin{cases} \forall i \in \{1, 2, \ldots, r-1\} : v^1_{i(v^1)} = v^2_{i(v^2)} \\ \& \\ v^1_{r(v^1)} > v^2_{r(v^2)}, \end{cases}$$

$$v^1 I^{\mathrm{L}} v^2 \iff \forall i \in \{1, 2, \ldots, r-1\} : v^1_{i(v^1)} = v^2_{i(v^2)},$$

and

$$v^1 R^{\mathrm{L}} v^2 \iff v^1 P^{\mathrm{L}} v^2 \ \ or\ \ v^1 I^{\mathrm{L}} v^2 \ \text{ for all } v^1, v^2 \in E^n.$$

We are now in the position of defining an axiom for extended Hammond equity. Take any two groups of generations $G_1$, $G_2$, which consist of finite number $n$ of successive generations. For any consumption path $X^1$, $X^2$, we have two $n$-dimensional vectors $(W_t(X^1))_{t \in G_1}$ and $(W_t(X^2))_{t \in G_2}$. With this notation we define an extension of Hammond equity relations in the case of sympathy to $n-1$ future generations. The strict extended Hammond relation, $\succ^{\mathrm{EH}}$, is defined by: for any $X^1, X^2 \in \mathcal{X}$, $X^1 \succ^{\mathrm{EH}} X^2$ if and only if there exist $t^r$ and $t^p$ such that

(i) $W_t(X^2) \geq W_t(X^1) \quad (t = t^r - (n-1), \ldots, t^r)$,
(ii) $W_t(X^1) \geq W_t(X^2) \quad (t = t^p - (n-1), \ldots, t^p)$,
(iii) $W_t(X^1) = W_t(X^2) \quad$ (otherwise), and
(iv) $(W_t(X^1))_{t = t^r - (n-1), \ldots, t^r} P^{\mathrm{L}} (W_t(X^2))_{t = t^p - (n-1), \ldots, t^p}$,

where $W_t(X^i) = W_0(X^i)$ for $t < 0$, $i = 1, 2$. The extended Hammond indifference relation, $\sim^{\mathrm{EH}}$, is defined by: for any $X^1, X^2 \in \mathcal{X}$, $X^1 \sim^{\mathrm{EH}} X^2$ if and only if $(W_t(X^1))_{t \in G_1} I^{\mathrm{L}} (W_t(X^2))_{t \in G_2}$ holds. The extended Hammond equity relation, $\succsim^{\mathrm{EH}}$, is defined by: for any $X^1, X^2 \in \mathcal{X}$

$$X^1 \succsim^{\mathrm{EH}} X^2 \iff X^1 \succ^{\mathrm{EH}} X^2 \ or\ X^1 \sim^{\mathrm{EH}} X^2.$$

This relation represents a concept of equity between two groups of successive generations, which is applied to the case where a change in the consumption of a generation causes a change in the utilities of the whole group. The reason why we need this type of requirement is that a change in the consumption of some generation under the feasibility of the economy brings increase in utility to a group of successive generations and decrease to another group that does not satisfy the conditions presupposed in the definition of the Hammond equity relation.

Now we provide five axioms. The first axiom simply requires non-emptiness of the choice set from the set of all feasible paths.

**Definition 1.** *A choice function $C$ satisfies nonempty choice from $\mathcal{X}$ (NE) iff $C(\mathcal{X}) \neq \varnothing$.*

The second axiom is a requirement that if a consumption path in a feasible set $S$ is extended Hammond superior to a path in the choice set $C(S)$, then it is also included in $C(S)$.

**Definition 2.** *A choice function $C$ satisfies inclusion of extended Hammond superior paths (IEH) iff $\forall X^1, X^2 \in \mathcal{X}\ \forall S \subseteq \mathcal{X}$:*

$$[X^1 \succsim^{\mathrm{EH}} X^2 \;\&\; X^1 \in S \;\&\; X^2 \in C(S)] \Rightarrow X^1 \in C(S).$$

The third axiom is a requirement that a path which is Pareto inferior to another path in a feasible set $S$ is excluded from the choice set $C(S)$.

