# To Envy or To Be Envied? Refinements of the Envy Test for the Compensation Problem

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## 1 Introduction

The envy test concept is an all-or-nothing notion, and this is problematic when there is no achievable envy-free option. The idea of ranking the "unfair" social states on the basis of how much envy they contain goes back at least to Feldman and Kirman (1974) and Varian (1976), but it is in Suzumura (1981a, b, 1983) that one finds a first systematic study of this issue. More recent contributions to this line of research include Chauduri (1986), Diamantaras and Thomson (1990), Tadenuma (2002), and Tadenuma and Thomson (1995).

One of the contexts where typically envy-free allocations are hard to achieve is when, as noticed in Pazner and Schmeidler (1974) for the production case and Fleurbaey (1994) for the distribution case, individuals have nontransferable personal characteristics to which the no-envy test nonetheless applies. This is sometimes called the "compensation problem," where one tries to compensate inequalities in personal characteristics by counteracting inequalities in transferable resources. This problem is rather different from standard problems of resource allocation because of the presence of nontransferable characteristics, which act as a constraint on redistribution. One typical example is when individuals have unequal levels of skills which give them unequal earning possibilities. Certain other characteristics, such as physical handicaps, may directly affect personal satisfaction and generate inequalities which call for redressing transfers. When inequalities in personal characteristics are huge, or when individuals disagree about the value of their respective characteristics, it may be very hard, or even impossible, to find transfers that eliminate envy between individuals. In Fleurbaey (1994) and Iturbe-Ormaetxe and Nieto (1996) one finds several suggestions about how to weaken the no-envy requirement in order to obtain nonempty solutions to the compensation problem. But most of these

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solutions fail to be nonempty on the whole domain, and a systematic use of rankings seems not to have been attempted yet in this branch of the literature. This chapter makes an attempt at filling this gap and examines several rankings, which may be of some interest.

Section 2 makes a brief survey of the compensation literature, proposing a few basic criteria for the evaluation of solutions. Section 3 examines rankings based on the number of envy relations, Section 4 deals with rankings that make use of additional information about the population's preferences, and Section 5 is devoted to rankings that involve the degree of envy as measured by the quantity of transfers that would be needed to suppress envy relations. It argues that such rankings are preferable to the others, and also establishes a correspondence between one such ranking and another based on the idea of rationalizing egalitarian competitive equilibria. Section 6 concludes the chapter.

## 2 A Brief Survey

The compensation problem can be described by the following simple model. The population is $N = \{1,...,n\}$ and every individual $i \in N$ is endowed with two kinds of characteristics: $y_i$, for which she is not responsible (circumstances), and $z_i$, for which she is. A profile of characteristics is $(y_N, z_N) = ((y_1,\ldots,y_n),(z_1,\ldots,z_n))$. The sets from which $y_i$ and $z_i$ are drawn, denoted $Y$ and $Z$, respectively, are assumed to have at least two elements.

Individual $i$'s well-being is denoted $u_i$ and is determined by a function $u$, which is the same for all individuals:

$$u_i = u(x_i, y_i, z_i),$$

where $x_i \in \mathbb{R}$ is the quantity of money transfer to which the individual is submitted. When $x_i < 0$, the transfer is a tax. The real-valued function $u$, defined either on $\mathbb{R} \times Y \times Z$ or $\mathbb{R}_+ \times Y \times Z$ depending on the cases (on which more below), is assumed to be continuous and increasing in $x_i$.

An allocation is denoted $x_N = (x_1,\ldots,x_n)$. The set of feasible allocations is denoted $X$. The precise definition of $X$ differs in different cases, but typically involves a condition $\sum_{i\in N} x_i \leq \Omega$ for some aggregate endowment $\Omega \in \mathbb{R}$. Given the fact that $u$ is increasing in $x_i$, this means that allocations such that $\sum_{i\in N} x_i = \Omega$ are all Pareto efficient. This considerably simplifies the analysis.

This is the simplest model in which the compensation problem can be studied, but other models have been studied. In particular, Fleurbaey and Maniquet (1996) and Pazner and Schmeidler (1974) have examined a production model in which agents differ in their productivity. For a general survey on the compensation problem, see Fleurbaey and Maniquet (2002).

Two general concepts of solutions will be useful here. Let $\mathcal{D}$ be the domain of economies $e = (y_N, z_N)$ under consideration. An *allocation rule* is a correspondence

$S : \mathcal{D} \rightarrow\rightarrow X$, such that for all $e \in \mathcal{D}$, $S(e) \subset X$ is the subset of allocations selected by $S$. A *social ordering function* is a mapping $R : \mathcal{D} \rightarrow \mathcal{R}_X$, where $\mathcal{R}_X$ is the set of complete orderings over $X$. The expression $x_N \, R(e) \, x'_N$ will mean that $x_N$ is weakly preferred to $x'_N$, and $P(e)$ and $I(e)$ will denote the corresponding strict preference and indifference relations, respectively. An *allocation rule derived from a social ordering function* is defined by selecting, for each economy, the maximal elements in $X$ for the social ordering defined by the social ordering function for this economy.

Two special cases will be of particular interest. The "distribution" case (Fleurbaey, 1994) is when $x_i$ has to be nonnegative, and there is a fixed amount $\Omega > 0$ to be distributed, that is, when (assuming no waste)

$$X = \left\{ x_N \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i = \Omega \right\}.$$

An interesting domain for this case is the domain $\mathcal{D}_1$ of economies satisfying, for all $i, j \in N$,

$$u\left(\frac{\Omega}{|N| - 1}, y_i, z_i\right) \geq u(0, y_j, z_i).$$

This domain is such that no individual considers his own $y_i$ to be a huge handicap compared to other values of $y_j$ in the population.[1]

The "TU" (transferable utility) case (Bossert, 1995) is when the well-being function is quasi-linear in $x$,

$$u_i = x_i + v(y_i, z_i),$$

$x_i$ is not bounded below,[2] and there is no external amount of money to be distributed, that is, when

$$X = \left\{ x_N \in \mathbb{R}^N \mid \sum_{i \in N} x_i = 0 \right\}.$$

The distribution case is relevant to situations in which the government has a fixed budget that can be used in order to provide targeted help to particular categories of people, such as disabled individuals, victims of a natural disaster, families with different needs. The TU case is not limited, but is especially relevant, to situations in which individual well-being is itself monetary. The most realistic applications of the TU case are offered by the federalism problem of organizing budget transfers between administrative units (local governments, sectorial administrations, social security agencies, etc.), which are partly responsible for their budget situation.

For the most part we focus here on the TU case, and only briefly mention the differences in results for the distribution case, when relevant.

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[1] Conditions of this kind are helpful in order to obtain the existence of envy-free allocations in the model where $y$ is an indivisible good that is transferable across agents. See, for example, Maskin (1987).

[2] More realistically, one could impose that $x_i + v(y_i, z_i)$ is bounded below (for instance by zero). We will not study this variant.

The compensation problem consists in neutralizing the impact of circumstances $y$ on well-being while not interfering with inequalities due to differences in responsibility characteristics $z$. The no-envy condition, due to Foley (1967) and Kolm (1972), is well suited to this purpose if it is applied as follows.

No-Envy: $\forall e \in \mathcal{D}, \forall x_N \in S(e), \forall i, j \in N, u(x_i, y_i, z_i) \geq u(x_j, y_j, z_i)$ .

