Harmless Homotopic Dictators

Nicholas Baigent

1 Introduction

This paper constructs continuous Paretian social welfare functions for which one agent is a homotopic dictator but another is, in a precise sense, almost all powerful. The significance of this arises from the widely differing views[1] that have been expressed about a theorem in Chichilnisky (1982) showing that, for all continuous Paretian social welfare functions there must be a homotopic dictator. What the analysis in this paper therefore shows is that Chichilnisky's theorem is not a genuine Arrow-type impossibility theorem in the sense that desirable properties are not shown to entail some undesirable concentration of power.

While this does not necessarily mean that Chichilnisky's theorem is not significant, at least it calls for a reappraisal. One possible argument for the significance of this theorem starts from the fact that a homotopic dictator is also a strategic manipulator. However, as argued below, this argument does not establish the independent significance of Chichilnisky's theorem. At best, its significance seems to be derivative.

Section 2 introduces the main concepts and definitions. Section 3 provides an informal overview drawing heavily on diagrams. Section 4 presents results and a final Section 5 concludes with a summary and a few remarks towards a reappraisal of Chichilnisky's theorem.

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[1] See Chichilnisky (1982), Heal (1997), Sen (1986) on the one hand and Baigent (2003), Lauwers (2000), MacIntyre (1998), Saari (1997) on the other hand.

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2 Concepts and Definitions

Consider parallel linear indifference curves on a two dimensional space of alternatives, and call the underlying preferences *linear preferences*. Figure 1 shows two indifference curves for each of two linear preferences. For a given linear preference, draw a vector of length 1 perpendicular to an indifference curve at an arbitrary alternative. Such vectors are called unit normals. Since they are independent of the arbitrary alternative, a linear preference may be represented by such a *unit normal*. Also, since each unit normal takes a point in the Euclidean plane to another point on a circle of radius 1, the set of all preferences may be taken as the set of points on a unit circle. For convenience, re-centre this circle at the origin as in Fig. 2. Thus, the set of all linear preferences will be taken as: $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} = 1\}$.

For all vectors, $x = (x_1, x_2) \in S^1$, its polar coordinates are $(1, \rho_x)$ where ρ_x is the distance around the circle S^1 from the vector $(1,0)$ to x in the positive (anticlockwise) direction as shown by a bold arc in Fig. 2.

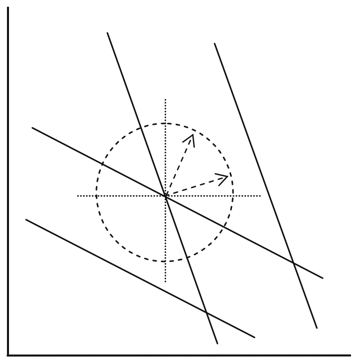


Fig. 1

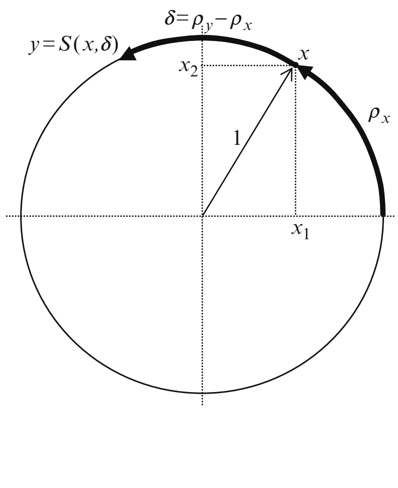


Fig. 2

Let $[0,2\pi]$ denote the closed interval of real numbers from 0 to 2π, and let $(0,2\pi)$ denote the open interval from 0 to 2π.[2] For all $x \in S^1$ and all $\delta \in [0,2\pi]$, let $s(x,\delta) \in S^1$ denote the point in S^1 that is a distance of δ around S^1 from x in an anticlockwise direction. Thus, for all $x,y \in S^1$, $s(x,\delta) = y$ if and only if $\rho_y - \rho_x = \delta$, see Fig. 2. That is, $s(x,\delta)$ determines an anticlockwise rotation from $x \in S^1$. Since the circumference of the unit circle is equal to 2π, it follows immediately that:

$$s(x,0) = s(x,2\pi) = x \tag{1}$$

For simplicity, consider the case of only two agents. A *social welfare function* is then a function $f : S^1 \times S^1 \to S^1$ that assigns a group linear preference $f(x,y) \in S^1$ to all pairs of individual linear preferences $(x,y) \in S^1 \times S^1$. Since the domain and range of a social welfare function are subsets of Euclidean spaces, when continuity is required it is taken in the usual sense for functions between subsets of Euclidean spaces.[3]

3 Overview

Continuous Paretian social welfare functions on a two-dimensional space of alternatives may be illustrated in a simple diagram. This diagram is used in this section to offer an informal presentation of the main point of the paper that is presented more precisely in Section 4.

