Topological Semantics of Justification Logic

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Abstract. The Justification Logic is a family of logical systems obtained from epistemic logics by adding new type of formulas $t{:}F$ which reads as *t is a justification for F*. The major epistemic modal logic S4 has a well-known Tarski topological interpretation which interprets $\Box F$ as the interior of F (a topological equivalent of the '*knowable part of F*'). In this paper we extend the Tarski topological interpretation from epistemic modal logics to justification logics which have both: knowledge assertions $\Box F$ and justification assertions $t{:}F$. This topological semantics interprets modality as the interior, terms t represent tests, and a justification assertion $t{:}F$ represents a *part of F which is accessible for test t*. We establish a number of soundness and completeness results with respect to Kripke topology and the real line topology for S4-based systems of Justification Logic.

Keywords: Justification Logic, Logic of Proofs, modal logic, topological semantics, Tarski.

1 Introduction

The Justification Logic is a family of logical systems originated from the Logic of Proofs LP (cf. [3,5,9,10]). These systems are obtained from epistemic modal logics by adding new type of formulas $t{:}F$ which read as

t is a justification for F.

Justification Logic overlaps mathematical logic, epistemology, λ-calculi, etc. The standard arithmetical provability semantics for LP was given in [3]. The epistemic Krike-style semantics for LP was offered in [16,17] and later extended to Justification Logic systems containing both epistemic modalities for "*F is known*" and justification assertions "*t is a justification for F*" ([9,10]). The major epistemic modal logic S4 which in the context of the Logic of Proofs may be regarded as a logic of explicit provability has a well-known Tarski's topological interpretation. Such a connection between topology and modal logic proved to be very fruitful

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for both domains. In particular, topology was used in [25] to prove Gödel's conjecture about a fair embedding of Intuitionistic Logic to Modal Logics. On the other hand, Modal Logic was used to describe the behavior of dynamic systems in real topology ([14]).

2 Background

The application of modal logic to topology dates back to Kuratowski [21] and Riesz in [32]. Let

$$\mathcal{T} = \langle \boldsymbol{X}, \mathbb{I} \rangle$$

be a topological space, where $\boldsymbol{X}$ is a set and $\mathbb{I}$ the interior operation. The following principles hold for all subsets Y and Z of $\boldsymbol{X}$:

1. $\mathbb{I}(Y \cap Z) = \mathbb{I}Y \cap \mathbb{I}Z$;
2. $\mathbb{I}Y = \mathbb{I}\mathbb{I}Y$;
3. $\mathbb{I}Y \subseteq Y$;
4. $\mathbb{I}\boldsymbol{X} = \boldsymbol{X}$.

These principles can be written as propositional modal formulas: Boolean operations are represented by the corresponding Boolean connectives, and the interior operator $\mathbb{I}$ by the modality $\Box$:

1. $\Box(A \wedge B) = \Box A \wedge \Box B$;
2. $\Box A \rightarrow \Box\Box A$;
3. $\Box A \rightarrow A$;
4. $\Box\top$.

These are the well-known postulates of the modal logic $\mathsf{S4}$. This corellation was noticed in the late 1930s by Tarski, Stone, and Tang. Neither Lewis' original motivation of modal logic ([22,23]), nor Gödel's provability interpretation of $\mathsf{S4}$ ([18]) were related to topology.

The **Tarski topological interpretation** of a propositional modal language naturally extends the set-theoretical interpretation of classical propositional logic. Given a topological space $\mathcal{T} = \langle \boldsymbol{X}, \mathbb{I} \rangle$ and a valuation (mapping) $*$ of propositional letters to subsets of $\boldsymbol{X}$, we can extend it to all modal formulas as follows:

$$\begin{gathered} \neg A = \boldsymbol{X} \setminus A^*; \\ (A \wedge B)^* = A^* \cap B^*; \\ (A \vee B)^* = A^* \cup B^*; \\ (\Box A)^* = \mathbb{I}A^*. \end{gathered} \tag{1}$$

A formula A is called valid in $\mathcal{T}$ (notation: $\mathcal{T} \Vdash A$) if

$$A^* = \boldsymbol{X}$$

for any valuation $*$. The set

$$\mathbf{L}(\mathcal{T}) := \{A \mid \mathcal{F} \Vdash A\}$$

is called the modal logic of $\mathcal{T}$.