Its main drawback, however, is that it is too demanding and is satisfied only on a very small domain. This is connected to the conflictual duality between two principles that it jointly encapsulates, namely, the compensation principle ("neutralize $y$") and the natural reward principle ("not interfere with $z$") (Fleurbaey, 1995). Here we focus on a small list of axioms embodying these principles. For the compensation principle:

Equal Well-Being for Equal Responsibility: $\forall e \in \mathcal{D}$, $\forall x_N \in S(e)$, $\forall i, j \in N$ such that $z_i = z_j$,

$$u(x_i, y_i, z_i) = u(x_j, y_j, z_j).$$

Equal Well-Being for Uniform Responsibility: $\forall e \in \mathcal{D}$, $\forall x_N \in S(e)$, if $\forall i, j \in N$, $z_i = z_j$, then

$$\forall i, j \in N,\ u(x_i, y_i, z_i) = u(x_j, y_j, z_j).$$

In the distribution case one can reformulate these axioms in terms of application of the leximin criterion, inequality being allowed when the better-off agent has a zero $x$. In the above formulation, however, they are nonempty on $\mathcal{D}_1$ (see Lemma 1 in Fleurbaey (1994)).

The dual "natural reward" axioms are the following:

Equal Treatment for Equal Circumstances: $\forall e \in \mathcal{D}$, $\forall x_N \in S(e)$, $\forall i, j \in N$ such that $y_i = y_j$,

$$x_i = x_j.$$

Equal Treatment for Uniform Circumstances: $\forall e \in \mathcal{D}$, $\forall x_N \in S(e)$, if $\forall i, j \in N$, $y_i = y_j$, then

$$\forall i, j \in N,\ x_i = x_j.$$

These four axioms appear to be very basic conditions, and a reasonable requirement for an allocation rule is that it should satisfy at least the two weak axioms ("uniform" case) and one of the strong axioms ("equal" case), knowing that the two strong axioms are incompatible (Fleurbaey, 1994).

Three allocation rules, conceived in terms of weakening the no-envy requirement, have been proposed in Fleurbaey (1994). One, inspired from Daniel (1975) and Feldman and Kirman (1974), selects the allocations with the smallest number of envy occurrences among the "balanced" allocations. A balanced allocation is such that for all $i \in N$, the number of agents he envies equals the number who envy him:

$$\left|\left\{j \in N \mid u(x_i, y_i, z_i) < u(x_j, y_j, z_i)\right\}\right| = \left|\left\{j \in N \mid u(x_i, y_i, z_j) > u(x_j, y_j, z_j)\right\}\right|.$$

Let $B(e) \subseteq X$ denote the subset of balanced allocations, and $E(x_N, e)$ denote the number of envy occurrences in $x_N$ :

$$E(x_N, e) = \left|\left\{(i, j) \in N \mid u(x_i, y_i, z_i) < u(x_j, y_j, z_i)\right\}\right|.$$

**Balanced and Minimal Envy ($S_{\mathbf{BME}}$):** $\forall e \in \mathcal{D}, \forall x_N \in X$,

$$x_N \in S_{\mathrm{BME}}(e) \Leftrightarrow x_N \in B(e) \text{ and } \forall x'_N \in B(e),\ E(x'_N, e) \geq E(x_N, e).$$

A second allocation rule, inspired by Chauduri (1986) and Diamantaras and Thomson (1990), tries to minimize the intensity of envy, this intensity being measured for every agent by the resource needed to make this agent non-envious:

$$I_i(x_N, e) = \min\{\delta \in \mathbb{R} \mid \forall j \in N \setminus \{i\},\ u(x_i + \delta, y_i, z_i) \geq u(x_j, y_j, z_i)\}.$$

The allocation rule is then defined as follows.

**Minimax Envy Intensity ($S_{\mathbf{MEI}}$):** $\forall e \in \mathcal{D}, \forall x_N \in X$,

$$x_N \in S_{\mathrm{MEI}}(e) \Leftrightarrow \forall x'_N \in F(e),\ \max_{i \in N} I_i(x'_N, e) \geq \max_{i \in N} I_i(x_N, e).$$

The third allocation rule makes use of all agents' opinions about the relative well-being of two agents. It tries to minimize the size of subsets of agents thinking that one agent is worse-off than another agent. It takes inspiration from "undominated diversity" (Parijs, 1990, 1995), which seeks to avoid situations in which one agent is deemed unanimously worse-off than another one, and is related to the family of solutions put forth by Iturbe-Ormaetxe and Nieto (1996), which generalizes van Parijs' idea and seeks to avoid such a unanimity among a subgroup of a given size and containing the worse-off agent. Let

$$N_i^m = \{G \subset N \mid |G| = m,\ i \in G\}.$$

**Minimal Unanimous Domination ($S_{\mathbf{MUD}}$):** $\forall e \in \mathcal{D}, \forall x_N \in X$,

$$x_N \in S_{\mathrm{MUD}}(e) \Leftrightarrow \exists m \in \{1, \ldots, n\},$$
$$\begin{cases} \text{(i) } \forall i, j \in N, \forall G \in N_i^m, \exists k \in G,\ u(x_i, y_i, z_k) \geq u(x_j, y_j, z_k), \\ \text{(ii) } \forall p < m, \forall x'_N \in F(e), \exists i, j \in N, \exists G \in N_i^p, \\ \qquad \forall k \in G,\ u(x'_i, y_i, z_k) < u(x'_j, y_j, z_k). \end{cases}$$

## 3 Ranking Envy Graphs

A difficulty with $S_{\mathrm{BME}}$ is that there do not always exist balanced allocations, so that the domain on which it is defined is restricted. Moreover, the various sufficient conditions of existence specified by Daniel (1975) and Fleurbaey (1994) are not very easy to interpret and apply. Necessary conditions have not been studied to the best of my knowledge.

A more substantial criticism is that one does not see why a lexicographic priority should be given to balancedness of allocations over the number of envy occurrences. If the only balanced allocation has everybody envying everybody, it might be better to prefer an unbalanced allocation with a much smaller number of envy relations.

In fact, the general idea underlying this allocation rule is to examine the graph of envy relations, with a double concern for symmetry and for minimizing the number of relations. A general approach to the problem of ranking envy graphs would probably be more suitable than a narrow focus on balanced allocations.

In Fig. 1, five graphs are represented for a population of four individuals. In case (a), individual 1 is envied by all the others; in case (b), individual 1 envies all the others; in case (c), a cycle of envy occurs; in case (d), an envy relation has been reversed in comparison to case (c); in case (e), this envy relation has been deleted. Although the number of envy relations is smaller in (a) and (b) than in (c), and the same in (d) as in (c), one should probably prefer (c), out of a concern for symmetry. Although (e) is not symmetric, it may not be worse than (c), because it has a strictly smaller graph of envy relations.

This is a difficult case since balancedness in (c) makes it look more equal than (e), but one must be careful to avoid the intuitive illusion that arrows from a transitive "better-off than" relation. In (e), there is no-envy between agents 1 and 2, and this is the relevant test of equality. The arrows from 2 to 1 via 3 and 4 do not mean that 2 is worse-off than 1. It is true that between two agents, reciprocal envy appears better than a one-way envy relation. But this does not necessarily extend to cycles of envy among more agents. Therefore, it is not unreasonable to consider that removing a nonreciprocal envy relation between two agents is always a good thing (given that the only information is the envy graph, a limitation that will be discussed below).

Defining a more precise preference order on envy graphs is a complex matter. Suzumura (1983) proposes a very natural ranking, which applies the reverse leximin criterion to the vector of individual envy indices, where an individual envy index is

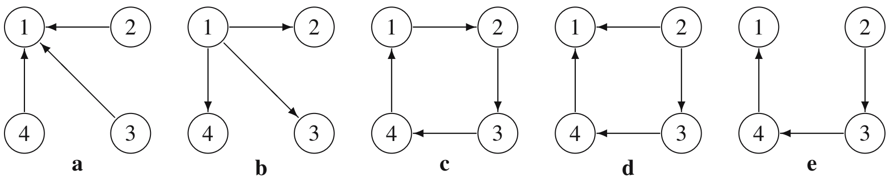


**Fig. 1** Envy graphs

simply the number of other agents this individual envies (i.e., the number of outgoing arrows in the graph). The reverse leximin criterion prefers a vector to another if its greatest component is smaller, or if the greatest components are equal in the two vectors but the second greatest component is smaller, and so on. It corresponds to the application of the standard leximin criterion to the opposite vectors.