**Definition 3.** *A choice function $C$ satisfies exclusion of Pareto inferior paths (EP) iff $\forall X^1, X^2 \in \mathcal{X}\ \forall S \subseteq \mathcal{X}$:*

$$[X^1 \succ^{\mathrm{P}} X^2 \;\&\; X^1 \in S] \Rightarrow X^2 \notin C(S).$$

The next two axioms are conditions of consistency for the choice sets. The fourth axiom is a requirement that any path in the choice set for a larger feasible set is also included in the choice set for a smaller feasible set if the path belongs to that set.

**Definition 4.** *A choice function $C$ satisfies condition $\alpha$ iff $\forall S^1, S^2 \subseteq \mathcal{X}, S^1 \subseteq S^2$ : $\forall X^1 \in S^1$:*

$$X^1 \in C(S^2) \Rightarrow X^1 \in C(S^1).$$

The fifth axiom is a requirement that if a path in the choice set for a smaller feasible set is included in the choice set for a larger feasible set, then any other path in the choice set for the smaller feasible set is also included in the choice set for the larger feasible set.

**Definition 5.** *A choice function $C$ satisfies condition $\beta$ iff $\forall S^1, S^2 \subseteq \mathcal{X}, S^1 \subseteq S^2$ : $\forall X^1 \in S^1, X^2 \in S^2$ :*

$$[X^1 \in C(S^1) \cap C(S^2) \;\&\; X^2 \in C(S^1)] \Rightarrow X^2 \in C(S^2).$$

## 4 Main Theorem

We are in the position to provide our main theorem about the characterization of the Rawlsian choice function.

**Lemma 1.** *Suppose that the utility of any generation t is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

$\gamma^i \rho_i$ *increases with i for* $i \leq n-1$, *and* $\rho_i$ *is non increasing in i. If a choice function C satisfies NE, EP, IEH, conditions* $\alpha$ *and* $\beta$, *then,*

$$C(\mathcal{X}) = C^{\mathrm{R}}(\mathcal{X}).$$

To prove this lemma we need Lemmas 2.–4..

**Lemma 2.** *Suppose that the utility of any generation t is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

$\gamma^i \rho_i$ *increases with i for* $i \leq n-1$, *and* $\rho_i$ *is nonincreasing in i. If a choice function C satisfies NE, EP, IEH,* $\alpha$, *and* $\beta$, *then* $W_0(X) = \min_t W_t(X)$ *for all* $X \in C(\mathcal{X})$.

*Proof.* By NE, $C(\mathcal{X}) \neq \varnothing$. Suppose that $X^* \in C(\mathcal{X})$ and that $W_0(X^*) \neq \min_t W_t(X^*)$. There are two cases to be considered: (i) there exists $\min_t W_t(X^*)$ and $\min_t W_t(X^*) < W_0(X^*)$; or (ii) there does not exist $\min_t W_t(X^*)$. In both cases, we can find some generation enjoying less welfare than generation 0. Let $t^m$ be such generation. For any $q \in (0,1)$, we can construct a feasible consumption path $X^1$ defined as follows:

$$\begin{cases} x_0^1 = x_0^* - \varepsilon \\ x_{t^m+n-1}^1 = x_{t^m+n-1}^* + q\varepsilon\gamma^{t^m+n-1} \\ x_t^1 = x_t^* \qquad (t \neq 0,\ t^m+n-1). \end{cases}$$

For sufficiently small $\varepsilon > 0$, we have the following:

$$\begin{cases} W_0(X^1) < W_0(X^*), \\ W_t(X^1) > W_t(X^*), \qquad (t^m \leq t \leq t^m+n-1), \\ W_0(X^1) > W_{t^m}(X^1), \\ W_t(X^1) = W_t(X^*), \qquad (0 < t < t^m \text{ or } t^m+n-1 < t). \end{cases}$$