The following classical result in this area is due to McKinsey and Tarski:

Theorem. ([24]) *Let $\mathcal{S}$ be a separable dense-in-itself metric space. Then* $\mathbf{L}(\mathcal{S}) = \mathsf{S4}$.

In particular, this yields that for each $n = 1, 2, 3, \ldots$,

$$\mathbf{L}(\mathbb{R}^n) = \mathsf{S4}.$$

Simplified proofs of this theorem were obtained in [12,26,34].

Kripke semantics can be regarded a special case of topological semantics. Indeed, given a Kripke frame (W, R), one can construct the topological space $(W, \mathbb{I})$ where

$$\mathbb{I}U := \{x \mid R(x) \subseteq U\},$$

so that validities in these two entities are the same. Hence, Kripke-completeness yields the topological completeness.

As we have mentioned above, the Justification Logic grew from the Logic of Proofs LP. A first incomplete sketch of the Logic of Proofs was made in Gödel's lecture of 1938 [19], which was not published until 1995 when the full Logic of Proofs was rediscovered independently in [2]. The Logic of Proofs LP ([2,3,4,6,15]) introduces the notion of *proof polynomials*, i.e., terms built from proof variables and constants by means of three operations:

- *application* "·", which given a proof s of an implication $F \to G$ and a proof t of its antecedent F provides a proof $s \cdot t$ of the succedent G;
- *sum* "+", which given proofs s and t returns a proof $s + t$ of everything proven by s or t;
- *proof checker* "!", which given a proof t of F verifies it and provides a proof $!t$ of the fact that t is indeed a proof of F.

LP is the classical logic with additional atoms

$$p{:}F$$

where p is a proof polynomial and F is a formula, with the intended reading

$$p \textit{ is a proof of } F\,.$$

As it was shown in [2,3], LP describes all valid principles of proof operators $t{:}F$

$$t \textit{ is a proof of } F \textit{ in Peano Arithmetic}$$

in its language. LP is able to realize the whole Gödel's S4 by recovering proof polynomials for provability assertions in any S4-derivation (realization theorem); this result provides a mathematical formalization of the Brouwer-Heyting-Kolmogorov semantics for intuitionistic logic IPC via well-known Gödel's translation of IPC into S4 [2,3,4,6,15]. The papers [1,8,28,29,30,31,33,35] studied joint logics of proofs and provability in a format that includes both provability assertions $\Box F$ and proof assertions $t{:}F$.

In [5,8,9,10] this approach has been extended to epistemic logic and applied for building mathematical models of justification, knowledge and belief. In particular, [9] introduced and studied the basic epistemic logic with justifications,

$$\mathsf{S4LP} = \mathsf{S4} \quad + \quad \mathsf{LP} \quad + \quad (t{:}F \rightarrow \Box F)\ .$$

Epistemic models for Justification Logics has been developed in [5,8,9,10, 16,17,27]. A Fitting model for S4LP is $(W, R, \mathcal{A}, \Vdash)$, where

- (W, R) is an S4-frame;
- $\mathcal{A}$ is an admissible evidence function: for each term t and formula F, $\mathcal{A}(t, F)$ is a subset of W. Informally, $\mathcal{A}(t, F)$ specifies a set of worlds where t is an admissible evidence for F. An evidence function is assumed to be monotonic:

  $$u \in \mathcal{A}(t, F) \text{ and } uRv \text{ yield } v \in \mathcal{A}(t, F)$$

  and has natural closure properties that agree with operations of S4LP;
- $\Vdash$ behaves in the standard Kripke style on Boolean connectives and $\Box$:
  - $u \Vdash P$ or $u \not\Vdash P$ is specified for each world u and each propositional variable P;
  - $u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$, $u \Vdash F \vee G$ iff $u \Vdash F$ or $u \Vdash G$, $u \Vdash \neg F$ iff $u \neg \Vdash F$;
  - $u \Vdash \Box F$ iff $v \Vdash F$ for all v such that uRv;
- $u \Vdash t{:}F$ iff $u \Vdash \Box F$ and $u \in \mathcal{A}(t, F)$.

In [8,17], S4LP is shown to be sound and complete with respect to this epistemic semantics.

3 Topological Semantics for Justifications

We start with offering a topological semantics for operation-free single-modality Justification Logics. It means we will work with the usual language of propositional modal logic enriched by a new construction $t : F$ where t is a **proof variable** and F is a formula.