The symbol $\geq_{\text{lex}}$ appearing in the definition below denotes the standard leximin criterion applied to vectors of real numbers. Namely, $x \geq_{\text{lex}} x'$ if the smallest component of $x$ is greater than the smallest component of $x'$, or they are equal and the second smallest component of $x$ is greater than the second smallest component of $x'$, and so on.

**Envious Count criterion ($\boldsymbol{R}_{\mathbf{EsC}}$):** Let

$$n_i(x_N) = \left|\left\{ j \in N \mid u(x_i, y_i, z_i) < u(x_j, y_j, z_i) \right\}\right|.$$

For all $x_N, x'_N \in X$, $x_N \, R_{\text{EsC}}(e) \, x'_N$ if and only if

$$-\left(n_i(x_N)\right)_{i\in N} \geq_{\text{lex}} -\left(n_i(x'_N)\right)_{i\in N}.$$

One can define a dual criterion to this one, that relies on the number of agents by whom a given agent is envied.

**Envied Count criterion ($\boldsymbol{R}_{\mathbf{EdC}}$):** Let

$$n'_i(x_N) = \left|\left\{ j \in N \mid u(x_i, y_i, z_j) > u(x_j, y_j, z_j) \right\}\right|.$$

For all $x_N, x'_N \in X$, $x_N \, R_{\text{EdC}}(e) \, x'_N$ if and only if

$$-\left(n'_i(x_N)\right)_{i\in N} \geq_{\text{lex}} -\left(n'_i(x'_N)\right)_{i\in N}.$$

The envious count and envied count criteria appear to be dual with respect to compensation and natural reward. One can indeed make the following observation. When two agents have the same responsibility characteristics, the worse-off will envy at least all the agents envied by the other plus the other agent himself, which means that his $n_i$ index is greater and the reverse leximin will give absolute priority to him. Similarly, when two agents have the same circumstances, those who envy one of them will systematically envy the agent with the greater $x$, and the one with the lower $x$ will envy him as well, so that his $n'_i$ will be greater and absolute priority will be put on him (i.e., absolute priority *against* him in this case, in order to reduce the number of agents envying him). This provides the intuition for the following result, which bears on the allocation rules derived from $R_{\text{EsC}}$ and $R_{\text{EdC}}$.

**Proposition 1.** *In the TU case, the allocation rule derived from the envious count criterion satisfies equal well-Being for equal responsibility and the allocation rule derived from the envied count criterion satisfies equal treatment for equal circumstances. This result does not extend to the distribution case.*

*Proof.* (1) Consider two agents $i, j \in N$ such that $z_i = z_j$ and an allocation $x_N$ such that $u_i > u_j$. Let $x'_N$ be such that $x'_i = x_i - (u_i - u_j)$ and $x'_k = x_k$ for all

$k \neq i$. Then $n_i(x'_N) = n_j(x'_N) = n_j(x_N) - 1$, while $n_k(x'_N) \leq n_k(x_N)$ for all $k \neq i, j$. Since $n_j(x_N) > n_i(x_N)$, the vector $(n_i(x'_N), n_j(x'_N))$ is better for the reverse leximin than $(n_i(x_N), n_j(x_N))$, and since $n_k(x'_N) \leq n_k(x_N)$ for all $k \neq i, j$, the whole vector $(n_k(x'_N))_{k\in N}$ is better than $(n_k(x_N))_{k\in N}$. The allocation $x'_N$ does not belong to $X$, but the allocation

$$x''_N = x'_N - \frac{1}{|N|}\sum_{i\in N} x'_i$$

does and is such that $n_k(x''_N) = n_k(x'_N)$ for all $k \in N$.

(2) The proof for the envied count criterion and equal treatment for uniform circumstances is similar. Let $y_i = y_j$ and $x_i > x_j$ in allocation $x_N$. Allocation $x'_N$ is defined by $x'_j = x_i$ and $x'_k = x_k$ for all $k \neq j$. The rest is very similar as above.

(3) Impossibility to extend to the distribution case is a corollary of the next proposition. □

However, these two criteria display no concern for balancedness. For instance, in the examples of Fig. 1, the vectors of indices $n_i(x_N)$ are, respectively, the following:
(a) (0,1,1,1)
(b) (3,0,0,0)
(c) (1,1,1,1)
(d) (0,2,1,1)
(e) (0,1,1,1)
As a consequence, the envious count criterion ranks the five graphs in the following decreasing order:

a e
c
d
b

Indifference between (a) and (e) is due to the fact that this ranking is not sensitive to the balancedness feature of graphs. It only counts the number of outgoing arrows and is indifferent to the direction of these arrows. A similar difficulty is obtained with the envied count criterion, which puts (a) at the bottom, but is indifferent between (b) and (e).[3]

A concern for balancedness can be incorporated by measuring individual situations with respect to envy in terms of an index that depends on $n_i$ and on $n'_i$. Let

$$d_i(x_N) = D(n_i(x_N), n'_i(x_N))$$

for a function $D$, the properties of which are discussed below. One can apply the reverse leximin criterion to such indices. The properties of the criterion will then depend on how $D$ ranks various $(n_i, n'_i)$ vectors. Figure 2 shows iso-curves for the $D$ function in the $(n_i, n'_i)$ space.

Panels (1) and (2) illustrate the two extreme cases of the envious count and envied count criteria:

[3] The vectors of indices $n'_i(x_N)$ are, respectively, (a) (3,0,0,0); (b) (0,1,1,1); (c) (1,1,1,1); (d) (2,0,1,1); and (e) (1,0,1,1).

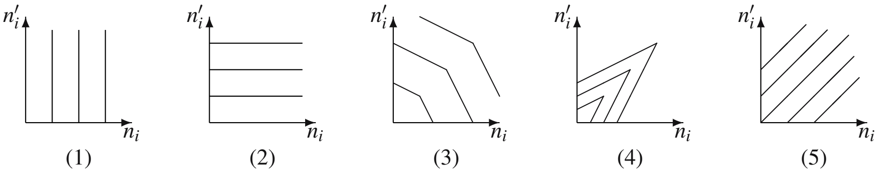


**Fig. 2** Iso-curves of $D$

(1) $D(n_i, n_i') = n_i$;
(2) $D(n_i, n_i') = n_i'$.

Panel (3) and (4) correspond to cases in which a concern for balancedness is introduced:

(3) $D(n_i, n_i') = 2\max\{n_i, n_i'\} + \min\{n_i, n_i'\}$.
(4) $D(n_i, n_i') = 2\max\{n_i, n_i'\} - \min\{n_i, n_i'\} = \max\{n_i, n_i'\} + |n_i - n_i'|$.

As far as the examples of Fig. 1 are concerned, formula (3) puts (e) above (c) whereas formula (4), displaying a greater concern for balancedness, puts (c) above (e). Panel (5) depicts the extreme case in which only balancedness matters:

(5) $D(n_i, n_i') = |n_i - n_i'|$.

It turns out that none of these criteria satisfies equal well-being for equal responsibility or equal treatment for equal circumstances in the distribution case. The next proposition shows that there is no hope to find better criteria along these lines. To keep things simple, attention is restricted to "reasonable" criteria that prefer an allocation with only one envy occurrence to any unbalanced allocation with more than $n$ envy occurrences, for $n$ great enough. This restriction seems unquestionable when dealing with criteria that rely only on envy graphs.