The Weak Pareto property of social welfare functions requires that the group preference rank one alternative strictly above another whenever every individual does. In Fig. 3, an indifference curve for each agent is given in bold for which a is ranked above b. This is also the case for the indifference curve of the group preference, shown by the dotted line. Indeed, for the social preferences illustrated, any

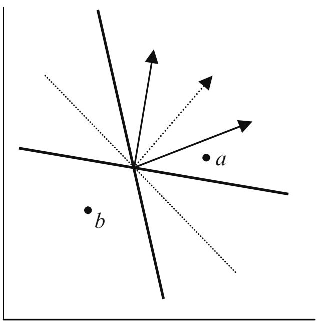
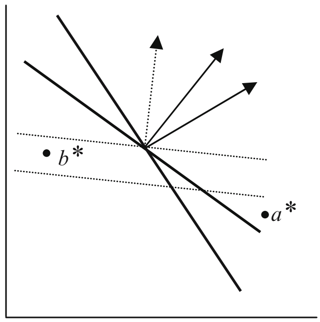


Fig. 3

[2] Though the same sort of parenthesis is used for both intervals of real numbers and vectors in $\mathbb{R}^2$, confusion is avoided by explicitly designating vectors, for example by writing, “the vector $(0,1)$”.

[3] That is, with respect to the relative topologies given the Euclidean topologies on $\mathbb{R}^4$ which contains the domain and $\mathbb{R}^2$ which contains the range.

Fig. 4



alternative ranked by both individual agents above another is also ranked above it by the group preference. In fact, this must be the case for all group preferences whose unit normal is contained in the cone spanned by the agents' unit normals. This is shown by the arrows in Fig. 3. Now, consider the case shown in Fig. 4. Both agents rank a^* above b^*, but the group ranks these alternatives in the opposite way. In this case, the unit normal for the group is not in the cone spanned by the agents' unit normals.

Alternatively but equivalently, for agents' preferences $x, y \in S^1$, the group preference must be on the shortest arc in S^1 from x to y. For example, in the case shown in Fig. 2, the group preference must be on the bold arc going anti-clockwise from x to y, and its distance δ' from x along this arc must satisfy $0 \leq \delta' \leq \delta$.

To illustrate a continuous weakly Paretian social welfare function, consider an arbitrary $x \in S^1$, and $f(x, s(x, \delta))$ as δ varies from 0 to 2π. This is shown in Fig. 5 in which values of δ are shown on the horizontal axis and the anticlockwise distance, $\rho_{f(x,s(x,\delta))} - \rho_x$, of the social preference from x is shown on the vertical axis.

The relevant details are all shown in the square with sides of length 2π, which is sub-divided into four sub-squares each with sides of length π. As δ goes from 0 to 2π on the horizontal axis, the height of the S-shaped curve, shown by a continuous line from the point $(0,0)$ to the point $(2\pi, 2\pi)$, shows the anticlockwise distance of the social preference around S^1 from x. At the point $(0,0)$ agents 1 and 2 both have preferences given by $x \in S^1$, and this is also the case at the point $(2\pi, 2\pi)$. At $\delta = \pi$, agent 2's preference is exactly opposite 1's preference in S^1 and exactly the same as the social preference since the S-curve goes through the point (π, π).

Now consider values of δ between 0 and π. In this case, the height of the S-curve is less than the height of the diagonal. This implies that in S^1, the anticlockwise distance from x to the social preference is less that that to 2's preference, thus satisfying the requirement of the cone restriction. This is also true for values of δ between π and 2π. In this case, the height of the S-curve is greater than the height of the diagonal. This implies that in S^1, the anticlockwise distance from x to the social preference is greater than that to 2's preference, and again the requirement of the cone restriction is satisfied.