An interpretation is specified for a topological space $\mathcal{T} = \langle \boldsymbol{X}, \mathbb{I} \rangle$ supplied with a *test function* $\mathcal{M}$ which maps a term t and a formula F to $\mathcal{M}(t, F) \subseteq \boldsymbol{X}$. The informal meaning of $\mathcal{M}$ is that $\mathcal{M}(t, F)$ represents a 'potentially accessible' region of $\boldsymbol{X}$ associated with F and t.

We assume that an evaluation $*$ works on propositional variables, Boolean connectives and modality $\Box$ according to the usual aforementioned Tarski interpretation (1). We will study several natural ways to extend $*$ on formulas $t{:}F$ and corresponding subsystems of S4LP. This approach was first discussed in [11].

We build our topological semantics for the Justification Logic language on the following formal and informal assumptions.

1. Our semantics is based on Tarski's topological semantics (1), e.g.,

$$(\Box F)^* = \mathbb{I}(F^*).$$

2. Justification terms are symbolic representations of *tests*. We postulate existence of a test function $\mathcal{M}$ which for each t and F specifies a set of points $\mathcal{M}(t,F)$ which we call

   the set of possible outcomes of a test t of a property F.

3. The $t{:}F$ will return a set of points where a test t confirms F. This reading will be supported by definitions (for different subsystems of $\mathsf{S4LP}$):

$$(t{:}F)^* \ = \ F^* \cap \mathcal{M}(t,F) \tag{2}$$

   or

$$(t{:}F)^* \ = \ \mathbb{I}(F^*) \cap \mathcal{M}(t,F). \tag{3}$$

   In case (2), test t supports F at all points where the possible outcome of t lies inside F. Case (3) corresponds to the "robust" understanding of testing: test t supports F at all points of the possible outcome of t which lie in the interior of F.
4. We first consider systems without operations on tests.

Now we introduce several systems of Justification Logic and simultaneously define their topological semantics in format $(\mathcal{T},\mathcal{M})$ where $\mathcal{T}=\langle \boldsymbol{X},\mathbb{I}\rangle$ is a topological space and $\mathcal{M}$ is a test function.

3.1 Basic Testing System $\mathsf{S4B_0}$

The most basic system in our list is

$$\mathsf{S4B}_0 \ = \ \mathsf{S4} \ + \ (t{:}F \rightarrow F).$$

In this system, there are no any assumptions about tests; they don't necessarily produce open sets of outcomes. The topological interpretation of $\mathsf{S4B}_0$ combines Tarski topological interpretation (1) for Booleans and modality $\Box$ with the interpretation of the justification assertions like in (2), i.e.,

$$(t{:}F)^* \ = \ F^* \cap \mathcal{M}(t,F).$$

3.2 Robust Testing System $\mathsf{S4B_1}$

The next system under consideration is

$$\mathsf{S4B}_1 \ = \ \mathsf{S4} \ + \ (t{:}F \rightarrow \Box F).$$

In $\mathsf{S4B}_1$, test sets are not necessarily open; however, the justification assertions are interpreted as "robust inclusion", i.e., case (3) :

$$(t{:}F)^* \ = \ \mathbb{I}(F^*) \cap \mathcal{M}(t,F)$$

3.3 Robust Open Testing System $\mathsf{S4B_2}$

Finally, we consider

$$\mathsf{S4B_2} \;=\; \mathsf{S4} \;+\; (t{:}F \rightarrow F) \;+\; (t{:}F \rightarrow \Box t{:}F).$$

This system corresponds to the full operation-free version of $\mathsf{S4LP}$. The test sets are assumed to be open, the justification assertions are interpreted in the robust sense (3):

$$(t{:}F)^* \;=\; \mathbb{I}(F^*) \cap \mathcal{M}(t, F)$$

3.4 Topological Soundness and Completeness

Theorem 1. All three systems $\mathsf{S4B_0}$, $\mathsf{S4B_1}$, and $\mathsf{S4B_2}$ are sound and complete with respect to the corresponding classes of topological models.

Proof. The soundness proofs are straightforward. In view of the Tarski topological interpretation of $\mathsf{S4}$ ([24]), it suffices to establish validity of non-$\mathsf{S4}$ principles of $\mathsf{S4B_0}$, $\mathsf{S4B_1}$, and $\mathsf{S4B_2}$ in the corresponding cases.