**Proposition 2.** *In the distribution case, no reasonable criterion based on envy graphs satisfies either equal well-being for equal responsibility or equal treatment for equal circumstances.*

*Proof.* In the distribution case, two agents $i, j$ can be in a situation in which no one envies the other when they have certain $x_i^*, x_j^*$, whereas at all other allocations at least one envies the other. In such a case, let us say that $i$ and $j$ are "locked" at $\left(x_i^*, x_j^*\right)$. Let us illustrate how this can happen. Let $u(x, y_i, z_i) = u(x, y_j, z_i)$ for all $x$, and $u(x, y_j, z_j) < u(x, y_i, z_j)$ for all $x \neq x^*$, while $u(x^*, y_j, z_j) = u(x^*, y_i, z_j)$. Then $i$ and $j$ are locked at $(x^*, x^*)$, since there is no envy at $(x^*, x^*)$, whereas for $(x_i, x_j)$ (with at least one different from $x^*$), $i$ envies $j$ if $x_i < x_j$ and $j$ envies $i$ if $x_i \geq x_j$.

Consider an $n$-agent population $\{1, \ldots, n\}$ where $z_1 = z_2$ and such that for all pairs of agents $i, j > 1$, $i$ and $j$ are locked at (1,1). Assume that $\Omega = n$, that agents $3, \ldots, n$ never envy agent 1 (whatever the allocation), that $u_1 > u_2$ at allocation $(1, \ldots, 1)$ and that 1 is envied by 2 at this allocation. Necessarily this is the only envy occurence in this allocation. At any other allocation in $X$, there will be at least

$n-2$ envy occurrences, because at least one of the agents $i > 1$ will have a different $x$ and this will create at least one envy occurrence between him and each one of the others.

Moreover, no allocation in which 1 and 2 do not envy each other is balanced. First note that in such an allocation $u(x_1, y_1, z_1) = u(x_2, y_2, z_2)$, since $z_1 = z_2$. If 2 envies another agent, then 1 envies this other agent as well, but 1 is not envied by 2 in such an allocation, and is never envied by $3, \ldots, n$ in all allocations. In this case 1's situation is unbalanced. If 2 does not envy any other agent, he must be envied by at least one agent $3, \ldots, n$ and his situation is unbalanced.

Therefore, for $n$ great enough, a reasonable criterion will prefer $(1, \ldots, 1)$ to any allocation in which 1 and 2 do not envy each other, and thereby violate equal well-being for equal responsibility.

For equal treatment for equal circumstances, assume $y_1 = y_2$, and all pairs of agents $i, j > 1$ are similarly locked together. Then, for certain preferences, the allocation $(0, 1, \ldots, 1)$ has only one envy occurrence, namely 1 envying 2. The rest of the argument is as above. □

This last result clearly suggests that the information contained in an envy graph is insufficient, and that can be interpreted as being due to the fact that this information is typically insufficient to pinpoint agents with identical $y$ or identical $z$.

## 4 Undominated Diversity and Beyond

In this section we turn to a setting with richer information. Recall the $S_{\mathrm{MUD}}$ allocation rule, which seeks to minimize the size of the set of agents who unanimously consider that $i$ is worse-off than $j$ (and $i$ is among them), for all pairs $(i, j)$. This allocation rule refines van Parijs' undominated diversity, which is too large in some cases (in particular, it accepts allocations with envy when envy-free allocations exist). It shares with it the drawback that it may happen to be empty in the distribution case. It is, however, nonempty in a rather wide class of situations.

**Lemma 1.** *The $S_{\mathrm{MUD}}$ is nonempty in the TU case. It is also nonempty in the distribution case on $\mathcal{D}_1$.*

*Proof.* **Distribution case**: Fleurbaey (1994, Prop. 10) proves that, if for all $i, j \in N$, there is $k \in N$ such that

$$u\Big(\frac{\Omega}{|N|-1}, y_i, z_k\Big) \geq u(0, y_j, z_k),$$

then $S_{\mathrm{MUD}}$ is nonempty. This assumption is satisfied on $\mathcal{D}_1$.

**TU case**: Consider first a modified version of the TU case in which the feasible set is $X^* = \left\{x_N \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i = \Omega\right\}$. Let

$$\Omega = (|N|-1) \max_{i,j,k \in N} \left(v(y_i, z_k) - v(y_j, z_k)\right).$$

With this value of $\Omega$, the above assumption is satisfied, so that there exists an allocation $x_N \in X^*$ such that for all $i, j \in N$, there is $k \in N$ such that

$$u_i(x_i, y_i, z_k) \geq u(x_j, y_j, z_k).$$

Let $\mu = \frac{1}{|N|}\sum_{i\in N} x_i$, and define $x_i' = x_i - \mu$ for all $i \in N$. The allocation $x_N'$ is such that $\sum_{i\in N} x_i' = 0$ and, by the quasi-linearity of $u$ in the TU case, it still holds that for all $i, j \in N$, there is $k \in N$ such that

$$u_i(x_i, y_i, z_k) \geq u(x_j, y_j, z_k).$$

□

In fact, the underlying idea of $S_{\text{MUD}}$ is again to rank graphs of envy relations. But, interestingly, instead of simply counting the arrows between individuals, the idea is to assign a number to every envy relation, which is equal to the number of individuals who share the envious' preferences. For instance, suppose $i$ envies $j$, and there are three other individuals who, with their own responsibility characteristics, would be better-off with $j$'s bundle of external resources and circumstances than with $i$'s. Then the envy arrow from $i$ to $j$ is assigned a value of four. When $i$ does not envy $j$, no arrow is drawn even if there are some other individuals who would be better-off with $j$'s bundle than with $i$'s. The absence of an arrow is equivalent to a value of zero.

In summary, for every ordered pair $(i, j)$, this procedure gives us a number, equal to zero if $i$ does not envy $j$, and equal to a positive integer between one and the population size otherwise. "Undominated diversity" is simply the rather special requirement that no pair has a number with the maximal value ($|N|$). The $S_{\text{MUD}}$ allocation rule applies the minimax criterion to the list of these numbers (i.e., it minimizes the greatest number), retaining in addition the requirement that no pair has number $|N|$.

A drawback of the minimax criterion is that it neglects the situation of envy relations with a less than maximal number and may therefore accept too much of envy. It appears much more reasonable to apply the reverse leximin criterion to the list of these numbers. Let us call this the "diversity" criterion, since it both extends and refines van Parijs' criterion, and takes account of the diversity of preferences in the population.

**Diversity criterion ($R_{\text{D}}$):** For any $x_N \in X$, $(i, j) \in N^2$, let

$$n_{ij}(x_N) = \begin{cases} 0 \text{ if } i \text{ does not envy } j, \\ \left|\left\{k \in N \mid u(x_j, y_j, z_k) > u(x_i, y_i, z_k)\right\}\right| \text{ otherwise.} \end{cases}$$

For all $x_N, x_N' \in X$, $x_N \, R_{\text{D}}(e) \, x_N'$ if and only if

$$-\left(n_{ij}(x_N)\right)_{i,j\in N} \geq_{\text{lex}} -\left(n_{ij}(x_N')\right)_{i,j\in N}.$$

An envy-free allocation corresponds to a list containing only zeros, and will be selected whenever it exists. Similarly, if there exist allocations satisfying the undominated diversity criterion, the selected allocations will be drawn from this subset. An interesting feature of the diversity criterion (already present in undominated diversity) is that it satisfies equal treatment for equal circumstances.

**Proposition 3.** *The allocation rule derived from the diversity criterion exactly selects the set of envy-free allocations whenever it is nonempty. The allocation rule derived from it satisfies equal well-being for uniform responsibility (on $\mathcal{D}_1$ for the distribution case) and equal treatment for equal circumstances (on $\mathcal{D}_1$ for the distribution case).*

*Proof.* (1) An envy-free allocation is such that $(n_{ij}(x_N))_{i,j\in N} = 0$, and this dominates any $-(n_{ij}(x'_N))_{i,j\in N} < 0$ for the leximin criterion. Therefore, the set of envy-free allocations is selected whenever it is nonempty.