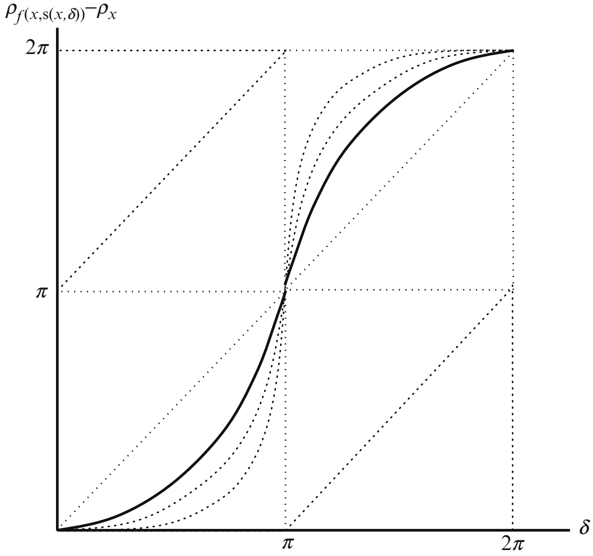


Fig. 5

Another crucial feature of the social welfare function illustrated by the S-curve is that the social preference is never the exact opposite of 2's preference. That is, the point in S^1 that gives the social preference is never exactly opposite the point that gives agent 2's preference. If it were, it would intersect the diagonals of the northwest or southeast sub-squares in Fig. 5, shown by dotted lines.

Now consider the case of a social welfare function that is illustrated by the diagonal of the square. In this case, as the preference of agent 2 rotates anticlockwise from x, the social preference also goes through exactly the same rotation. That is, the preferences of society and agent 2 are always identical. If this is the case for all possible preferences $x \in S^1$ that agent 1 could have, then this social welfare function is *dictatorial* and agent 2 is the *dictator*.

Finally, a crucial role is played by two continuous deformations of the S-curve. In one of these, the continuous S-curve is continuously deformed into the diagonal. Just continuously raise the S-curve for all $\delta \in (0,\pi)$ and lower it for all $\delta \in (\pi,2\pi)$. Such pairs of functions that can be continuously deformed into each other are called *homotopic* functions. Thus, the social welfare function illustrated by the S-curve in Fig. 5 and the social welfare function illustrated by the diagonal are homotopic. Furthermore, agent 2 is then called a *homotopic dictator* for the social welfare function illustrated by the S-curve. Indeed, there must be a homotopic dictator by Chichilnisky's theorem.

The other important observation is that the social welfare function illustrated by the S-curve may also be continuously deformed as shown by the broken lines in Fig. 5. For this continuous deformation, for all $\delta \in (0,\pi)$, the heights of the curves shown by the broken lines decrease towards 0 and for all $\delta \in (\pi,2\pi)$, the heights of the curves shown by the broken lines increase towards 2π. For this class of deformations of the S-curve, apart from its end points, only the point (π,π) remains constant. In other words, it may be concluded that, *if agents do not have opposite preferences, the group preference may be made arbitrarily close to the preference of agent* 1*, even though agent* 2 *remains a homotopic dictator*. It is only if agents have opposite preferences that agent 2 is necessarily asymmetrically powerful.

4 Results

This section makes precise concepts that are used informally in Sect. 3 and the results given in this section justify the conclusion of the Sect. 3.

Projection functions on $S^1 \times S^1$ are continuous social welfare functions that have a special role. They are functions $p_i : S^1 \times S^1 \to S^1, i = 1,2$, such that, for all $x, y \in S^1$, $p_1(x,y) = x$ and $p_2(x,y) = y$. For the social welfare function p_i, $i = 1,2$, i is called the *dictator*. Note that if agent 2 is a *dictator* then, for all $x \in S^1$ and all $\delta \in [0,2\pi]$, $f(x,s(x,\delta)) = s(x,\delta)$.[4]

The concept of homotopic dictatorship first requires the concept of homotopic functions. For arbitrary continuous functions F,G from A to B, F and G are *homotopic* if and only if there is a continuous function $h : A \times [0,1] \to B$ such that, for all $a \in A$, $h(a,0) = F(a)$ and $h(a,1) = G(a)$. Thus, homotopic functions F and G may be continuously deformed into each other. For a social welfare function $f : S^1 \times S^1 \to S^1$, agent $i, i = 1,2$, is a *homotopic dictator* if and only if f and p_i are homotopic. A dictator is a homotopic dictator but not necessarily vice versa.