Principle $t{:}F \rightarrow F$ is valid in both (2) and (3) since in each case $(t{:}F)^*$ is a subset of F^*. Principle $t{:}F \rightarrow \Box F$ is valid in (3) since $(t{:}F)^*$ is a subset of $(\Box F)^*$, which is the interior of F^*. Finally, $t{:}F \rightarrow \Box t{:}F$ is valid in (3) since the test sets are open hence $(t{:}F)^*$ are all open and coincide with their interiors.

Completeness proofs go via epistemic models which are then converted into topological spaces with topology induced by the Kripke accessibility relation.

We consider the case of $\mathsf{S4B_2}$, the remaining cases are receiving a similar treatment. Let us first establish the completeness of $\mathsf{S4B_2}$ with respect to the class of Fitting models $(W, R, \mathcal{A}, \Vdash)$ without operations on justifications.

We follow the standard canonical model construction.

- W is the set of all maximal consistent sets in $\mathsf{S4B_2}$. We denote elements of W as Γ, Δ, etc.;
- $\Gamma R \Delta$ iff $\Gamma^\sharp \subseteq \Delta$, where $\Gamma^\sharp = \{\Box F \mid \Box F \in \Gamma\}$;
- $\mathcal{A}(s, F) = \{\Gamma \in W \mid s{:}F \in \Gamma\}$;
- $\Gamma \Vdash p$ iff $p \in \Gamma$.

Let us check that $(W, R, \mathcal{A}, \Vdash)$ is indeed an $\mathsf{S4B_2}$-model. It is immediate from the definitions that the accessibility relation R is reflexive and transitive. The admissible evidence function $\mathcal{A}$ is monotonic. Indeed, suppose $\Gamma \in \mathcal{A}(t, F)$ and $\Gamma R \Delta$. Then $t{:}F \in \Gamma$, $\Box t{:}F \in \Gamma$, $\Box t{:}F \in \Delta$, and $t{:}F \in \Delta$, i.e. $\Delta \in \mathcal{A}(t, F)$.

Lemma 1 (Truth Lemma). For every formula F, $\Gamma \Vdash F$ iff $F \in \Gamma$.

Proof. Induction on F. The base case in given in the definition of the canonical model. The Boolean and modality cases are standard. Let us consider the case when F is $t{:}G$.

Let $t{:}G \in \Gamma$ and $\Gamma R \Delta$. Then $\Box t{:}G \in \Gamma$, $\Box t{:}G \in \Delta$, $t{:}G \in \Delta$ (since $\Box t{:}G \rightarrow t{:}G$), and $G \in \Delta$ (since $t{:}G \rightarrow F$). By the Induction Hypothesis, $\Delta \Vdash G$.

Furthermore, by the definition of the admissible evidence function, $\Gamma \in \mathcal{A}(t, G)$, hence $\Gamma \Vdash t{:}G$.

If $t{:}G \notin \Gamma$, then $\Gamma \notin \mathcal{A}(t, G)$, hence $\Gamma \not\Vdash t{:}G$.

Let us now finish the proof of completeness of $\mathsf{S4B}_2$ with respect to $\mathsf{S4B}_2$-models. Suppose $\mathsf{S4B}_2 \not\vdash F$. Then the set $\{\neg F\}$ is consistent, and hence included into some maximal consistent set Γ. Naturally, $F \notin \Gamma$. By the Truth Lemma, $\Gamma \not\Vdash F$.

Now we convert a given countermodel $\mathcal{K} = (W, R, \mathcal{A}, \Vdash)$ for F into an appropriate topological space and find an interpretation under which F does not hold. A Kripke topological space $\mathcal{T}_{\mathcal{K}}$ associated with $\mathcal{K}$ is a topological space with the carrier W and open sets which are all subsets of W closed upward under R:

$$Y \textit{ is open iff for all } u \in Y\textit{, if } uRv \textit{ then } v \in Y.$$

To make $\mathcal{T}_{\mathcal{K}}$ a topological $\mathsf{S4B}_2$-model it remains to define a test function

$$\mathcal{M}(t, F) = \mathcal{A}(t, F).$$

Given a Fitting model $\mathcal{K} = (W, R, \mathcal{A}, \Vdash)$ for $\mathsf{S4B}_2$ we can also define a topological interpretation $*$ of $\mathsf{S4B}_2$-language in $\mathcal{T}_{\mathcal{K}}$:

$$p^* = \{u \in W \mid u \Vdash p\} \text{ for a propositional letter } p.$$

Any interpretation $*$ is extended to all $\mathsf{S4B}_2$-formulas in the standard way:

- $(A \vee B)^* = A^* \cup B^*$;
- $(\neg A)^* = W \setminus A^*$;
- $(\Box A)^* = \mathbb{I}(A^*)$;
- $(t{:}A)^* = \mathbb{I}(A^*) \cap \mathcal{M}(t, A)$.