(2) When $z_i = z_j$ for all $i, j \in N$, the allocation that equalizes well-being across all agents is the only envy-free efficient allocation and is therefore selected whenever it is feasible, which is always true in the TU case, and on $\mathcal{D}_1$ in the distribution case. This proves the satisfaction of equal well-being for uniform responsibility.

(3) That it satisfies equal treatment for equal circumstances is a consequence of the fact that $n_{ij}(x_N) = |N|$ if $x_i < x_j$ while $y_i = y_j$ and that, in the TU case as well as in the distribution case on $\mathcal{D}_1$, by Lemma 1 there always exist (undominated diversity) allocations with $\max(n_{ij}(x_N))_{i,j\in N} < |N|$. In the distribution case, out of $\mathcal{D}_1$, an allocation satisfying equal treatment for equal circumstances is not always selected. Consider a situation with uniform $z$ and $y_1 = y_2$ in which the leximin-utility allocation is $x_N = (3,3,0,\ldots,0)$, while $u(4,y_2,z) = u(0,y_i,z)$ for all $i \neq 1$. Then the allocation $x'_N = (2,4,0,\ldots,0)$ is preferred when $|N| > 3$, because it has

$$(n_{ij}(x'_N))_{i,j\in N} = \begin{pmatrix} 0 & |N| & |N| & \cdots & |N| \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

compared to

$$(n_{ij}(x_N))_{i,j\in N} = \begin{pmatrix} 0 & 0 & |N| & \cdots & |N| \\ 0 & 0 & |N| & \cdots & |N| \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

□

The allocation rule associated with $R_{\mathrm{D}}$ is more satisfactory than those of the previous section, and this can be linked to the richer information used by $R_{\mathrm{D}}$.

The diversity criterion is clearly on the side of the natural reward principle, and one may wonder if a dual criterion can be defined that would embrace the compensation principle instead. The dual criterion does exist, and refers to all the values of

$y_k$ instead of the values of $z_k$. For a given $y_k$, it evaluates the situation of an individual $i$ by computing the value of $x$ that would give him, with circumstances $y_k$, the same utility as in the contemplated allocation. Now consider two individuals $i$ and $j$. This computation amounts to imagining a situation that is equivalent in terms of well-being but with equal circumstances ($y_k$) for both individuals. The ideal allocation should then be egalitarian between them, and any inequality observed in $x$ for this imaginary situation does reflect a problem. Now, different computations made with different $y_k$ may yield different answers and this criterion takes account of this possible diversity.

Formally, for any $x_N \in X$, $i,k \in N$, let $x_{ik}(x_N)$ be defined by

$$u(x_i, y_i, z_i) = u(x_{ik}(x_N), y_k, z_i).$$

In the distribution case, it may happen that $u(x_i, y_i, z_i) > u(x, y_k, z_i)$ for all $x \geq 0$, or that $u(x_i, y_i, z_i) < u(x, y_k, z_i)$ for all $x \geq 0$. We focus here, for this case, on the domain $\mathcal{D}_2$ such that for all $i, j \in N$, one has $u(0, y_i, z_i) = u(0, y_j, z_i)$ and there exists $x \geq 0$ such that $u(\Omega, y_i, z_i) < u(x, y_j, z_i)$. On this domain, $x_{ik}(x_N)$ is always well defined.

**Compensation Diversity criterion ($R_{\mathrm{CD}}$):** For any $x_N \in X$, $(i, j) \in N$, let

$$m_{ij}(x_N) = \begin{cases} 0 \text{ if } i \text{ does not envy } j, \\ \left|\left\{k \in N \mid x_{jk}(x_N) > x_{ik}(x_N)\right\}\right| \text{ otherwise.} \end{cases}$$

For all $x_N, x'_N \in X$, $x_N \, R_{\mathrm{CD}}(e) \, x'_N$ if and only if

$$-\left(m_{ij}(x_N)\right)_{i,j \in N} \geq_{\mathrm{lex}} -\left(m_{ij}(x'_N)\right)_{i,j \in N}.$$

The following statement establishes the connection between this criterion and the compensation principle.

**Proposition 4.** *The allocation rule derived from the compensation diversity criterion exactly selects the set of envy-free allocations whenever it is nonempty (on $\mathcal{D}_2$ for the distribution case). The allocation rule derived from it satisfies equal well-being for equal responsibility (on $\mathcal{D}_2$ for the distribution case) and equal treatment for uniform circumstances.*

*Proof.* (1) Notice that $x_{ii}(x_N) \equiv x_i$. When $i$ envies $j$, one has

$$u(x_i, y_i, z_i) < u(x_j, y_j, z_i),$$

implying that if

$$u_i = u(x_{ij}(x_N), y_j, z_i),$$

as is always obtained in the TU case and on $\mathcal{D}_2$ in the distribution case, then $x_{ij}(x_N) < x_j = x_{jj}(x_N)$ and therefore $m_{ij}(x_N) > 0$. An envy-free allocation is such that $(m_{ij}(x_N))_{i,j \in N} = 0$, and this dominates any $-(m_{ij}(x'_N))_{i,j \in N} < 0$ for the leximin criterion. Therefore, the set of envy-free allocations is selected whenever it is nonempty.

(2) When $y_i = y_j$ for all $i, j \in N$, the allocation $x_N = 0$ is the only envy-free efficient allocation and is therefore selected. This proves the satisfaction of equal treatment for uniform circumstances.

(3) That it satisfies equal well-being for equal responsibility is a consequence of the fact that $m_{ij}(x_N) = |N|$ if $u_i < u_j$ while $z_i = z_j$ and that, in the TU case as well as in the distribution case on $\mathcal{D}_2$, by a dual to Lemma 1, there always exist allocations with $\max\left(m_{ij}(x_N)\right)_{i,j \in N} < |N|$. □

An interesting difference between diversity and compensation diversity is worth noting. When $i$ envies $j$, this is recorded by the diversity criterion on the basis of $z_i$, that is, the preferences of the *envious* agent:

$$u(x_i, y_i, z_k) < u(x_j, y_j, z_k) \text{ for } k = i,$$

whereas with compensation diversity, this is recorded with $y_j$, that is, the circumstances of the *envied* agent:

$$u(x_i, y_i, z_i) = u(x_{ik}(x_N), y_k, z_i) < u(x_j, y_j, z_i) = u(x_{jk}(x_N), y_k, z_i) \text{ for } k = j.$$

As in the previous section with the envious count and envied count criteria, whether one focuses on the envious or on the envied may contribute to determining whether the criterion falls on the compensation side or on the natural reward side. A similar configuration will again be obtained in the next section.

## 5 From Envy Intensity to Walras

Although the diversity criteria improve on the envy count criteria of Section 3, they may still be criticized for the restricted information they rely upon. They rank allocations on the basis of a rather poor information, namely, the graphs of envy relations (and of similar preference relations for the diversity criteria). Allocations are made of distributions of resources, which provide a much finer scale for the measurement of envy situations. It is quite unjustifiable to ignore this information and simply focus on zero-one markers of presence or absence of envy relations. In particular, the envy count and diversity criteria are indifferent between any pair of allocations with the same graph, even if one allocation may have much less inequality, that is, a smaller degree of envy, than the other. They may also prefer an allocation with fewer relations of envy but with a very high degree of envy in these relations to another allocation with more envy occurrences but which is in fact much closer to an envy-free situation. In conclusion, looking at the graphs of envy relations, even augmented by $n_{ij}$ numbers, is probably not a very good idea.