Then, by the definition of the extended Hammond relation,

$$X^1 \succsim^{\mathrm{EH}} X^*. \tag{11}$$

In making the path $X^1$ from $X^*$ there remains an amount of the consumption good $(1-q)\varepsilon$. If we increase the consumption by $\delta > 0$ for each generation by dividing the amount $(1-q)\varepsilon$, the equality

$$\delta(1+\gamma^{-1}+\gamma^{-2}+\cdots)=\delta\sum_{t=0}^{\infty}\gamma^{-t}=(1-q)\varepsilon$$

must hold. Hence, we can construct a feasible consumption path $X^2$ defined as follows:

$$x_t^2 = x_t^1 + (1-q)\frac{\varepsilon}{\sum_{t=0}^{\infty}\gamma^{-t}}$$

for all $t \geq 0$. Then

$$X^2 \succ^{\mathrm{P}} X^1. \tag{12}$$

By (12) and condition EP,

$$X^1 \notin C(\{X^*, X^1, X^2\}). \tag{13}$$

Now we show $X^* \notin C(\{X^*, X^1, X^2\})$. Suppose on the contrary that $X^* \in C(\{X^*, X^1, X^2\})$. Then we obtain $X^1 \in C(\{X^*, X^1, X^2\})$ by (11) and condition IEH. Applying condition $\alpha$ to this relation,

$$X^1 \in C(\{X^*, X^1\}). \tag{14}$$

Since $X^* \in C(\mathcal{X})$, we have

$$X^* \in C(\{X^*, X^1\}), \tag{15}$$

with the help of $\{X^*, X^1\} \subseteq \mathcal{X}$ and condition $\alpha$. Equations (14) and (15) together imply

$$\{X^*, X^1\} = C(\{X^*, X^1\}). \tag{16}$$

Under the assumption $X^* \in C(\{X^*, X^1, X^2\})$, (16) with condition $\beta$ implies

$$X^1 \in C(\{X^*, X^1, X^2\}),$$

which contradicts (13). Therefore

$$X^* \notin C(\{X^*, X^1, X^2\}) \tag{17}$$

must hold.

On the other hand, $X^* \in C(\mathcal{X})$ implies that $X^* \in C(\{X^*, X^1, X^2\})$ with the help of condition $\alpha$. This contradicts (17). Hence $W_0(X) = \min_t W_t(X)$ for any $X \in C(\mathcal{X})$. □

**Lemma 3.** *Suppose that the utility of any generation $t$ is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

*$\gamma^i \rho_i$ increases with $i$ for $i \leq n-1$, and $\rho_i$ is nonincreasing in $i$. If a choice function $C$ satisfies NE, EP, IEH, conditions $\alpha$, and $\beta$, then generation 0 in $X^* \in C(\mathcal{X})$ has the largest welfare among all feasible consumption paths where generation 0 has the least welfare among all the generations. That is,*

$$W_0(X^*) = \max_{X \in \mathcal{D}_0} W_0(X)$$

*for any $X^* \in C(\mathcal{X})$, where $\mathcal{D}_0 = \{X \in \mathcal{X} \mid W_0(X) = \min_t W_t(X)\}$.*

*Proof.* By NE, $C(\mathcal{X}) \neq \varnothing$. Suppose, on the contrary, that $X^* \in C(\mathcal{X})$ and that there is $X^{**} \in \mathcal{D}_0$ such that $W_0(X^*) < W_0(X^{**})$. Let $X^{***} = \arg\max_{X \in \mathcal{D}_0} W_0(X)$. Then $W_0(X^*) < W_0(X^{**}) \leq W_0(X^{***})$. Hence, without loss of generality, we assume $X^{***} = X^{**}$. By the feasibility condition and assumptions of $U$, $(x_0^{**}, \ldots, x_{n-1}^{**})$ is the unique solution of the problem:

$$\max_{x_0,\ldots,x_{n-1}} \sum_{t=0}^{n-1} \rho_t U(x_t)$$
$$\text{s.t.} \quad \sum_{t=0}^{n-1} \gamma^{-t} x_t = \hat{x} \sum_{t=0}^{n-1} \gamma^{-t}.$$

Therefore, $(x_0^{**}, \ldots, x_{n-1}^{**})$ is characterized by

$$\mathrm{MRS}(x_i^{**}, x_j^{**}) = \frac{\rho_i U'(x_i^{**})}{\rho_j U'(x_j^{**})} = \gamma^{j-i},$$

for any $0 \leq i < j \leq n-1$.

Two cases should be distinguished.

*Case* 1: $x_i^* < x_i^{**}$.