Next, the cone restriction is made precise. For two agents the satisfaction of the cone restriction is equivalent to the Weak Pareto property, though for more than two agents it is strictly weaker though still sufficient for Chichilnisky's theorem.

For all $x \in S^1$ and $\delta \in [0,2\pi]$, the *closed circular cone* spanned by x and $s(x,\delta)$ is defined as follows:

$$C(x,s(x,\delta)) = \begin{cases} \{y \in S^1 : y = s(x,\delta'), 0 \le \delta' \le \delta\} \text{ if } 0 \le \delta < \pi, \\ \{y \in S^1 : y = s(x,\delta'), \delta \le \delta' \le 2\pi\} \text{ if } \pi < \delta \le 2\pi. \end{cases} \tag{2}$$

A social welfare function $f : S^1 \times S^1 \to S^1$ satisfies the *cone restriction* if and only if, for all $x \in S^1$ and $\delta \in [0,2\pi] \setminus \{\pi\}$, $f(x,s(x,\delta)) \in C(x,s(x,\delta))$. That is, as long as the agents do not have opposite preferences, the social preference is on the shortest arc between them. Note that if agents have opposite preferences so that $\delta = \pi$, the cone restriction does not restrict the social preference. Finally, as noted already, a social welfare function has the *Weak Pareto* property if and only if it satisfies the cone restriction.

The class of social welfare functions that are illustrated in Fig. 5 may now be defined as follows. For all real numbers t, $t \ge 1$:

[4] Dictatorship of agent 1 would require $f(x,s(x,\delta)) = x$.

$$f_t(x,s(x,\delta)) = \begin{cases} s(x,\pi^{1-t}\delta^t) \text{ if } \delta \in [0,\pi], \\ s(x,2\pi - \pi^{1-t}(2\pi-\delta)^t) \text{ if } \delta \in [\pi,2\pi], \end{cases} \tag{3}$$

$f_t : S^1 \times S^1 \to S^1$ are easily shown to be continuous, and their properties are established by the following results.

Proposition 1. *For all $t \geq 1$ and all $x \in S^1$: (i) $f_t(x,s(x,0)) = x$; (ii) $f_t(x,s(x,2\pi)) = x$ and (iii) $f_t(x,s(x,\pi)) = s(x,\pi)$.*

Proof. (i) Substituting $\delta = 0$ into the first part of (3) and then using (1) gives $f_t(x,s(x,0)) = s(x,0) = x$. A similar argument substituting $\delta = 2\pi$ into (3) and again using (1) proves (ii). For (iii), substitute $\delta = \pi$ into both parts of (3) gives what is required. For example, substituting into the first part gives $f_t(x,s(x,\pi)) = s(x,\pi^{1-t}\pi^t) = s(x,\pi)$. □

Proposition 2. *For all $t \geq 1$, $f_t : S^1 \times S^1 \to S^1$ satisfies the cone restriction.*

Proof. There are four cases to consider.

(i) $\delta = 0$: Substituting $\delta = 0$ into (2) gives $C(x,s(x,0)) = \{x\}$. Using (1) and (3) now gives $f_t(x,s(x,0)) = x$, so that $f_t(x,s(x,0)) \in C(x,s(x,0))$.

(ii) $\delta = 2\pi$: A similar argument as used in (i) but beginning by substituting $\delta = 2\pi$ into (2) leads to $f_t(x,s(x,2\pi)) \in C(x,s(x,2\pi))$.

(iii) $\delta \in (0,\pi)$: (3) implies that $f_t(x,s(x,\delta)) = s(x,\pi^{1-t}\delta^t)$. Therefore satisfying the cone restriction in this case requires that $0 \leq \pi^{1-t}\delta^t \leq \delta$ from (2). Since π and δ are both positive, $0 < \pi^{1-t}\delta^t$. Since $\delta < \pi$, it follows that $\pi^{1-t}\delta^t < \delta^{1-t}\delta^t = \delta$.