The $S_{\text{MEI}}$ allocation is based on a finer information and indeed suggests an alternative approach. For every allocation and for every pair of individuals $(i, j)$, compute the number $t_{ij}$ as the smallest amount of external resources such that giving this to $i$ in addition to what he receives in this allocation would prevent him from envying $j$. If $i$ already does not envy $j$, this number is typically negative, meaning that one can diminish $i$'s resources without making him envy $j$. And one always has $t_{ii} = 0$. Let $t_{ij}$ be called the degree of $i$'s envy toward $j$. The $S_{\text{MEI}}$ allocation rule as defined above amounts to retaining the greatest $t_{ij}$ for every $i$ (ignoring $t_{ii}$) as a measure of his greatest degree of envy (or smallest degree of non-envy if it is negative), and to apply the minimax criterion to the vector of these numbers.[4] This is a rather natural solution, but Fleurbaey (1994, Prop. 9) notices that it satisfies neither equal well-being for equal responsibility nor equal treatment for equal circumstances. This suggests looking for another way to rank distributions $(t_{ij})_{i,j \in N}$.

We examine two other, a priori less intuitive, options which may ultimately be more satisfactory. The first is similar to the above but incorporates $t_{ii}$ in the computation of the greatest degree of envy, so that this number is always nonnegative, and applies the summation operation rather than the minimax. The second solution computes, for every individual, the greatest degree of envy among those who might envy him, and then applies the summation operator. In both cases, the social objective is to minimize the value of these sums. The first is focused on the degree of envy from the standpoint of the envious (the transmitters), while the second takes the viewpoint of those who are envied (the receivers).

For any allocation $x_N \in X$, any pair of agents $i, j \in N$, let $t_{ij}(x_N)$ be the smallest value of $t$ such that

$$u(x_i + t, y_i, z_i) \geq u(x_j, y_j, z_i)$$

and $d_{ij}(x_N)$ be the smallest value of $d$ such that

$$u(x_i, y_i, z_i) \geq u(x_j - d, y_j, z_i).$$

Tadenuma and Thomson (1995), in the context of transferable indivisibles, have considered the two notions of $t_{ij}$ and $d_{ij}$. The definition of $d_{ij}(x_N)$ should be slightly modified in the distribution case when

$$u(x_i, y_i, z_i) < u(0, y_j, z_i),$$

in which case one can propose to compute $d_{ij}(x_N)$ as the smallest value of $x_j + d$ for $d$ such that

$$u(x_i + d, y_i, z_i) \geq u(0, y_j, z_i).$$

In the TU case, one simply has

$$t_{ij}(x_N) = d_{ij}(x_N) = x_j + v(y_j, z_i) - x_i - v(y_i, z_i).$$

In the distribution case, one may have $t_{ij}(x_N)$ or $d_{ij}(x_N)$ undefined if

$$\lim_{x \to +\infty} u(x, y_i, z_i) < u(x_j, y_j, z_i) \text{ or } u(x_i, y_i, z_i) > \lim_{x \to +\infty} u(x, y_j, z_i).$$

[4] In fact, the presentation of $S_{\text{MEI}}$ in Sect. 2 was a little simplistic since Fleurbaey (1994) already introduces the $t_{ij}$ and applies the reverse leximin criterion to the vector $(\max_{j \neq i} t_{ij})_{i \in N}$, instead of the minimax.

To avoid this problem, we may restrict our attention to the domain $\mathcal{D}_0$ such that for all $i, j \in N$, there exists $x \geq 0$ such that

$$u(x, y_i, z_i) \geq u(\Omega, y_j, z_i) \text{ and } u(x, y_j, z_i) \geq u(\Omega, y_i, z_i).$$

In all cases, the following three statements are equivalent: (1) $t_{ij}(x_N) > 0$; (2) $d_{ij}(x_N) > 0$; and (3) $i$ envies $j$. One always has $t_{ii}(x_N) \equiv d_{ii}(x_N) \equiv 0$.

The two social ordering functions are formally defined as follows.

**Envious Intensity ($\boldsymbol{R}_{\mathbf{EsI}}$):** For all $x_N, x'_N \in X$, $x_N \, R_{\mathrm{EsI}}(e) \, x'_N$ if and only if

$$\sum_{i \in N} \max_{j \in N} t_{ij}(x_N) \leq \sum_{i \in N} \max_{j \in N} t_{ij}(x'_N).$$

**Envied Intensity ($\boldsymbol{R}_{\mathbf{EdI}}$):** For all $x_N, x'_N \in X$, $x_N \, R_{\mathrm{EdI}}(e) \, x'_N$ if and only if

$$\sum_{j \in N} \max_{i \in N} d_{ij}(x_N) \leq \sum_{j \in N} \max_{i \in N} d_{ij}(x'_N).$$

The quantity $\max_{j \in N} t_{ij}(x_N)$ measures how much must be added to $x_i$ for envious $i$ to get rid of envy, while $\max_{i \in N} d_{ij}(x_N)$ measures how much must be deducted from $x_j$ for envied $j$ not to be envied any more.

The allocation rules derived from these rankings appear to have more interesting properties than $S_{\mathrm{MEI}}$.

**Proposition 5.** *The allocation rules derived from the envious and envied intensity criteria both exactly select the set of envy-free allocations whenever it is nonempty (on $\mathcal{D}_0$ for the distribution case). The allocation rule derived from envious intensity satisfies equal well-being for uniform responsibility (on $\mathcal{D}_1$ for the distribution case) and equal treatment for equal circumstances (on $\mathcal{D}_0$ for the distribution case). The allocation rule derived from envied intensity satisfies equal well-being for equal responsibility (on $\mathcal{D}_2$ for the distribution case) and equal treatment for uniform circumstances.*

*Proof.* (1) For all $x_N \in X$, all $i \in N$,

$$\max_{j \in N} t_{ij}(x_N) \geq t_{ii}(x_N) \equiv 0,$$

and $\max_{j \in N} t_{ij}(x_N) > 0$ if and only if $i$ is envious, so that one has

$$\sum_{i \in N} \max_{j \in N} t_{ij}(x_N) = 0$$

if and only if $x_N$ is envy-free. The same can be said about $\sum_{j \in N} \max_{i \in N} d_{ij}(x_N)$. Therefore, the set of envy-free allocations is selected by either allocation rule whenever it is nonempty.

(2) By the same argument as in Proposition 3, step 2 (respectively, Proposition 4, step 2), this implies that the allocation rule derived from envious intensity

(respectively, envied intensity) satisfies equal well-being for uniform responsibility (respectively, equal treatment for uniform circumstances).

(3) Now let us turn to envious intensity and equal treatment for equal circumstances. Consider two agents $i, j$ such that $y_i = y_j$, and suppose, by way of contradiction, that there is an allocation $x_N$ minimizing $\sum_{k\in N} \max_{l\in N} t_{kl}(x_N)$, with $x_i > x_j$. The fact that $x_i > x_j$ implies that $t_{ji}(x_N) > 0$ and that, for all $k \neq j$,

$$\max_{l\in N} t_{kl}(x_N) \geq t_{ki}(x_N) > t_{kj}(x_N).$$

Take $\delta$ such that

$$0 < \delta < \frac{1}{|N|} \min_{k\in N} \left( \max_{l\in N} t_{kl}(x_N) - t_{kj}(x_N) \right)$$

and $\delta < (x_i - x_j) / |N|$ . Construct a new allocation such that $x'_k = x_k - \delta$ for all $k \neq j$, and $x'_j = x_j + (|N| - 1)\,\delta$. Notice that one still has $x'_i > x'_j$ and therefore, for all $k \neq j$,

$$\max_{l\in N} t_{kl}(x'_N) > t_{kj}(x'_N).$$

Consider $k, l \neq j$. One has

$$u(x_k + t_{kl}(x_N), y_k, z_k) \geq u(x_l, y_l, z_k).$$

One also has either

$$u(x_k - \delta + t_{kl}(x'_N), y_k, z_k) = u(x_l - \delta, y_l, z_k) < u(x_l, y_l, z_k),$$

implying $t_{kl}(x_N) > -\delta + t_{kl}(x'_N)$, or $t_{kl}(x'_N) = -(x_k - \delta)$, implying $t_{kl}(x_N) \geq -\delta + t_{kl}(x'_N)$ since one always has $t_{kl}(x_N) \geq -x_k$. One therefore has

$$\max_{l\in N} t_{kl}(x'_N) \leq \max_{l\in N} t_{kl}(x_N) + \delta$$

for all $k \neq j$.