From the definitions it is immediate that $t{:}G \to G$ holds at this model. Note that due to monotonicity of the admissible evidence function $\mathcal{A}$, for each t and F the test sets $\mathcal{M}(t, G)$ are open in $\mathcal{T}_{\mathcal{K}}$. Therefore $t{:}G \to \Box t{:}G$ also holds at the model.

Lemma 2 (The Main Lemma)

$$u \Vdash G \quad \Leftrightarrow \quad u \in G^*$$

Proof. Induction on G. The base case when G is atomic is covered by the definition. The Boolean connective case is straightforward.

Let G be $\Box B$. Suppose $u \Vdash \Box B$, then for all $v \in W$ such that uRv, $v \Vdash A$ as well. By the Induction Hypothesis, $v \in B^*$ for all $v \in W$ such that uRv. This yields that the whole open cone $O_u = \{v \mid uRv\}$ is a subset of B^*. Therefore, $u \in \mathbb{I}(B^*) = (\Box B)^*$.

Suppose $u \in (\Box B)^* = \mathbb{I}(B^*)$. Since $\mathbb{I}(B^*)$ is open, $v \in \mathbb{I}(B^*)$ hence $v \in B^*$ for all v such that uRv. By the Induction Hypothesis, $v \Vdash B$ for all v such that uRv. Therefore, $u \Vdash \Box B$.

Let G be $t{:}B$. Suppose $u \Vdash t{:}B$. Then, by definition, $u \in \mathcal{A}(t, B)$ and $v \Vdash B$ for all v such that uRv. By the definition of a test function, $u \in \mathcal{M}(t, B)$. By the Induction Hypothesis, $v \in B^*$ for all v such that uRv, which means that $u \in \mathbb{I}(B^*)$. Hence $u \in \mathbb{I}(B^*) \cap \mathcal{M}(t, B)$, i.e., $u \in (t{:}B)^*$.

Suppose $u \in \mathbb{I}(B^*) \cap \mathcal{M}(t, B)$. Then $u \in \mathcal{M}(t, B)$ hence $u \in \mathcal{A}(t, B)$. Furthermore, $u \in \mathbb{I}(B^*)$. Like in the case $G = \Box B$, we conclude that $u \Vdash \Box B$. Altogether this yields $u \Vdash t{:}B$.

To conclude the proof of Theorem 1 consider a Fitting $\mathsf{S4B}_2$-model, where $u \nVdash F$. By the Main Lemma, $u \notin F^*$, hence F is not valid in the topological $\mathsf{S4B}_2$-model $\mathcal{T}_{\mathcal{K}}$.

3.5 Completeness with Respect to Real Topology

Theorem 2. $\mathsf{S4B}_0$, $\mathsf{S4B}_1$, $\mathsf{S4B}_2$ are complete with respect to the real topology $\mathbb{R}^n$.

Proof. We will use the following main lemma from recent refinements of the Tarski Theorem from [12,26,34] :

Lemma 3. There is an open and continuous map π from $(0, 1)$ onto the Kripke topological space corresponding to a finite rooted Kripke frame.

Such a map π preserves truth values of modal formulas at the corresponding points. It suffices now to refine the proof of Theorem 1 to produce a finite rooted Fitting counter-model for F and to define the test function $\mathcal{M}'(t, G)$ on $(0, 1)$ as

$$\mathcal{M}'(t, G) = \pi^{-1}\mathcal{M}(t, G).$$

The resulted topological model is a $(0, 1)$-countermodel for F. This construction yields completeness with respect to the real topology $\mathbb{R}^n$, for each $n = 1, 2, 3, ...$

4 Future Work

The next natural steps in this direction could be introducing operations on tests. It also looks promising to introduce tests in systems of topological reasoning about knowledge [13] and Dynamic Topological Systems [7,20].

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