(iv) $\delta \in (\pi,2\pi)$: (3) implies that $f_t(x,s(x,\delta)) = s(x,2\pi - \pi^{1-t}(2\pi-\delta)^t)$. Therefore satisfying the cone restriction in this case requires that $\delta \leq 2\pi - \pi^{1-t}(2-\delta)^t \leq 2\pi$ from (2). Since $\delta \in (\pi,2\pi)$, it follows that $0 < 2\pi - \delta < \pi$. Using the argument in (iii) with $\delta' = 2\pi - \delta$ instead of δ, it follows that $\pi^{1-t}(2\pi-\delta)^t < 2\pi - \delta$ or, rearranging, $\delta < 2\pi - \pi^{1-t}(2-\delta)^t$ which is part of what is required. For the other part, note that $\pi^{1-t}(2\pi-\delta)^t > 0$ since both π and $2\pi - \delta$ are positive. Therefore, $2\pi - \pi^{1-t}(2\pi-\delta)^t < 2\pi$ and this completes the proof. □

Corollary of Propositions 1 and 2: For all $\delta \in [0,2\pi]$, $f_t(x,s(x,\delta)) \neq -s(x,\delta+\pi)$.

That is, the social preference is never the exact opposite of 2's preference.

Proposition 3. *For all $\delta \in (0,\pi) \cup (\pi,2\pi)$, $\lim_{t\to\infty} f_t(x,s(x,\delta)) = x$.*

Proof. There are two cases to consider.

(i) $\delta \in (0,\pi)$. In this case (3) implies $f_t(x,s(x,\delta)) = s(x,\pi^{1-t}\delta^t)$, so that $\lim_{t\to\infty} f_t(x,s(x,\delta)) = \lim_{t\to\infty} s(x,\pi^{1-t}\delta^t)$. From continuity, $\lim_{t\to\infty} s(x,\pi^{1-t}\delta^t) = s(x,\lim_{t\to\infty}\pi^{1-t}\delta^t)$. Since $\pi^{1-t}\delta^t = \pi(\delta/\pi)^t$, $\lim_{t\to\infty}\pi^{1-t}\delta^t = 0$ since $\lim_{t\to\infty}\pi(\delta/\pi)^t = \pi\lim_{t\to\infty}(\delta/\pi)^t$ and $|(\delta/\pi)| < 1$. Therefore, using (1), $\lim_{t\to\infty} f_t(x,s(x,\delta)) = \lim_{t\to\infty} s(x,\pi^{1-t}\delta^t) = s(x,\lim_{t\to\infty}\pi^{1-t}\delta^t) = s(x,0) = x$.

(ii) $\delta \in (\pi,2\pi)$. In this case, (3) implies $f_t(x,s(x,\delta)) = s(x,2\pi - \pi^{1-t}(2\pi-\delta)^t)$. $\lim_{t\to\infty} s(x,2\pi - \pi^{1-t}(2\pi-\delta)^t) = s(x,\lim_{t\to\infty}(2\pi - \pi^{1-t}(2\pi-\delta)^t))$ from continuity, and furthermore, $\lim_{t\to\infty}(2\pi - \pi^{1-t}(2\pi-\delta)^t) = 2\pi - \lim_{t\to\infty}(\pi^{1-t}(2\pi-\delta)^t)$.

Also $\pi^{1-t}(2\pi - \delta)^t = \pi\left(\frac{2\pi-\delta}{\pi}\right)^t$ and $\lim_{t\to\infty}\pi\left(\frac{2\pi-\delta}{\pi}\right)^t = \pi\lim_{t\to\infty}\left(\frac{2\pi-\delta}{\pi}\right)^t$. Therefore, since $|(\frac{2\pi-\delta}{\pi})^t| < 1$, it follows that $\lim_{t\to\infty}\left(\frac{2\pi-\delta}{\pi}\right)^t = 0$, and this implies that $s(x, \lim_{t\to\infty}(2\pi - \pi^{1-t}(2\pi-\delta)^t)) = s(x, 2\pi)$. Therefore, $f_t(x, s(x,\delta)) = s(x, 2\pi) = x$ which completes the proof. □

Since, for all $t, t \geq 1$, $f_t : S^1 \times S^1 \to S^1$ is continuous and satisfies the cone restriction, it follows from Chichilnisky's theorem that either agent 1 or agent 2 must be a *homotopic dictator*. The final result shows that the homotopic dictator is agent 2.