Now consider $j$ and $k \neq j$ envied by $j$ in $x'_N$ (at least $i$ is envied by $j$). One has

$$u(x_j + (|N| - 1)\,\delta + t_{jk}(x'_N), y_j, z_j) = u(x_k - \delta, y_k, z_j) < u(x_k, y_k, z_j)$$

and

$$u(x_k, y_k, z_j) \leq u(x_j + t_{jk}(x_N), y_j, z_j),$$

implying

$$(|N| - 1)\,\delta + t_{jk}(x'_N) < t_{jk}(x_N).$$

For any $k \neq j$ that is not envied by $j$ in $x'_N$, one has

$$t_{jk}(x'_N) \leq 0 < x_i - x_j - |N|\,\delta \leq \max_{l\in N} t_{jl}(x_N) - |N|\,\delta.$$

Therefore,

$$\max_{l\in N} t_{jl}(x'_N) < \max_{l\in N} t_{jl}(x_N) - (|N|-1)\,\delta.$$

Summing up over all agents, one obtains

$$\sum_{k\in N} \max_{l\in N} t_{kl}(x'_N) < \sum_{k\in N} \max_{l\in N} t_{kl}(x_N) + (|N|-1)\,\delta - (|N|-1)\,\delta,$$

contradicting the assumption that $x_N$ minimizes $\sum_{i\in N} \max_{j\in N} t_{ij}(x_N)$.

(4) Finally, envied intensity and equal well-being for equal responsibility. Consider two agents $i, j$ such that $z_i = z_j$, and suppose, by way of contradiction, that there is an allocation $x_N$ minimizing $\sum_{k\in N} \max_{l\in N} d_{lk}(x_N)$, with $u_i > u_j$. The fact that $u_i > u_j$ implies that $d_{ji}(x_N) > 0$ and that, for all $k \in N$,

$$\max_{l\in N} d_{lk}(x_N) \geq d_{jk}(x_N) > d_{ik}(x_N).$$

Take $\delta > 0$ such that

$$u(x_i - (|N|-1)\,\delta, y_i, z_i) > u(x_j + \delta, y_j, z_j).$$

Construct a new allocation such that $x'_i = x_i - (|N|-1)\,\delta$ and for all $k \neq i$, $x'_k = x_k + \delta$. One still has $u'_i > u'_j$ and therefore, for all $k \in N$,

$$\max_{l\in N} d_{lk}(x'_N) > d_{ik}(x'_N).$$

Consider $k, l \neq i$. One has (in the TU case as well as in the distribution case for the domain $\mathcal{D}_2$)

$$u(x_l, y_l, z_l) = u(x_k - d_{lk}(x_N), y_k, z_l),$$

and

$$u(x_l + \delta, y_l, z_l) = u(x_k + \delta - d_{lk}(x'_N), y_k, z_l) > u(x_l, y_l, z_l),$$

implying $\delta - d_{lk}(x'_N) > -d_{lk}(x_N)$, that is, $d_{lk}(x'_N) < d_{lk}(x_N) + \delta$. One therefore has

$$\max_{l\in N} d_{lk}(x'_N) \leq \max_{l\in N} d_{lk}(x_N) + \delta$$

for all $k \neq i$.

Now consider $i$ and $k \neq i$. One has

$$u(x_k + \delta, y_k, z_k) = u(x_i - (|N|-1)\,\delta - d_{ki}(x'_N), y_i, z_k) > u(x_k, y_k, z_k)$$

and

$$u(x_k, y_k, z_k) = u(x_i - d_{ki}(x_N), y_i, z_k),$$

implying

$$d_{ki}(x'_N) < d_{ki}(x_N) - (|N|-1)\,\delta.$$

Therefore,

$$\max_{l\in N} d_{li}(x'_N) < \max_{l\in N} d_{li}(x_N) - (|N|-1)\,\delta.$$

Summing up over all agents, one obtains

$$\sum_{k\in N} \max_{l\in N} d_{lk}(x'_N) < \sum_{k\in N} \max_{l\in N} d_{lk}(x_N) + (|N|-1)\,\delta - (|N|-1)\,\delta,$$

contradicting the assumption that $x_N$ minimizes $\sum_{k\in N} \max_{l\in N} d_{lk}(x_N)$. □

The envied intensity criterion is a little mysterious because it is not written in terms of indices of personal situations (being envied is not a characteristic of one's situation but rather a token of the others' situations), contrary to the envious intensity that transparently measures how envious every agent is and constructs a synthetic measure of this. The envied intensity criterion, however, can be related to a more orthodox social ordering function. Let $(q_i)_{i\in N} \in \mathbb{R}^N$ be a vector of prices for $y_N$, in a virtual market in which agents could buy bundles $(x,y)$. The budget constraint for $i \in N$ on this market is such that a bundle $(x, y_j)$ is affordable if

$$x + q_j = I_i,$$

where $I_i$ denotes $i$'s personal wealth. Let $e_i$ denote $i$'s expenditure function:

$$e_i(u_i, q_N) = \min\left\{x + q_j \mid (x, j) \in \mathbb{R}\times N \text{ and } u(x, y_j, z_i) \geq u_i\right\}.$$

We now define the Egalitarian Walras social ordering function. Let $Q$ be the subset of $q_N$ such that $\sum_{i\in N} q_i = 0$.

**Egalitarian Walras ($R_{\text{EW}}$):** For all $x_N, x'_N \in X$, $x_N\, R_{\text{EW}}(e)\, x'_N$ if and only if

$$\max_{q_N\in Q} \min_{i\in N} e_i(u(x_i, y_i, z_i), q_N) \geq \max_{q_N\in Q} \min_{i\in N} e_i(u(x'_i, y_i, z_i), q_N).$$

This social ordering function is an adaptation to this model of a function introduced in Fleurbaey and Maniquet (2008) for the fair division context in order to rationalize the egalitarian competitive equilibrium. Notice that this social ordering function can be used to rank all allocations, not just the feasible ones.

Consider the virtual market in which circumstances $y$ are tradable. This is just the model of allocation of large indivisibles as studied, for example, in Svensson (1983), with a number of indivisible goods equal to the number of agents. One can easily extend the definition of the above social ordering function in order to consider possibilities of permutations in $y_N$ : $(x_N, y_N)\; R_{\text{EW}}(e)\; (x'_N, y'_N)$ if and only if

$$\max_{q_N\in Q} \min_{i\in N} e_i(u(x_i, y_i, z_i), q_N) \geq \max_{q_N\in Q} \min_{i\in N} e_i(u(x'_i, y'_i, z_i), q_N).$$

In this market a competitive equilibrium is an allocation $(x_N, y_N)$ associated to a price vector $q$ such that for all $i \in N$, $(x_i, y_i)$ maximizes $u(x, y_j, z_i)$ over the set of bundles $(x, y_j)$ satisfying the budget constraint $x + q_j = I_i$. It is egalitarian if $I_i = I_j$ for all $i, j \in N$.