The assumptions of $U$ assure that

$$\mathrm{MRS}(x_i^*, x_j^*) = \frac{\rho_i U'(x_i^*)}{\rho_j U'(x_j^*)} > \gamma^{j-i}.$$

Consider $X^1$ and $X^2$ defined as follows:

$$\begin{cases} x_i^1 = x_i^* + q\varepsilon, \\ x_j^1 = x_j^* - \gamma^{j-i}\varepsilon, \\ x_t^1 = x_t^* \qquad (t \neq i, j), \end{cases}$$

and for all $t \geq 0$

$$x_t^2 = x_t^1 + (1-q)\frac{\varepsilon}{\sum_{t=0}^{\infty} \gamma^{-t}},$$

where $0 < q < 1$, $\varepsilon > 0$. Then

$$X^2 \succ^{\mathrm{P}} X^1. \tag{18}$$

By (18) and condition EP,

$$X^1 \notin C(\{X^*, X^1, X^2\}). \tag{19}$$

For sufficiently small $\varepsilon$,

$$\begin{cases} W_t(X^1) > W_t(X^*), & (t = i-(n-1), \dots, i), \\ W_t(X^1) < W_t(X^*), & (t = j-(n-1), \dots, j), \\ W_t(X^1) = W_t(X^*), & (otherwise) \end{cases}$$

hold so that $(W_t(X^1))_{t=j-(n-1),\dots,j} P^{\mathrm{L}} (W_t(X^1))_{t=i-(n-1),\dots,i}$ by the continuity of $W$. Hence we have $X^1 \succsim^{\mathrm{EH}} X^*$ by the extended Hammond equity.

Now we show $X^* \notin C(\{X^*, X^1, X^2\})$. Suppose on the contrary that $X^* \in C(\{X^*, X^1, X^2\})$. Then we obtain $X^1 \in C(\{X^*, X^1, X^2\})$ by (11) and condition IEH. Applying condition $\alpha$ to this relation,

$$X^1 \in C(\{X^*, X^1\}). \tag{20}$$

Since $X^* \in C(\mathcal{X})$, we have

$$X^* \in C(\{X^*, X^1\}) \tag{21}$$

with the help of $\{X^*, X^1\} \subseteq \mathcal{X}$ and condition $\alpha$. Equations (20) and (21) together imply

$$\{X^*, X^1\} = C(\{X^*, X^1\}). \tag{22}$$

Under the assumption $X^* \in C(\{X^*, X^1, X^2\})$, (16) with condition $\beta$ implies

$$X^1 \in C(\{X^*, X^1, X^2\}),$$

which contradicts (19). Therefore

$$X^* \notin C(\{X^*, X^1, X^2\}) \tag{23}$$

must hold.

On the other hand, $X^* \in C(\mathcal{X})$ implies that $X^* \in C(\{X^*, X^1, X^2\})$ with the help of condition $\alpha$. This contradicts (23). Hence this case cannot be true.

*Case* 2: $x_i^* > x_i^{**}$.

The assumptions of $U$ assure that

$$\mathrm{MRS}(x_i^*, x_j^*) = \frac{\rho_i U'(x_i^*)}{\rho_j U'(x_j^*)} < \gamma^{j-i}.$$

Define $X^1$ and $X^2$ as follows:

$$\begin{cases} x_i^1 = x_i^* - \varepsilon, \\ x_j^1 = x_j^* + q\gamma^{j-i}\varepsilon, \\ x_t^1 = x_t^* \qquad (t \neq i, j), \end{cases}$$

and for all $t \geq 0$

$$x_t^2 = x_t^1 + (1-q)\frac{\varepsilon}{\sum_{t=0}^{\infty}\gamma^{-t}},$$

where $0 < q < 1$, $\varepsilon > 0$. Then, by the same reasoning as *Case* 1, we come to the contradiction that $X^* \in C(\{X^*, X^1, X^2\})$ and $X^* \notin C(\{X^*, X^1, X^2\})$. Hence this case cannot be true either.

By *Cases* 1 and 2, we have a contradiction for any $0 \leq i < j \leq n-1$ if $\mathrm{MRS}(x_i^*, x_j^*) \neq \gamma^{j-i}$. Therefore, generation 0 in $X^* \in C(\mathcal{X})$ has the largest welfare among all feasible consumption paths where generation 0 has the least welfare among all the generations. □

The next lemma shows a sufficient condition for a consumption path to be infeasible. The idea of the proof is due to Lemma 1 in Arrow (1973).