Proposition 4. *For all $t, t \geq 1$, f_t and p_t are homotopic.*

Proof. First, it will be shown that $f_t(x, s(x,\delta)) \neq s(x,\delta)$. If $\delta = \pi$ then this follows from part (iii) of Proposition 1. If $\delta \neq \pi$ and $f_t(x, s(x,\delta)) = -s(x,\delta)$ then the cone restriction would not be satisfied, contrary to Proposition 3. Therefore, for all $x \in S^1$ and all $\delta \in [0, 2\pi]$, $f_t(x, s(x,\delta)) \neq -s(x,\delta)$. Since $s(x,\delta) = p_2(x, s(x,\delta))$, it follows that $f_t(x, s(x,\delta)) \neq -p_2(x, s(x,\delta))$. Given this, the following homotopy between f_t and p_2 is well defined. For all $x \in S^1$, all $\delta \in [0, 2\pi]$ and all $\lambda \in [0,1]$:

$$h_t(x, s(x,\delta), \lambda) = \frac{\lambda f_t(x, s(x,\delta)) + (1-\lambda)p_2(x, s(x,\delta))}{||\lambda f_t(x, s(x,\delta)) + (1-\lambda)p_2(x, s(x,\delta))||}.$$

Recall from the definitions of f_t and p_2 that all values of these functions lie in the unit circle, S^1, and thus unit norms. It is then straightforward to check that, for all $x \in S^1$ and $\delta \in [0, 2\pi]$, $h_t(x, s(x,\delta), 1) = f_t(x, s(x,\delta))$ and $h_t(x, s(x,\delta), 0) = p_2(x, s(x,\delta))$, and also that h_t is continuous as required. □

Propositions 3 and 4 justify and make precise the claim that concludes Sect. 2. Namely, *if agents do not have opposite preferences, the group preference may be made arbitrarily close to the preference of agent 1, even though agent 2 is a homotopic dictator.*

5 Conclusion

One possible reservation about the analysis in this paper is that it is limited to two agents. However, given the nature of the issue, it is only necessary to establish the conclusion for a simple case, and this has been accomplished. Indeed, Chichilnisky's theorem is not an Arrow-type impossibility result in the sense that it shows that desirable properties entail an undesirable concentration of power.

It may be argued that a homotopic dictator is also a strategic manipulator in the sense of being able to get any particular social preference, for all preferences of other agents. This is indeed the case. It can be seen from Fig. 5 and easily checked from (3), that, for all $t, t \geq 1$, and all $x \in S^1$, $f_t(x, s(x, [0, 2\pi])) = S^1$. Thus, for any possible preference, agent 2 can choose a possibly different preference so that the former is the social preference. This does concentrate a certain sort of power in agent 2. However, if strategic manipulation is of concern, then conditions for its existence can be given directly, and there seems to be no purpose served by tying it to an analysis of homotopic dictatorship.

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References

Baigent, N. (2003). Topological theories of social choice. In K. Arrow, A. Sen, & K. Suzumura (Eds.), *Handbook of social choice and welfare: Vol. 2.* Amsterdam: Elsevier

Chichilnisky, G. (1982). The topological equivalence of the Pareto condition and the existence of a dictator. *Journal of Mathematical Economics*, *9*, 223–234

Heal, G. (1997). Social choice and resource allocation: A topological perspective. *Social Choice and Welfare*, *14*, 147–160

Lauwers, L. (2000). Topological social choice. *Mathematical Social Sciences*, *40*, 1–39

MacIntyre, I. D. A. (1998). Two-person and majority continuous aggregation in 2-good space in social choice: A note. *Theory and Decision*, *44*, 199–209

Saari, D. G. (1997). Informational geometry of social choice. *Social Choice and Welfare*, *14*, 211–232

Sen, A. K. (1986). Social choice theory. In K. J. Arrow & M. D. Intriligator (Eds.), *Handbook of mathematical economics, vol. iii* (pp. 1073–1181). Amsterdam: North-Holland