Let $\Pi(y_N)$ denote the set of permutations of $y_N$ and let $(x_N, y_N, q_N) \in X \times \Pi(y_N) \times Q$ be any allocation and price vector. If for all $i \in N$,

$$x_i + q_i = e_i(u(x_i, y_i, z_i), q_N),$$

then this is a competitive equilibrium. More generally, one always has

$$e_i(u(x_i, y_i, z_i), q_N) \leq x_i + q_i$$

for all $i \in N$, with at least one strict inequality if this is not a competitive equilibrium. By construction one has

$$\sum_{i \in N} (x_i + q_i) = \Omega,$$

implying that one always has

$$\max_{q_N \in Q} \min_{i \in N} e_i(u(x_i, y_i, z_i), q_N) \leq \frac{\Omega}{|N|},$$

with equality if and only if $(x_N, y_N, q_N)$ is an egalitarian competitive equilibrium. This shows that the Egalitarian Walras social ordering function rationalizes the egalitarian competitive equilibrium (in the sense that it exactly selects the set of egalitarian equilibria whenever it is non-empty), which, in the particular context of indivisibles, coincides with the set of envy-free and efficient allocations (Svensson, 1983).

Let us now focus on the distribution case for the domain $\mathcal{D}_2$ and on the TU case. In these two cases one can simply define $d_{ij}(x_N)$ by the equation

$$u(x_j - d_{ij}(x_N), y_j, z_i) = u_i.$$

One then computes

$$e_i(u(x_i, y_i, z_i), q_N) = \min_{j \in N} (x_j - d_{ij}(x_N) + q_j).$$

One therefore has

$$\begin{aligned}
\min_{i \in N} e_i(u(x_i, y_i, z_i), q_N) &= \min_{i,j \in N} (x_j - d_{ij}(x_N) + q_j) \\
&= \min_{j \in N} \left( x_j + q_j - \max_{i \in N} d_{ij}(x_N) \right),
\end{aligned}$$

implying that for all $x_N \in X$, $q_N \in Q$,

$$\min_{i \in N} e_i(u(x_i, y_i, z_i), q_N) \leq \frac{1}{|N|} \sum_{j \in N} \left( x_j + q_j - \max_{i \in N} d_{ij}(x_N) \right).$$

Since $\sum_{j \in N} (x_j + q_j) = \Omega$, this can be simplified into

$$\min_{i\in N} e_i(u(x_i,y_i,z_i),q_N) \leq \frac{\Omega}{|N|} - \frac{1}{|N|}\sum_{j\in N}\max_{i\in N} d_{ij}(x_N).$$

Now let, for all $j \in N$,

$$q_j = -\left(x_j - \max_{i\in N} d_{ij}(x_N)\right) + \frac{\Omega}{|N|} - \frac{1}{|N|}\sum_{k\in N}\max_{i\in N} d_{ik}(x_N).$$

By construction $q_N \in Q$. Moreover, for all $j \in N$,

$$x_j + q_j - \max_{i\in N} d_{ij}(x_N) = \frac{\Omega}{|N|} - \frac{1}{|N|}\sum_{k\in N}\max_{i\in N} d_{ik}(x_N),$$

implying that

$$\begin{aligned}\min_{i\in N} e_i(u(x_i,y_i,z_i),q_N) &= \min_{j\in N}\left(x_j + q_j - \max_{i\in N} d_{ij}(x_N)\right)\\ &= \frac{\Omega}{|N|} - \frac{1}{|N|}\sum_{k\in N}\max_{i\in N} d_{ik}(x_N).\end{aligned}$$

Since we have seen above that this is an upper bound for $\min_{i\in N} e_i(u(x_i,y_i,z_i),q_N)$ when $q_N$ varies, one actually has

$$\max_{q_N\in Q}\min_{i\in N} e_i(u(x_i,y_i,z_i),q_N) = \frac{\Omega}{|N|} - \frac{1}{|N|}\sum_{j\in N}\max_{i\in N} d_{ij}(x_N),$$

which establishes an exact equivalence (for a fixed value of $\Omega$) between the egalitarian Walras and the envied intensity criteria in the TU case and on the domain $\mathcal{D}_2$ for the distribution case. It is somewhat surprising that a maximin social ordering function can be equivalent to a purely additive criterion. But one observes in the above computations that the maximin criterion of the egalitarian Walras function is what triggers the focus on the maximal intensity retained in the computation of $\max_{i\in N} d_{ij}(x_N)$ for the envied intensity criterion.

**Proposition 6.** *The egalitarian Walras and the envied intensity criteria are equivalent in the TU case. In the distribution case, they are equivalent on the domain* $\mathcal{D}_2$.

Regarding envious intensity, one can similarly establish an equivalence with the social ordering function that evaluates an allocation $x_N$ by computing

$$\max_{q\in Q}\min_{i\in N}(x_i + q_i - \max_{j\in N} t_{ij}(x_N)).$$

This amounts to measuring the agents' wealth $x_i + q_i$ and deducting from it their maximal degree of envy. One notices here that the duality between compensation and natural reward appears to be related to the duality of consumer theory. Applied to the market for (transferable) indivisibles, this social ordering function also rationalizes the equal competitive equilibrium. Contrary to egalitarian Walras, it does not satisfy the Pareto principle in that context and therefore appears less interesting.

## 6 Conclusion

Six main no-envy rankings have been examined in this chapter, in the context of the compensation problem. Apart from the intrinsic interest of these solutions, which still deserves further assessment, the main conceptual insight obtained here may be that the well-know duality in the compensation problem between the compensation principle and the natural reward principle is related to a duality between focussing on the envied and focussing on the envious. But this connection is not simple since, for instance, the compensation principle is satisfied by the envious count criterion and the envied intensity criterion, that is, two criteria focussing on a different side of the envy relation. The key observation underlying these two facts is that when $z_i = z_j$ and $u_i > u_j$, agent $j$ envies all the agents envied by $i$ with greater intensity than $i$, implying that, for all $k \neq i$, $i$ is never such that $d_{ik} = \max_{l \in N} d_{lk}$, and that, since $j$ also envies $i$ in top of the others envied by $i$, one has $n_j > n_i$.

A gap that this chapter may highlight is that there is a lack of axiomatic framework for the study of social ordering functions in the compensation problem. The evaluation of rankings that has been performed here was concerned with satisfying axioms of allocation rules and therefore focused on the subsets selected by the contemplated rankings. It is not very difficult to formulate axioms for rankings that bear a close relation to the axioms presented here. For instance, one can think of the following variants of the above axioms, applying to a social ordering function $R$:

Transfer for Equal Responsibility: $\forall e \in \mathcal{D}$, $\forall x_N, x'_N \in X$, $\forall i, j \in N$ such that $z_i = z_j$, if $x'_i - x_i = x_j - x'_j$ and

$$u\left(x'_i, y_i, z_i\right) > u\left(x_i, y_i, z_i\right) \geq u\left(x_j, y_j, z_j\right) > u\left(x'_j, y_j, z_j\right)$$

while $x'_k = x_k$ for all $k \neq i, j$, then $x_N \, P(e) \, x'_N$.

Transfer for Equal Circumstances: $\forall e \in \mathcal{D}$, $\forall x_N, x'_N \in X$, $\forall i, j \in N$ such that $y_i = y_j$, if $x'_i - x_i = x_j - x'_j$ and

$$x'_i > x_i \geq x_j > x'_j$$

while $x'_k = x_k$ for all $k \neq i, j$, then $x_N \, P(e) \, x'_N$.

Of all the social ordering functions studied in this chapter, only envious intensity and envied intensity come close to satisfying such axioms – and this illustrates again the usefulness of a fine measure of the degree of envy – but they satisfy only weak versions of the axioms, involving a weak preference $x_N \, R(e) \, x'_N$. Some leximin version of these two rankings should be invented in order to cope with this problem. This is rather easy for the envied intensity criterion, for which a maximin interpretation (egalitarian Walras) has already been provided. For envious intensity, the solution is less obvious, and in particular the maximin ranking underlying $S_{\text{MEI}}$ appears to be of no help in this matter. A more systematic study of these issues is left for another occasion.

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