**Lemma 4.** *Suppose that the utility of any generation t is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

$\gamma^i \rho_i$ *increases with i for* $i \leq n-1$*, and* $\rho_i$ *is nonincreasing in i. If a consumption path X satisfies that* $\sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln} \geq \sum_{s=0}^{n-1}\gamma^{-s}x_s^{\mathrm{R}}$ *for all* $l \in \mathcal{Z}_+$ *and* $\sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln} > \sum_{s=0}^{n-1}\gamma^{-s}x_s^{\mathrm{R}}$ *for some* $l' \in \mathcal{Z}_+$*, then X is infeasible.*

*Proof.* By the feasibility condition, the relation between $k_{ln}$ and $k_{(l+1)n}$ can be written as

$$k_{(l+1)n} = \gamma^n \left(k_{ln} - \sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln}\right). \tag{24}$$

On the other hand, a Rawlsian maximal consumption path satisfies the condition that $k_n = k_0$. Hence

$$k_0 = \gamma^n \left(k_0 - \sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln}^{\mathrm{R}}\right). \tag{25}$$

By (24) and (25),

$$k_{(l+1)n} - k_0 = \gamma^n \left[(k_{ln} - k_0) - \left(\sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln} - \sum_{s=0}^{n-1}\gamma^{-s}x_{s+ln}^{\mathrm{R}}\right)\right].$$

For simplicity of description, define $h_l$ and $a_l$ as follows:

$$h_l = \sum_{s=0}^{n-1} \gamma^{-s} x_{s+ln} - \sum_{s=0}^{n-1} \gamma^{-s} x^{\mathrm{R}}_{s+ln},$$
$$a_l = \gamma^{-ln}(k_{ln} - k_0).$$

Then, $\gamma^{(l+1)n} a_{l+1} = \gamma^n(\gamma^{ln} a_l - h_l)$ iff $a_{l+1} = a_l - \gamma^{-ln} h_l$. Since $a_0 = 0$ by definition, $a_l = \sum_{u=0}^{l-1} \gamma^{-nu} h_u$ is true. Now, by the assumptions of lemma and definitions of $h_l$ and $a_l$, the following inequality holds:

$$\limsup_{l\to\infty} \sum_{u=0}^{l-1} \gamma^{-nu} h_u > 0.$$

Then, for some $\varepsilon > 0$, there is sufficiently large $\bar{l}$ such that $a_l < -\varepsilon \quad \text{for} \quad l \geq \bar{l}$. Since $\gamma^{-ln} k_0 < \varepsilon$ for any sufficiently large $l$, there exists some $l' \in \mathcal{Z}_+$ such that $a_{l'} < -\gamma^{l'n} k_0$ . Hence $k_{l'n} < 0$, and $X$ is infeasible. □

Now we provide the proof of lemma 1. with these lemmas.

*Proof of lemma 1.:*

By NE, $C(\mathcal{X}) \neq \varnothing$. Let $X^*$ be any consumption path in $C(\mathcal{X})$. By Lemma 3. and Theorem 1, $W_0(X^*) \geq W_0(X^{\mathrm{R}})$. Suppose that $W_0(X^*) > W_0(X^{\mathrm{R}})$. Then $W_{ln}(X^*) > W_0(X^{\mathrm{R}})$ for all $l \in \mathcal{Z}_+$ by Lemma 2.. Since $(x^{\mathrm{R}}_0, \ldots, x^{\mathrm{R}}_{n-1})$ is the unique solution of the maximization problem (9) and (10) and the consumption path $X^{\mathrm{R}}$ is an infinite repetition of $(x^{\mathrm{R}}_0, \ldots, x^{\mathrm{R}}_{n-1})$ by Theorem 1,

$$\sum_{s=0}^{n-1} \gamma^{-s} x^*_{s+ln} > \sum_{s=0}^{n-1} \gamma^{-s} x^{\mathrm{R}}_s$$

holds for all $l \in \mathcal{Z}_+$. Hence, $X^*$ is infeasible by Lemma 4., which is a contradiction. Therefore, we have $W_0(X^*) = W_0(X^{\mathrm{R}})$.

Since, for all $l \in \mathcal{Z}_+$,

$$W_{ln}(X^*) \geq \min W_t(X^*) = W_0(X^*) = W_0(X^{\mathrm{R}})$$

holds,

$$\sum_{s=0}^{n-1} \gamma^{-s} x^*_{s+ln} \geq \sum_{s=0}^{n-1} \gamma^{-s} x^{\mathrm{R}}_s, \tag{26}$$

for all $l \in \mathcal{Z}_+$. By Theorem, 1 $(x^{\mathrm{R}}_0, \ldots, x^{\mathrm{R}}_{n-1})$ is the unique maximum of the problem (9) and (10). Hence equality (26) holds only when $(x^*_{ln}, \ldots, x^*_{(l+1)n-1}) = (x^{\mathrm{R}}_0, \ldots, x^{\mathrm{R}}_{n-1})$. Therefore, $X^* = {}_{\mathrm{rep}}(x^{\mathrm{R}}_0, \ldots, x^{\mathrm{R}}_{n-1}) = X^{\mathrm{R}}$. □

We provide the converse of Lemma 1..

**Lemma 5.** *Suppose that the utility of any generation t is given by*

$$W_t(X) = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

*$\gamma^i \rho_i$ increases with i for $i \leq n-1$, and $\rho_i$ is nonincreasing in i. The Rawlsian choice function $C^{\mathrm{R}}$ satisfies NE, IEH, EP, $\alpha$, and $\beta$.*

*Proof.* NE: As noted in Sect. 2, Arrow (1973) showed $C^{\mathrm{R}}(\mathcal{X}) \neq \varnothing$.

IEH: Suppose that (i) $S \subseteq \mathcal{X}$, (ii) $X^1 \in S$, (iii) $X^2 \in C^{\mathrm{R}}(S)$, and (iv) $X^1 \succsim^{\mathrm{EH}} X^2$. By (iv), $\min_t W_t(X^1) \geq \min_t W_t(X^2)$. Then $X^1 \in C^{\mathrm{R}}(S)$, so that IEH holds.

EP: Suppose that (i) $S \subseteq \mathcal{X}$, (ii) $X^1 \in S$, and (iii) $X^1 \succ^{\mathrm{P}} X^2$. Then, by (iii), $\min_t W_t(X^1) > \min_t W_t(X^2)$. So $X^2 \notin C^{\mathrm{R}}(S)$ and EP holds.

Condition $\alpha$: Suppose that (i) $S^1 \subseteq S^2 \subseteq \mathcal{X}$, (ii) $X^1 \in C^{\mathrm{R}}(S^2)$, and (iii) $X^1 \in S^1$. By (i) and (ii), $\min_t W_t(X^1) \geq \min_t W_t(X)$ for all $X \in S^1$. Hence, we obtain $X^1 \in C^{\mathrm{R}}(S^1)$, and condition $\alpha$ holds.

Condition $\beta$: Suppose that (i) $S^1 \subseteq S^2 \subseteq \mathcal{X}$, (ii) $X^1, X^2 \in C^{\mathrm{R}}(S^1)$. By (ii), $\min_t W_t(X^1) = \min_t W_t(X^2)$. Therefore, we have $X^1 \in C^{\mathrm{R}}(S^2) \iff X^2 \in C^{\mathrm{R}}(S^2)$. $\square$

With Lemmas 1. and 5., we finally come to the following characterization theorem.

**Theorem 1.** *Suppose that the utility of any generation t is given by*

$$W_t = \sum_{i=0}^{n-1} \rho_i U(x_{t+i}),$$

*that $\gamma^i \rho_i$ increases with i for $i \leq n-1$, and that $\rho_i$ is nonincreasing in i. Then, (i) the Rawlsian choice function $C^{\mathrm{R}}$ satisfies NE, EP, IEH, $\alpha$, and $\beta$; and (ii) if a choice function C satisfies NE, EP, IEH, $\alpha$, and $\beta$, then $C(\mathcal{X}) = C^{\mathrm{R}}(\mathcal{X})$.*

## 5 Related Examples

As for independence of the axioms we must examine in five cases whether there exists a choice function that satisfies all but one axiom. In the following, we show only three examples. Examination of other cases is an important task in our future research. The next three examples are related to NE, IEH, and EP, respectively. The first example is trivial.

*Example 1.* The empty choice function, $C^0(S) = \varnothing$, satisfies EP, IEH, $\alpha$, and $\beta$, but it violates NE.

We show that a myopic choice function satisfies the other axioms than IEH.

*Example 2.* A myopic choice function, $C^{\mathrm{M}}(S) = \arg\max_{X \in S} W_0(X)$, satisfies NE, EP, $\alpha$, and $\beta$, but it violates IEH.

First, we show that $C^{\mathrm{M}}$ satisfies *EP*. Suppose that for any $X^1$, $X^2$, and any $S$, $[X^1 \succ^{\mathrm{P}} X^2$ and $X^1 \in S]$ hold. Then $W_0(X^1) > W_0(X^2)$. By the definition of $C^{\mathrm{M}}$, if $X^1$ is feasible, $C^{\mathrm{M}}(S)$ does not contain $X^2$. Therefore, $C^{\mathrm{M}}$ satisfies EP.

Second, suppose that $C^{\mathrm{M}}$ satisfies the hypothesis of condition $\alpha$. Then, by the definition of $C^{\mathrm{M}}$, generation 0 has the maximal welfare on $X^1$ in $S^2$. Hence, clearly it does so in $S^1$ $(\subseteq S^2)$.

Third, suppose that $C^{\mathrm{M}}$ satisfies the hypothesis of $\beta$. Then by the definition of $C^{\mathrm{M}}$, generation 0 has the same welfare on both $X^1$ and $X^2$ and therefore the conclusion of $\beta$ holds.

Now, consider two consumption paths, $X^1$ and $X^2$, such that $W(X^1) = (2,0,0,0,0,...)$ and $W(X^2) = (1,1,0,0,0,...)$. IEH requires $X^2 \in C(\{X^1,X^2\})$, but $\{X^1\} = C(\{X^1,X^2\})$ by definition. Therefore, IEH does not hold.

A trivial choice function satisfies the other axioms than EP.

*Example 3.* A trivial choice function, $C^{\mathrm{T}}(S) = S$, satisfies NE, IEH, $\alpha$, and $\beta$, but it violates EP.

$C^{\mathrm{T}}$ always contains all feasible consumption paths. So the conclusions of IEH, $\alpha$, and $\beta$ hold for any feasible set and any utility function, respectively. Therefore, $C^{\mathrm{T}}$ satisfies IEH, $\alpha$, and $\beta$.

On the other hand, $C^{\mathrm{T}}$ violates EP.

## 6 Concluding Remarks

This chapter has provided an axiomatic characterization of the Rawlsian choice function in the Arrow–Dasgupta economy. Properties of the maximin consumption path have been examined by Arrow (1973) and Dasgupta (1974a, b), and it was shown that the maximin principle generates a saw-tooth shaped path. We make use of the axioms of non-emptiness, the Pareto principle, extended Hammond equity, conditions $\alpha$ and $\beta$ to characterize the Rawlsian choice function. Pareto principle and Hammond equity are familiar to the characterization of the maximin principle in an intragenerational economy. Extended Hammond equity is an extension thereof in a dynamic economy with sympathetic preferences to future generations. Our versions of these conditions are exclusion of Pareto inferior paths and inclusion of extended Hammond superior paths. Conditions $\alpha$ and $\beta$ are also familiar to the characterization of consistent choice functions.

Our characterization of the Rawlsian choice function is partial, in that Theorem 2 does not provide a complete axiomatization. We have shown that the Rawlsian choice function satisfies the above five axioms, and that the choice function satisfying these axioms generates the same choice set as that by the Rawlsian choice function when the opportunity set is the whole set of feasible paths. A full characterization of the Rawlsian choice function is a good research agenda in the field of intergenerational equity, which is left for future study.

The other remaining problems to be solved along this line of research are as follows. First, we must classify the family of choice functions that satisfies NE, IEH, $\alpha$, $\beta$, and a weaker axiom of Pareto principle than EP. Since this family contains the Rawlsian choice function, we should explore whether there exists any other eligible one than the trivial choice function. Second, we must verify whether any other choice function than the myopic one that satisfies NE, EP, $\alpha$, and $\beta$. Third, we should scrutinize the possibility whether other consistency axioms characterize the Rawlsian choice function.

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